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EXTENSION OF VASICEK MODEL TO THE MODELLING OF INTEREST RATE

[*1]Udoye, A. M., [2]Akinola, L. S. and [3]Ogbaji, E. O.

[1,2]Department of Mathematics, Federal University, Oye-Ekiti, Ekiti State, Nigeria.
[3]Department of Mathematics & Statistics, Federal University, Wukari, Taraba State, Nigeria.

[*]Corresponding Author's email: adaobi.udoye@fuoye.edu.ng

ABSTRACT
Interest rate modelling is an interesting aspect of stochastic processes. It has been observed that interest rates fluctuates at random times, hence the need for its modelling as a stochastic process. In this paper, we apply Vasicek model, Itô's lemma and least-square regression method to model an interest rate known as Nigerian interbank offered rate. The Vasicek model is a mean-reverting model, such property makes it suitable for the modelling of a given interest rate. The model is applied to the Nigerian interbank offered rate to derive its dynamics that captures features of the interest rate. These features include the ability of the interest rate to revert back to long-term mean rate and its random movements. The parameter values of the Vasicek model are determined using data from the Nigerian interbank offered rate. The estimated parameter values are applied to derive the dynamics of the interest rate. The dynamics provides easier way of pricing interest rate derivatives in order to avoid unnecessary risks in financial markets.

Keywords: Vasicek model, Interest Rate, Itô lemma, Least-square regression

INTRODUCTION
The Vasicek model was introduced in 1977 as a mean-reverting short rate model. This means that the model has a feature of reverting back to a certain mean after some time. It has a lot of applications especially in financial markets. There are other short rate models, namely, Hull-White model, Merton model, Cox-Ingersoll-Ross model, etc. Each model has its own features but Vasicek model accommodates the tendency of interest rate reverting back to its long-term mean unlike other models and thus, it is very useful in modelling interest rates in interbank transactions. Interbank offered rates refer to interest rates for which banks obtain loans in the interbank markets between overnight and 12 months. Such rates include London Interbank Offered Rate (LIBOR), Euro Interbank Offered Rate (EURIBOR) and Nigerian Interbank Offered Rate (NIBOR). The LIBOR represents the rate that prime banks intend to pay on United States (US) dollar deposited inside and outside US banks. It can also be seen as a reference rate depending on the mean interest rate in which the banks can borrow in the London interbank markets. EURIBOR is the recommended rate for the Euro region, while, NIBOR is the short term loaning rates of main banks in the Nigerian interbank market quoted as annual rates. Many individuals have worked on the applications of the different short rate models. Myška (2007) emphasized that interest rates have value at certain limited range and highlighted on the calibration of Hull-White and Vasicek model. Epstein et al. (1999) applied a non-probabilistic, non-linear interest rate and Vasicek model in deriving the price of a convertible bond, that is, coupon-influenced bond where the holder gets coupon payments at predetermined periods. Park et al. (2006) discussed Monte-Carlo simulation method for bond option pricing using Vasicek and CIR models. Yin et al. (2018) discussed a corporate bond-pricing model of credit rating risk using Vasicek model. Filipovic and Willems (2019) obtained a structure based on multivariate polynomial jump-diffusions to price the term structures of dividends and interest rates. Thus, from literature, Vasicek interest rate model has been studied and extended to different aspects having property of mean reversion but has not been applied in the modelling of NIBOR driven by a Wiener process. In this paper, we give a detailed derivation of the solution of the Vasicek model, using least-square regression method; we obtain the short rate dynamics for NIBOR. In section 2, we discuss important tools that are necessary in achieving our result. In section 3, we obtain the dynamics for NIBOR by applying the Vasicek model and least-square regression method. The result is achieved using NIBOR data between January, 2007 and December, 2017 obtained from the website of Central Bank of Nigeria.

Mathematical Tools
We consider important tools which include Itô lemma (Cont and Tankov, 2004) to be applied in order to establish our result. In the following section, $r_t = r(t)$ denotes an interest rate at time $t$and E denotes expectation.

The Vasicek Model
The Vasicek (1977) model is given by

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t$$

where $\alpha,\ \beta, \sigma$ and $W_t$denote velocity of mean reversion of the interest rate, long-term mean level of the interest rate, diffusion rate of the short rate and Wiener process respectively. dr_t is the first differential coefficient of r_t .**Itô Lemma.** Let $Y_t, t \geq 0$ be a given stochastic process with drift $b$such that

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dW_t$$

where $b(t, Y_t)$ and $\sigma(t, Y_t)$ are continuous non-anticipating processes with

$$E\left[\int_0^T \sigma_t^2 dt\right] < \infty.$$

Then, if $F:[0,T]\times R\rightarrow R$ is a $C^{1,2}$ function (that is, two times continuously differentiable function), it implies that

$$dF(t,Y_t)=\left(\frac{\partial F(t,y)}{\partial t}+b(t,Y_t)\frac{\partial F(t,y)}{\partial y}+\frac{1}{2}\left(\sigma(t,Y_t)\right)^2\right)dt+\sigma(t,Y_t)\frac{\partial F(t,y)}{\partial y}dW_t$$

where $Y_t=y$.

RESULTS AND DISCUSSION

We consider the Vasick model whose dynamics is given by

$$dr_t=\delta(\beta-r_t)dt+\sigma dW_t \qquad (1)$$

where $\delta,\ \beta$ and σ denote speed of mean reversion of the interest rate, long-term mean level of the interest rate and diffusion rate of the short rate, respectively. W_t represents Wiener process.

Theorem 1. Let the dynamics of a short rate r_t model be given by equation (1), then

$$r_t=r_0e^{-\delta t}+\beta\left(1-e^{-\delta t}\right)+\sigma e^{-\delta t}\int_0^t e^{\delta s}dW_s. \qquad (2)$$

Proof. Let $F(t,r_t)=ye^{\delta t}$, $\frac{\partial F(t,r)}{\partial t}=\delta re^{\delta t}$, $\frac{\partial F(t,r)}{\partial r}=e^{\delta t}$, $\frac{\partial^2 y}{\partial r^2}=0$.

Then, using the Itô's Lemma,

$$dr_te^{\delta t}=\delta\beta e^{\delta t}dt+\sigma e^{\delta t}dW_t. \qquad (3)$$

Integrating equation (3) with respect to t, we obtain

$$r_te^{\delta t}-r_0=\delta\beta\int_0^t e^{\delta s}\,ds+\sigma\int_0^t e^{\delta s}\,dW_s.$$

Therefore,

$$r_t=r_0e^{-\delta t}+\beta\left(1-e^{-\delta t}\right)+\sigma e^{-\delta t}\int_0^t e^{\delta s}dW_s$$

as given by (2).

Theorem 2. Let the dynamics of a short rate r_t model be given by equation (2), the variance of the short rate satisfies

$$Var[r_t]=\frac{\sigma^2}{2\delta}\left(1-e^{-2\delta t}\right). \qquad (4)$$

Proof: The variance 'Var' of the short rate is given by $Var[r_t]=E[r_t-E(r_t)]^2$.

From equation (2),

$$E[r_t]=r_0e^{-\delta t}+\beta\left(1-e^{-\delta t}\right). \qquad (5)$$

Thus,

$$Var[r_t]=\left(\sigma e^{-\delta t}\int_0^t e^{\delta s}dW_s\right)^2=\sigma^2e^{-2\delta t}\int_0^t e^{2\delta s}ds$$
$$=\frac{\sigma^2}{2\delta}\left(1-e^{-2\delta t}\right).$$

Theorem 3. The solution to the short rate dynamics given by equation (2) can be written as

$$r_t=r_0e^{-\delta t}+\beta\left(1-e^{-\delta t}\right)+\frac{\sigma^2}{2\delta}\left(1-e^{-2\delta t}\right). \qquad (6)$$

Proof. The proof follows from equations (4) and (5).

The parameter values of the short rate model are obtained from the Nigerian interbank offered rate (NIBOR) data between January 2007 and December 2017 collected from the website of Central Bank of Nigeria, using least-square regression and maximum likelihood method (Van den Berg, 2011).

Using least-square regression,

$$r_i=a+hr_{i-1}+\epsilon,\ \ i=1,2,\dots,n. \qquad (7)$$

Let

$$\epsilon=\sum_i(r_i-\bar{r}_i)^2=\sum_i[r_i-(a+hr_{i-1})]^2.$$

Then,

$$\frac{\partial\varepsilon}{\partial a}=2\sum_i[r_i-(a+hr_{i-1}](-1)=-2\sum_i r_i+2an+2h\sum_i r_{i-1}=0.$$

Hence,

$$a=\frac{\sum_i r_i}{n}-h\frac{\sum_i r_{i-1}}{n}=\mu_{r_i}-h\mu_{r_{i-1}}$$

where μ_{r_i}represents the mean of r_i.

$$\frac{\partial\varepsilon}{\partial h}=2\sum_i[r_i-a-hr_{i-1}](-r_{i-1})=2h\sum_i r_{i-1}^2+2a\sum_i r_{i-1}-2\sum_i r_ir_{i-1}=0.$$
$$h\sum_i r_{i-1}^2=\sum_i r_ir_{i-1}-a\sum_i r_{i-1}=\sum_i r_ir_{i-1}-(\mu_{r_i}-h\mu_{r_{i-1}})\sum_i r_{i-1}$$
$$=\sum_i r_ir_{i-1}-\mu_{r_i}\sum_i r_{i-1}+h\mu_{r_{i-1}}\sum_i r_{i-1}.$$

Thus,

$$h\left(\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}\right) = \sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}.$$

Hence,

$$h = \frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}} = \frac{\sum_i (r_i - \mu_{r_i})(r_{i-1} - \mu_{r_{i-1}})}{\sum_i (r_{i-1} - \mu_{r_{i-1}})^2}.$$

From equation (2), let

$$r_i = r_{i-1}e^{-\delta\Delta t} + \beta(1 - e^{-\delta\Delta t}) + \sigma\int_0^t e^{-\delta(t-s)}\, dW_s. \qquad (8)$$

From equations (7) and (8),

$$h = e^{-\delta\Delta t}, \;\; a = \beta(1 - e^{-\delta\Delta t})$$

implies that

$$\beta = \frac{a}{1-e^{-\delta\Delta t}} = \frac{a}{1-h} \text{ and } \varepsilon = \sigma\int_0^t e^{-\delta(t-s)}\, dX_s.$$

Moreover,

$$\begin{aligned}\delta &= -\frac{\ln h}{\Delta t} = -\frac{1}{\Delta t}\ln\frac{\sum_{i=1}^n (r_i - \mu_{r_i})(r_{i-1} - \mu_{r_{i-1}})}{\sum_{i=1}^n (r_{i-1} - \mu_{r_{i-1}})^2} \\ &= -\frac{1}{\Delta t}\ln\frac{\sum_{i=1}^n [r_i r_{i-1} - r_i\mu_{r_{i-1}} - r_{i-1}\mu_{r_i} + \mu_{r_i}\mu_{r_{i-1}}]}{\sum_{i=1}^n [{r_{i-1}}^2 - 2r_{i-1}\mu_{r_{i-1}} + {\mu_{r_{i-1}}}^2]} \\ &= -\frac{1}{\Delta t}\ln\frac{\sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i\mu_{r_{i-1}} - \sum_{i=1}^n r_{i-1}\mu_{r_i} + \sum_{i=1}^n \mu_{r_i}\mu_{r_{i-1}}}{\sum_{i=1}^n {r_{i-1}}^2 - 2\sum_{i=1}^n r_{i-1}\mu_{r_{i-1}} + n\,{\mu_{r_{i-1}}}^2} \\ &= -\frac{1}{\Delta t}\ln\frac{r_{xy} - \mu_x r_y - \mu_y r_x + n\mu_x\mu_y}{r_{xx} - 2\mu_x r_x + n\mu_x^2} = -\frac{1}{\Delta t}\ln\frac{r_{xy} - \beta r_y - \beta r_x + n\beta^2}{r_{xx} - 2\beta r_x + n\beta^2}.\end{aligned}$$

Furthermore, from equations (6) and (7),

$$\begin{aligned}\beta &= \frac{a}{1-h} = \frac{\sum_{i=1}^n [r_i - r_{i-1}h]}{n[1-h]} = \frac{\sum_{i=1}^n [r_i - r_{i-1}e^{-\delta\Delta t}]}{n[1 - e^{-\delta\Delta t}]} = \frac{\mu_{r_i} - h\mu_{r_{i-1}}}{1 - \left(\frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right)} \\ &= \frac{\mu_{r_i} - \left(\frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right)\mu_{r_{i-1}}}{1 - \left(\frac{\sum_i r_i r_{i-1} - \mu_{r_i}\sum_i r_{i-1}}{\sum_i r_{i-1}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right)} \\ &= \mu_{r_i} - \left(\frac{\mu_{r_{i-1}}\sum_i r_i r_{i-1} - \mu_{r_{i-1}}\mu_{r_i}\sum_i r_{i-1}}{\sum_i {r_{i-1}}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}}\right) \div \frac{\sum_i {r_{i-1}}^2 - \mu_{r_{i-1}}\sum_i r_{i-1} - \sum_i r_i r_{i-1} + \mu_{r_i}\sum_i r_{i-1}}{\sum_i {r_{i-1}}^2 - \mu_{r_{i-1}}\sum_i r_{i-1}} \\ &= \frac{\mu_{r_i}\sum_i {r_{i-1}}^2 - \mu_{r_i}\mu_{r_{i-1}}\sum_i r_{i-1} - \mu_{r_{i-1}}\sum_i r_i r_{i-1} + \mu_{r_{i-1}}\mu_{r_i}\sum_i r_{i-1}}{\sum_i {r_{i-1}}^2 - \mu_{r_{i-1}}\sum_i r_{i-1} - \sum_i r_i r_{i-1} + \mu_{r_i}\sum_i r_{i-1}} \\ &= \frac{\mu_{r_i}\sum_i {r_{i-1}}^2 - \mu_{r_{i-1}}\sum_i r_i r_{i-1}}{\sum_i {r_{i-1}}^2 - \mu_{r_{i-1}}\sum_i r_{i-1} - \sum_i r_i r_{i-1} + \mu_{r_i}\sum_i r_{i-1}} \\ &= \frac{\sum_i r_i \sum_i {r_{i-1}}^2 - \sum_i r_{i-1}\sum_i r_i r_{i-1}}{n\left[\sum_i {r_{i-1}}^2 - \frac{\sum_i r_{i-1}}{n}\sum_i r_{i-1} - \sum_i r_{i-1}r_i + \frac{\sum_i r_i}{n}\sum_i r_{i-1}\right]} \\ &= \frac{\sum_i r_i \sum_i {r_{i-1}}^2 - \sum_i r_{i-1}\sum_i r_i r_{i-1}}{n[\sum_i {r_{i-1}}^2 - \sum_i r_{i-1}r_i] - [\sum_i r_{i-1}\sum_i r_{i-1} - \sum_i r_i \sum_i r_{i-1}]}.\end{aligned}$$

Hence,

$$\beta = \frac{r_y r_{xx} - r_{xy}r_x}{n(r_{xx} - r_{xy}) - (r_x^2 - r_x r_y)}.$$

Therefore, the values of the parameters are obtained as

$$r_x = \sum_{i=1}^{n=131} r_{i-1} = 18.8897, \quad r_{xx} = \sum_{i=1}^{n=131} r_{i-1}^2 = 3.1001, \quad r_{xy} = \sum_{i=1}^{n=131} r_i r_{i-1} = 2.9357,$$

$$r_y = \sum_{i=1}^{n=131} r_i = 18.9212, \quad r_{yy} = \sum_{i=1}^{n=131} r_i^2 = 3.1094.$$

Whence, the long-term mean rate is obtained as

$$\beta = \frac{r_y r_{xx} - r_{xy}r_x}{n(r_{xx} - r_{xy}) - (r_x^2 - r_x r_y)} = 0.1447$$

whereas the speed to which the interest rate reverts back to the mean is obtained as

$$\delta = -\frac{1}{\Delta t}\ln\frac{r_{xy} - \beta r_y - \beta r_x + n\beta^2}{r_{xx} - 2\beta r_x + n\beta^2} = 0.5959$$

and the fixed time step $\Delta t = 1$.
The volatility is obtained as

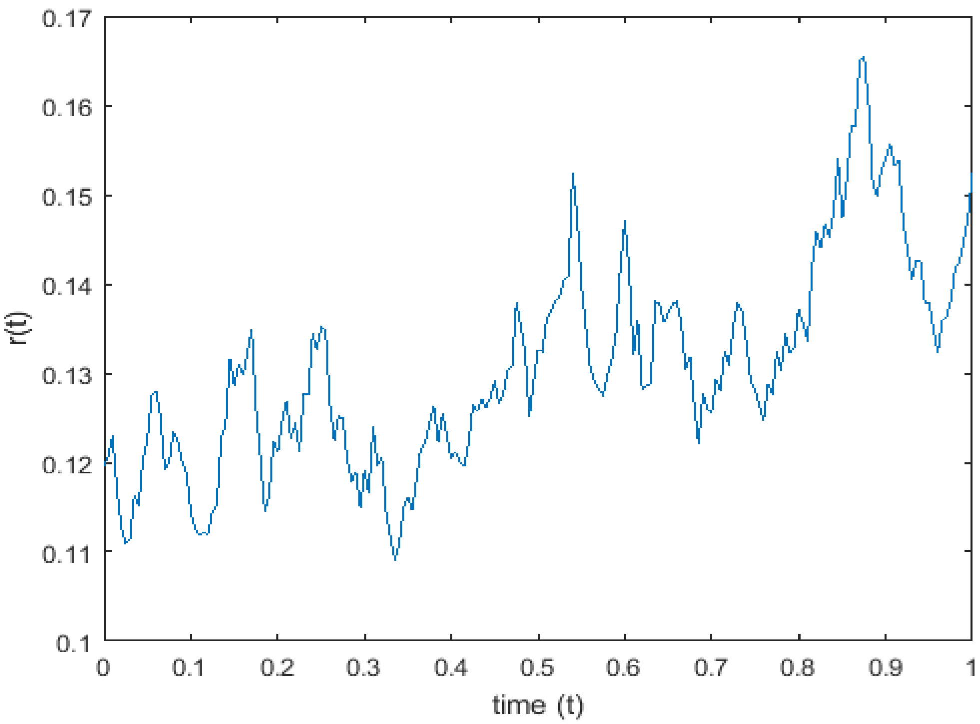
$$\sigma = \sqrt{\frac{\frac{1}{n}\left(r_{yy} - 2r_{xy}e^{-\delta\Delta t} - r_{xx}(e^{-\delta\Delta t})^2 - K + n\beta^2(1 - e^{-\delta\Delta t})^2\right)}{1 - (e^{-\delta\Delta t})^2}} = 0.0585$$

where $K = 2\beta\left(1 - e^{-\delta\Delta t}\right)\left(r_y - r_x e^{-\delta\Delta t}\right)$.

Therefore, the dynamics of NIBOR using Vasicek model driven by Wiener process is given by

$$dr_t = 0.5959(0.1447 - r_t)dt + 0.0585dW_t. \qquad (9)$$

Plotting (8) in MATLAB, we obtain the figure below that shows how the NIBOR rate $r(t)$ evolves over time t.



The dynamics of the NIBOR gives the interpretation that the interest rate reverts to the long-term mean rate of 0.1447 with a speed of mean reversion given by 0.5959, while its volatility rate is given by 0.0585. The above makes it easier when pricing interest rate derivatives. For example, the price of a zero-coupon bond at time t with maturity T is given by

$$P(t,T) = \exp\left(-\int_t^T r_u du\right),$$

and from Theorem 1, we have

$$r_t = r_0 e^{-0.5959t} + 0.1447(1 - e^{-0.5959t}) + 0.0585e^{-\delta t}\int_0^t e^{0.5959s}dW_s.$$

CONCLUSION

Interest rates are unstable, they move randomly over a given interval of time. We have been able to obtain the dynamics for the Nigerian interbank offered rate using the Vasicek model via Itô lemma and least-square regression method. The dynamics is very useful in the pricing of interest rate derivatives such as bonds, stocks.

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APPENDIX

30 Days NIBOR data

Jan-07	13.17	Jan-08	12.99	Jan-09	14.91	Jan-10	12.84
Feb-07	11.96	Feb-08	12.76	Feb-09	18.07	Feb-10	11.27
Mar-07	11.37	Mar-08	12.12	Mar-09	18.92	Mar-10	7.85
Apr-07	10.57	Apr-08	12.78	Apr-09	15.25	Apr-10	5.13
May-07	11.01	May-08	13.15	May-09	15.91	May-10	8.03
Jun-07	11.56	Jun-08	13.46	Jun-09	19.84	Jun-10	5.95
Jul-07	12.85	Jul-08	13.06	Jul-09	19.66	Jul-10	6.51
Aug-07	12.49	Aug-08	15.34	Aug-09	14.29	Aug-10	8.2
Sep-07	12.38	Sep-08	16.76	Sep-09	13.78	Sep-10	8.57
Oct-07	12.4	Oct-08	15.63	Oct-09	13.35	Oct-10	11.13
Nov-07	12.55	Nov-08	17.98	Nov-09	13.75	Nov-10	11.67
Dec-07	12.89	Dec-08	15.85	Dec-09	13.45	Dec-10	11.5
Jan-11	10.15	Jan-12	15.44	Jan-13	13.1	Jan-14	11.21
Feb-11	11.19	Feb-12	15.61	Feb-13	12.79	Feb-14	12.3
Mar-11	11.47	Mar-12	15.57	Mar-13	11.07	Mar-14	13.03
Apr-11	12.51	Apr-12	15.39	Apr-13	11.97	Apr-14	12.03
May-11	11.67	May-12	14.61	May-13	12.94	May-14	12.42
Jun-11	13.15	Jun-12	15.79	Jun-13	12.81	Jun-14	12.17
Jul-11	11.45	Jul-12	16.12	Jul-13	11.57	Jul-14	12.42
Aug-11	10.79	Aug-12	19.18	Aug-13	14.65	Aug-14	12.97
Sep-11	11.74	Sep-12	14.55	Sep-13	17.74	Sep-14	12.37
Oct-11	15.74	Oct-12	13.52	Oct-13	12.01	Oct-14	12.6
Nov-11	17	Nov-12	13.43	Nov-13	12.03	Nov-14	13.07
Dec-11	16.74	Dec-12	13.13	Dec-13	11.85	Dec-14	15.79
Jan-15	13.7	Jan-16	13.72	Jan-17	10.64		
Feb-15	15.47	Feb-16	15.19	Feb-17	26.78		
Mar-15	15.89	Mar-16	15.89	Mar-17	19.61		
Apr-15	15.17	Apr-16	15.02	Apr-17	52.74		
May-15	14.61	May-16	14.64	May-17	27.75		
Jun-15	15.45	Jun-16	15.38	Jun-17	26.03		
Jul-15	14.32	Jul-16	14.34	Jul-17	17.9		
Aug-15	17.16	Aug-16	17.08	Aug-17	26.38		
Sep-15	15.52	Sep-16	15.52	Sep-17	19.11		
Oct-15	13.05	Oct-16	13.59	Oct-17	39.24		
Nov-15	12.02	Nov-16	12.02	Nov-17	20.02		
Dec-15	9.13	Dec-16	9.13	Dec-17	16.32		