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MODIFIED LAGUERRE COLLOCATION BLOCK METHOD FOR SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper, the derivation of block procedure for linear multi-step methods (K=2) using the Laguerre polynomials as the basis functions was considered. Discrete methods was given which were used in block and implemented for solving the initial value problems, being the continuous interpolation derived and collocated at grid points. The derived scheme was used to solve some second order ordinary differential equations (ODEs) in order to show their validity and accuracy .The numerical results obtained shows that the proposed methods are efficient in solving second order ordinary differential equations.

Keywords: Linear Multi-step Method, Laguerre Polynomials, Second Order Ordinary Differential Equations, Basis Function, Grid Point.
Classification: Numerical

INTRODUCTION

The analytical solutions of ordinary differential equations of the form:

$$f\left(x, y, y', y'', ..., y^{(n)}\right) = 0, \; y(x) = y_0, \; y'(x) = y_1, ..., \; y^{(n-1)}(x) = y_n \qquad (1)$$

have been difficult to obtain, especially when the equation is nonlinear. In order to obtain an approximate solution to the problem, numerical methods are resorted to. Also, solutions to differential equations which always occur in series form have been possibly obtained in closed forms as a result of numerical methods. Hence, various numerical techniques such as finite difference methods (FDM), integral transform methods (ITM) and linear multi-step methods (LMM) have been developed by many researchers to provide approximate solutions to both ordinary and partial differential equations. For instance, the power series expansion solution in the form of convergent series was considered in (Abualnaja, 2015) to obtain analytical solutions of partial differential equations. Also, in Awoyemi (2001), iterative methods were developed for solving initial value problems using approximate solutions with series form in terms of parametric form. The methods are always developed based on the nature and the type of the differential equation under consideration.

Ordinary differential equations of higher orders always give rise to discrete variable methods known as the single step and multi-step methods which are otherwise known as linear multi-step methods.

The single step methods such as Euler's and Runge-Kutta methods among others are purposely developed to solve first order (IVP) of ordinary differential equations. Solving these kinds of differential equations by implementing the single-step methods require that the differential is reduced to a system of first order initial value problems (IVP). However, eminent researchers like (Awoyemi, 2003; Chu & Hamilton, 1987; El-Ajou, Abu & Momani, 2015) have attempted to solve the differential equation using (LMM) without reducing to a system of a first order ODEs.

Moreover, El-Ajou *et al.* (2015) proposed (LMM) for the first order differential equations in the predictor-corrector mode, using the power series as basis function. Continuous (LMM) ensures easy approximation of solution at all points of the integration interval (Fatunla, 1994), based on the collocation method, Isamil, Ken, and Othman (2009) proposed a two-step hybrid method for the solution of a first order (IVP) at chosen grid points which was implemented on the hybrid predictor-corrector mode. Other researchers who also studied the hybrid method include Jator (2007), Jator and Li (2009). The development of a class of methods called block method is an outcome of improving the numerical solution of (IVPs) of ordinary differential equations. It was first proposed by Kayode (2004), afterwards, many scholars have been working on the implementation of the block methods for the numerical solution of ordinary differential equations. Among them are Lambert (1973), Milne (1953), Okedayo, Owolanke, Amumeji, and Ogunbamike (2018), Omar and Suleiman (1999, 2003).

In the recent years, Onumanyi, Sirisena, and Dauda (2001); Sararfyan (1990) proposed five-step and four-step (LMM) respectively to obtain a (FDM) as a block for the direct solution of first order (IVP), also, block technique with (LMM) adopting Legendre polynomials as basis function was recently developed by Onumanyi *et al.* (2001), but the block schemes were not included. Thus, in this work, Laguerre polynomial is considered as basis function.

Abualnaja (2015) developed a block procedure with linear multi-steps using Legendre polynomials but did not include the block schemes. Okedayo *et al.* (2018) made an extension to Abualnaja (2015) work, using Legendre polynomial as a basis function to derive some block methods for the solution of first order ordinary differential equation. Okedayo *et al.* (2018) also worked on the same principle using Laguerre polynomials as a basis function to solve the same problems. Thus, in this paper, Laguerre polynomial is used as a basis function to derive some block methods for the solution of second order ordinary differential equation, which is an extension of Okedayo *et al.* (2018).

Definition: Linear Multi-step method

A LMM with k-step size have the form

$$y_n = \sum_{i=0}^{n} c_i x_{n+i} + h\sum_{i=0}^{k} \beta_i f_{n+i}$$

Where c_i and β_j are constants, y_n is the numerical solution at $x = x_n, f_n = f(x_n, y_n)$. If $\beta_k \neq 0$, the LMM becomes implicit scheme, otherwise explicit, (Yahaya & Badmus, 2009).

Definition: Zero Stability

The LMM (1) is said to satisfy the root conditions if all the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple

Derivation of the Method

In the derivation of the method, we consider the approximate solution of the form

$$y_k(x) = \sum_{i=0}^{k} c_i \psi_i(x), \;\; x_n \le x \le x_{n+k}. \quad (2)$$

$$= c_0 + c_1 x_n + \ldots + c_n x_n^n$$

Then we perturb the equation above, we get:

$$\sum_{i=2}^{k} c_i \psi_i^{"}(x) = f\left(x, y, y^{"}\right) + \lambda L_k(x) \quad (3)$$

Where $\psi_i(x) = x^i, \; i = 0,1,\ldots,k$ and $L_k(x)$ is the Laguerre polynomial of degree k, which is defined on the interval $[-1,1]$, and can be determined with the aid of the recurrence formula:

$$L_{i+1}(x) = (-1)^{n+1} e^x \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x} x^{n+1}\right) \quad (4)$$

Where the first four polynomials are:

$$L_0(x) = 1, \; L_1(x) = (x-1), \; L_2(x) = \left(x^2 - 4x + 2\right), \; L_3(x) = x^3 + 9x^2 + 18x - 6,$$
$$L_4(x) = \frac{1}{24}\left(x^4 - 16x^3 + 72x^2 - 96x + 24\right) \quad (5)$$

In other to use these polynomials on the interval $[x_n, x_{n+k}]$, we define the Shifted Laguerre polynomials by introducing the change of variable:

$$x = \frac{2\bar{x} - \left(x_{n+k} + x_n\right)}{x_{n+k} + x_n}, \quad k = 1,2,3 \quad (6)$$

Specification of the method

For the derivation at $k = 2$,

CASE K=2:

we take the polynomial $L_2(x) = x^2 - 4x + 2$ and use (6), then collocate this equation at $x_n, x_{n+1} and\, x_{n+2}$, we obtain

$$L_2(x_n) = \frac{2x_n - (x_{n+1} - x_n)}{x_{n+1} - x_n} \qquad (7)$$

Taking $\bar{x} = x_n,\ x_{n+1} = x_n + h,\ x_{n+2} = x_n + 2h$

$$L_2(x_n) = \frac{2x_n - (x_{n+1} + x_n)}{x_{n+1} + x_n} = -1$$

$$L_2(x_n) = (-1)^2 - 4(-1) + 2 = 1 + 4 + 2 = 7$$

$$L_2(x_{n+1}) = \frac{2x_{n+1} - (x_{n+1} - x_n)}{x_{n+1} - x_n} = 0$$

$$L_2(x_{n+1}) = 1 - 4 + 2 = -2$$

$$L_2(x_{n+2}) = \frac{2x_{n+2} - (x_{n+1} - x_n)}{x_{n+1} - x_n} = 3$$

$$L_2(x_{n+2}) = 9 - 12 + 2 = -1$$

$\psi_1''(x) = 0,\ \psi_2''(x) = 2,\ \psi_3''(x) = 6x$. Then equation (4) will reduce to the form:

$$2c_2 + 6c_3x_n = f(x, y, y') + \lambda L_2(x) \qquad (8)$$

Collocating (7) at x_{n+i}, $i = 0,1,2$ and interpolate (2) at $x = x_n$ a system of 5 equations is obtain c_i, $(i = 0,1,2,3)$ and parameter λ,

$$L_0(x) = 1,\ L_1(x) = (x-1),\ L_2(x) = (x^2 - 4x + 2),\ L_3(x) = x^3 + 9x^2 + 18x - 6,$$

$$L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$

$$a := y(n) = c(0) + c(1)x(n) + c(2)x(n)^2 + c(3)x(n)^3;$$

$$y_n = c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 \qquad (9)$$

$$b := f(n) = c(1) + 2c2x(n) + 3c(3)x(n)^2;$$

$$f_n = c_1 + 2c_2x_n + 3c_3x_n^2 \qquad (10)$$

$$d := p(n) = 2c(2) + 6c(3)x(n) - 7\lambda$$

$$p_n = 2c_2 + 6c_3x_n - 7\lambda \qquad (11)$$

$e := p(n+1) = 2c(2) + 6c(3)x(n) + 6c(3)h - 2\lambda$; where $x(n+1) = x(n) + h$

$$p_{n+1} = 2c_2 + 6c_3x_n + 6c_3h - 2\lambda \qquad (12)$$

$$g := p(n+2) = 2c(2) + 6c(3)x(n) + 12c(3)h + \lambda$$

$$p_{n+2} = 2c_2 + 6c_3x_n + 12c_3h + \lambda \qquad (13)$$

Subtract (13) from (12)

$$j := e - g$$

$$p_{n+1} - p_{n+2} = -6c_3h - 3\lambda \qquad (14)$$

Subtract (12) from (11)

$$k := \exp and(d - e)$$

$$p_n - p_{n+1} = -5\lambda - 6c_3 h \qquad (15)$$

Subtract (11) from (10)

$$l := solve(\{k - j\}, \lambda);$$

$$\left\{\lambda = -\frac{1}{2}p_n + p_{n+1} - \frac{1}{2}p_{n+2}\right\} \qquad (16)$$

Substitute (16) into (15)

$$m := solve\left(\left\{p_n - p_{n+1} = -5\left(-\frac{1}{2}p_n + p_{n+1} - \frac{1}{2}p_{n+2}\right) - -6c_3\right\}, c[3]\right);$$

$$\left\{c_3 = \frac{1}{12} - \frac{3p_n - 8p_{n+1} + 5p_{n+2}}{h}\right\} \qquad (17)$$

Substitute (16) and (17) into (11)

$$\left\{c_2 = -\frac{1}{4}\frac{5p_n h + 3x_n p_n - 8x_n p_{n+1} + 5x_n p_{n+2} - 14p_{n+1}h + 7p_{n+2}h}{h}\right\} \qquad (18)$$

Substitute (16), (17) and (18) into (10)

$$\left\{c_1 = \frac{1}{4}\frac{1}{h}\left(4f_n h + 10x_n p_n h + 3x_n^2 p_n - 8x_n^2 p_{n+1} + 5x_n^2 p_{n+2} - 28x_n p_{n+1}h + 14x_n p_{n+2}h\right)\right\} \qquad (19)$$

Substitute (17), (18) and (19) into (9) and using (2),

$$\left\{c_0 = \frac{1}{12}\frac{1}{h}\left(-12y_n h + 12x_n f_n h + 15x_n^2 p_n h + 3x_n^3 - 8x_n^2 p_{n+1} + 5x_n^3 p_{n+2} - 42x_n^2 p_{n+1}h + 21x_n^2 p_{n+2}h\right)\right\}$$

(20)

We now use (2),

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Substitute (17), (18), (19) and (20) into (2) we have,

$$y_{n+1} = -h^2 p_n + \frac{17}{6}h^2 p_{n+1} - \frac{4}{3}h^2 p_{n+2} + y_n + f_n h \qquad (21)$$

Where p_n, p_{n+1}, p_{n+2} is the second derivative.

The block scheme of case K=2 is therefore:

$$\begin{aligned}
y_{n+1} &= -h^2 p_n + \frac{17}{6}h^2 p_{n+1} - \frac{4}{3}h^2 p_{n+2} + y_n + f_n h \\
y_{n+2} &= y_n + 2f_n h - 3h^2 p_n + \frac{26}{3}h^2 p_{n+1} - \frac{11}{3}h^2 p_{n+2} \\
y_{n+3} &= y_n + 3f_n h - \frac{9}{2}h^2 p_n + \frac{27}{2}h^2 p_{n+1} - \frac{9}{2}h^2 p_{n+2} \\
y_{n+4} &= y_n - 4h^2 p_n + \frac{40}{3}h^2 p_{n+1} - \frac{4}{3}h^2 p_{n+2} + 4f_n h
\end{aligned} \qquad (22)$$

Analysis of the method

The necessary and sufficient conditions for LMM to be convergent are that it must be consistent and zero stable.

Order, Error Constant and Consistency of the Methods

The schemes developed above belong to the class of the Linear Multi-step Method (LMM) of the form:

$$\sum_{j=0}^{k}\alpha_j(x)y(x_{n+j})=h\sum_{j=0}^{k}\beta_j(x)f(x_{n+j}) \tag{23}$$

Equation (23) is a method associated with a linear difference operator

$$L[y(x);h]=\sum_{j=0}^{k}\left(\alpha_j y(x+jh)=h\beta_j y'(x+jh)\right) \tag{24}$$

Where $y(x)$ is continuously differentiable on the interval $[a,b]$, and the Taylor series expansion about the point x is expressed as

$$L[y(x);h]=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+\ldots+c_mh^my^m(x) \tag{25}$$

Confirming [12], scheme (21) is said to be of order P if $C_0=C_1=C_2=\ldots=C_p=0$ and the error constant is $C_{p+1}\neq 0$.

Hence, we establish that (21) is of the following order:

When $k=2,\ P=3$ and $C_{p+1}=0.041667$

Stability Analysis

The scheme for K=2 is expressed as:

$$A^{(0)}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ B^{(0)}=\begin{pmatrix}\frac{17}{6}&-\frac{4}{3}&0&0\\\frac{26}{3}&-\frac{1}{3}&0&0\\\frac{27}{2}&-\frac{9}{2}&0&0\\\frac{40}{3}&-\frac{4}{3}&0&0\end{pmatrix},\ A^{(1)}=\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix},\ B^{(1)}=\begin{pmatrix}0&0&0&-1\\0&0&0&-3\\0&0&0&-\frac{9}{2}\\0&0&0&-4\end{pmatrix}$$

The first characteristics polynomial of the scheme is:

$$\rho(\pi)=\det\left[\pi A^{(0)}-A^{(1)}\right]=\det\left[\begin{pmatrix}\pi&0&0&0\\0&\pi&0&0\\0&0&\pi&0\\0&0&0&\pi\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right]=\det\begin{pmatrix}\pi&0&0&-1\\0&\pi&0&-1\\0&0&\pi&-1\\0&0&0&\pi-1\end{pmatrix}$$

$$=\begin{vmatrix}\pi&0&0&-1\\0&\pi&0&-1\\0&0&\pi&-1\\0&0&0&\pi-1\end{vmatrix}=0$$

$$\pi^3(\pi-1)=0$$

$$\pi_1=\pi_2=\pi_3=0\ \ or\ \ \pi_4=1$$

Numerical Experiments

In order to confirm the accuracy and efficiency of the scheme, we now consider two non-linear initial value problems: Tables 1 and 2**.** Where F is the function, N is the number of terms, YEXT is Exact solution of Y, F' is second derivative of function F and YLAG is Laguerre solution of Y.

Example 1. $y''(x)=\frac{(y')^2}{2y}-2y$ $\qquad y\left(\frac{\pi}{6}\right)=\frac{1}{4}, y'\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ (Adewale, Olaide & Sunday, 2013)

Exact solution: $y(x)=(\sin\ x)^2$

Table 1: Results and errors of problem 1 for K=2.

K=2						
N	**X**	**F**	**YEXT**	**F'**	**YLAG**	**|YEXT-YLAG|**
0	0.52360	0.86603	0.25000	0.09375	0.25000	0.00000
1	0.57360	0.91162	0.29448	0.12236	0.29342	0.00106
2	0.62360	0.94810	0.34101	0.15326	0.34022	0.00079
3	0.67360	0.97511	0.38913	0.18500	0.38862	0.00051
4	0.72360	0.99237	0.43836	0.21585	0.43814	0.00022
5	0.77360	0.99972	0.48820	0.24396	0.48827	0.00007
6	0.82360	0.99708	0.53816	0.26751	0.53852	0.00036
7	0.87360	0.98448	0.58774	0.28482	0.58838	0.00064
8	0.92360	0.96204	0.63645	0.29453	0.63736	0.00091
9	0.97360	0.92999	0.68379	0.29570	0.68495	0.00116
10	1.02360	0.88865	0.72929	0.28796	0.73159	0.00230

Example 2. $y''-x(y')^2=0$ $\qquad y(0)=1, y'(0)=\frac{1}{2}$ [2]

Exact solution: $y(x)=1+\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Table 2: Results and errors of problem 2 for K=2

K=2						
N	**X**	**F**	**YEXT**	**F'**	**YLAG**	**|YEXT-YLAG|**
0	0.00000	0.50000	1.00000	0.00000	1.00000	0.00000
1	0.05000	0.47561	1.02501	0.01131	1.02501	0.00001
2	0.10000	0.45238	1.05004	0.02046	1.04881	0.00123
3	0.15000	0.43023	1.07514	0.02777	1.07269	0.00245
4	0.20000	0.40909	1.10034	0.03347	1.09669	0.00364
5	0.25000	0.38889	1.12566	0.03781	1.12084	0.00482
6	0.30000	0.36957	1.15114	0.04097	1.14515	0.00599
7	0.35000	0.35106	1.17682	0.04314	1.16967	0.00715
8	0.40000	0.33333	1.20273	0.04444	1.19443	0.00830
9	0.45000	0.31633	1.22892	0.04503	1.21946	0.00946
10	0.50000	0.30000	1.25541	0.04500	1.24494	0.01047

CONCLUSION

In this paper, a class of implicit block collocation scheme for the direct solution of initial value problems of general second order ordinary differential equations was developed using Laguerre Collocation approach. The collocation technique yielded a consistent and zero stable implicit block multi-step methods with continuous coefficients and the method is implemented without the need for developing correctors. The derived modified Laguerre block scheme was used to solve some second order ordinary differential equation given in the table as YLAG which is Laguerre solution of Y and the exact solution of the ODEs given as YEXT., this was done in order to show their validity and accuracy of the derived scheme .The numerical results obtained shows that the proposed methods are efficient in solving second order ordinary differential equations.

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