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A COMPARATIVE STUDY OF ORTHOGONAL POLYNOMIALS FOR NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we adopt the general method of interpolation and collocation in Linear Multistep Methods in deriving some numerical schemes for solving second order ordinary differential equations. Different choices of the interpolating function in the form of shifted Legendre, shifted Chebyshev and Lucas polynomials with the same interpolation and collocation points are considered in order to establish uniformity or otherwise of the derived schemes for the various polynomials. Furthermore, probable disparities in the derived schemes for varied choices of interpolation and collocation points are also investigated. Results indicate that all the polynomials yield exactly the same schemes for the same choice of interpolation and collocation points but different schemes for different choices of interpolation and collocation points. However, numerical examples considered showed that all the derived schemes performed exactly in the same manner in terms of accuracy, regardless of the choices of interpolation or collocation points. Nevertheless, the derived schemes perform admirably better when compared with existing methods in literature.

Keywords: Interpolation, collocation, orthogonal polynomials, block method

INTRODUCTION

In recent times, systematic solution algorithm for solving general forms of ordinary differential equations have been developed. The derivation of such algorithms are variably achieved with the modification of existing methodology in the literature. Approximating algorithms to differential equations came in various forms and derivation methodology such as B-Spline Collocation Method, Finite Element Method, Finite Variation Method, Differential Transform Method, Laplace Transform Method, Fourier Transform Method, Linear Multistep Method, Adomian Decomposition Method as well as He's Homotopy Perturbation. These aforementioned methods have been implemented simultaneously or modified successively, just as the case of linear multistep methods. Evidently, continuous research aimed at deriving solution methods with finer properties and better approximations led to the emergence of employing special polynomials as an approximant to solutions of differential equations.

Diverse method(s) of obtaining approximate solution to general or special equations in the form of

$$y^{(n)} = f(x, y, y', ..., y^{(n-1)}),\ y(x_0) = y_0,\ y'(x_0) = y'_0,\ ...,\ y^{(n-1)}(x_0) = y_0^{(n-1)} \tag{1}$$

have been derived for various classes of linear multistep methods. Amongst these are the methods of Fatunla (1988), Ayinde and Ibijola (2015), Onumanyi *et al* (1994), Butcher (1993), James *et al* (2012), Sunday *et al* (2012), Mohammed and Yahaya (2010), Skwame *et al* (2012), Ajileye *et al* (2018). Ogunride *et al* (2020), Isah *et al* (2012) and Salawu *et al* (2020), where (1) is of first order. Further development led to approximate solution to (1) where $n = 2$, see (Adesanya *et al* (2009), Awoyemi *et al* (2012), Awoyemi and Kayide (2005), D'Ambrosio *et al* (2009), Fatunla (1991), Fudziah *et al* (2009), Jator and Li (2009), Yahaya and Badmus (2009) and Adoghe and Omole (2018)). Similarly, extensions has been made to solving (1) with cases where $n = 3$ and $n = 4$, see (Mohammed and Adeniyi (2014), Ogunware *et al* (2018), Yakubu *et al* (2011), Awoyemi *et al* (2014), Anake *et al* (2013), Adesanya *et al* (2012) and Adoghe and Omole (2019)) and (Adoghe and Omole (2019), Adeyeye and Omar (2019) and Luke *et al* (2020)) respectively.

To achieve higher efficiency of derived methods of solution, it is expected that the approximating polynomial agrees with the solution of a given differential equation at good number of points within a finite interval within which the solution to such differential equation is being sort. To this end, researchers considered alternative polynomials, as a replacement to power series as an interpolating function. Solution to (1) was approximated by a method with a combination of Chebyshev polynomial and exponential function as in Ogunride *et al* (2020). Similarly, Isah *et al* (2012) and Salawu *et al* (2020) considered shifted Chebyshev and shifted Legendre polynomials respectively.

Remarkable experimental outcome from the use of these oscillating polynomials as basis function for first and second order form of (1) suggested that such polynomials could be employed for higher order problems. As seen in Luke *et al* (2020), Lucas polynomial as basis function was considered, with finer numerical results.

Following these developments, we wish to exploit the diverse possibilities of choices of these special classes of polynomials. We attempt to discover their meeting points, comparative performances and possible uniformity of results, with the aim of recommending the best polynomial for use as basis function when solving ordinary differential equations.

The rest part of these research is outlined thus; derivation methodology, analysis of derived schemes based on choices of basis polynomials as well as interpolation and collocation points, numerical experiment, discussion of results and conclusion.

METHOD DERIVATION

Derivation of a Block of Uniform Discrete Schemes.

Consider a linear form of (1) with $n = 2$, being a general second order ordinary differential equation, expressed as

$$y'' = f(x, y, y'),\ y(x_0) = y_0,\ y'(x_0) = y'_0,\ x_0 \le x \le x_n \tag{2}$$

The solution to (2) is assumed to exist, at least on the assumption that f is continuous. In addition, given some specified interval, we assume that the solution to (2) agrees with some polynomial given as

$$y(x) = \sum_{i=1}^{w} a_i \varphi_i(x) \tag{3}$$

at finitely many points within the interval.

(3) shall be considered for three cases of the function $\varphi_i(x)$ viz:

1. When $\varphi_i(x)$ is of Lucas polynomial type
2. When $\varphi_i(x)$ is of shifted Legendre polynomial type
3. When $\varphi_i(x)$ is of shifted Chebyshev polynomial type

where w is a sum of interpolation points and collocation points less one for all cases.

For the interpolation and collocation derivation technique so adopted, (3) is uniformly interpolated at points x_n and x_{n+2} while the second derivative of (3) is collocated at the points $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4}, thus deriving a 4-step block method.

Interpolating and collocating the basis function (3) yields a system of 7 equations in 7 unknowns $a_i's$, the solution for the unknowns are substituted back into the basis function (3) to give the respective continuous formulation of the proposed 4 step method for each of the cases in the general form of

$$y(x) = \sum_{j=1}^{n} \alpha_j(x,h) y_{n+j} + \sum_{j=1}^{n} \beta_j(x,h) f_{n+j} \tag{4}$$

where $\alpha_j(x,h)$ and $\beta_j(x,h)$ are continuous in x.

The first and second derivative of (4) with respect to x is given as

$$y'(x) = \sum_{j=1}^{n} \alpha'_j(x,h) y_{n+j} + \sum_{j=1}^{n} \beta'_j(x,h) f_{n+j} \tag{5}$$

$$y''(x) = f(x, y, y') = \sum_{j=1}^{n} \alpha''_j(x,h) y_{n+j} + \sum_{j=1}^{n} \beta''_j(x,h) f_{n+j} \tag{6}$$

For each of the cases specified above, the functions $\alpha(x,h)$ and $\beta(x,h)$ in (4) are given as;

$$\alpha_0 = \frac{x_{n+2} - x_n}{2h},\ \alpha_1 = 0,\ \alpha_2 = \frac{x_n - x_{n+2}}{2h} \text{ and } \alpha_3 = 0$$

$$\beta_0 = \frac{1}{1440h^4}\left(2(x_n - x)^6 + 30h(x_n - x)^5 + 175h^2(x_n - x)^4 + 500h^3(x_n - x)^3 + 720h^4(x_n - x)^2 + 424h^5(x_n - x)\right)$$

$$\beta_1 = \frac{1}{1440h^4}\left((x_n - x)^6 + \frac{27}{2}h(x_n - x)^5 + 65h^2(x_n - x)^4 + 120h^3(x_n - x)^3 - 144h^5(x_n - x)\right)$$

$$\beta_2 = -\frac{1}{120h^4}\left((x_n - x)^6 + 12h(x_n - x)^5 + \frac{95}{2}h^2(x_n - x)^4 + 60h^3(x_n - x)^3 - 20h^5(x_n - x)\right)$$

$$\beta_3 = \frac{1}{180h^4}\left((x_n - x)^6 + \frac{21}{2}h(x_n - x)^5 + 35h^2(x_n - x)^4 + 40h^3(x_n - x)^3 - 16h^5(x_n - x)\right)$$

$$\beta_4 = \frac{1}{720h^4}\left((x_n - x)^6 + 9h(x_n - x)^5 + \frac{55}{2}h^2(x_n - x)^4 + 30h^3(x_n - x)^3 - 12h^5(x_n - x)\right)$$

The proposed block of discrete schemes is obtained when (4) is evaluated at x_{n+4} , x_{n+3} and x_{n+1} while (5) is evaluated at $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} , giving rise to uniform block of schemes for the cases 1, 2 and 3, which are expressed as

$$\left.\begin{aligned}
y_{n+4} - 2y_{n+2} + y_n &= \frac{h^2}{15}\left(f_n + 16f_{n+1} + 26f_{n+2} + 16f_{n+3} + f_{n+4}\right)\\
2y_{n+3} - 3y_{n+2} + y_n &= \frac{h^2}{480}\left(17f_n + 252f_{n+1} + 402f_{n+2} + 52f_{n+3} - 3f_{n+4}\right)\\
y_{n+2} - 2y_{n+1} + y_n &= \frac{h^2}{480}\left(19f_n + 204f_{n+1} + 14f_{n+2} + 4f_{n+3} - f_{n+4}\right)\\
y_{n+2} - y_n &= 2hy_n' + \frac{h^2}{90}\left(53f_n + 144f_{n+1} - 30f_{n+2} + 16f_{n+3} - 3f_{n+4}\right)\\
y_{n+2} - y_n &= 2hy_{n+1}' - \frac{h^2}{360}\left(39f_n + 70f_{n+1} - 144f_{n+2} + 42f_{n+3} - 7f_{n+4}\right)\\
y_{n+2} - y_n &= 2hy_{n+2}' - \frac{h^2}{90}\left(5f_n + 104f_{n+1} + 78f_{n+2} - 8f_{n+3} + f_{n+4}\right)\\
y_{n+2} - y_n &= 2hy_{n+3}' - \frac{h^2}{360}\left(31f_n + 342f_{n+1} + 768f_{n+2} + 314f_{n+3} - 15f_{n+4}\right)\\
y_{n+2} - y_n &= 2hy_{n+4}' - \frac{h^2}{90}\left(3f_n + 112f_{n+1} + 126f_{n+2} + 240f_{n+3} + 59f_{n+4}\right)
\end{aligned}\right\} \qquad (7)$$

The derived block (8) is a simultaneous numerical integrator of second order initial value problems of special second order ordinary differential equations, which is same for either the choice of Lucas, Shifted Legendre or Shifted Chebyshev polynomial.

Derivation of Uniform and Non-Uniform Order Block of Discrete Schemes

For each case of orthogonal polynomial, (3) is considered with different choices of interpolation and collocation points;

$\varphi_i(x)$ of Lucas Polynomial type

In this case, (3) is of Lucas polynomial type with the choice of x_{n+1}, x_{n+3} , x_{n+4} as interpolation points and $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ as collocation points. A continuous formulation of a four-step method is obtained in the form of (4).

A set of discrete schemes is derived when the continuous formulation, its first and second derivatives are evaluated at { x_n, x_{n+2} }, { $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ } and { x_{n+4} };the result is presented in (8).

$$\left.\begin{aligned}
&3y_{n+4}+14y_{n+2}-54y_{n+1}+37y_n=h^2\left(\frac{31}{222}f_{n+3}+\frac{1}{2}f_{n+2}+\frac{67}{74}f_{n+1}+\frac{17}{222}f_n\right)\\
&3y_{n+4}+17y_{n+3}-37y_{n+2}+19y_{n+1}=h^2\left(\frac{11}{148}f_{n+3}+\frac{5}{12}f_{n+2}+\frac{23}{444}f_{n+1}-\frac{1}{444}f_n\right)\\
&\frac{10}{37}y_{n+4}+\frac{7}{74}y_{n+3}-\frac{27}{74}y_{n+1}=h^2\left(\frac{5201}{13320}f_{n+3}+\frac{47}{120}f_{n+2}+\frac{5741}{4440}f_{n+1}-\frac{4399}{13320}f_n\right)+hy'_n\\
&80y_{n+4}-675y_{n+3}+595y_{n+1}=h^2\left(72f_{n+3}-666f_{n+2}-414f_{n+1}+18f_n\right)-1110hy'_{n+1}\\
&1680y_{n+4}+4140y_{n+3}-5820y_{n+1}=h^2\left(2881f_{n+3}+777f_{n+2}-1257f_{n+1}+119f_n\right)+13320hy'_{n+2}\\
&720y_{n+4}-2745y_{n+3}+2025y_{n+1}=h^2\left(1906f_{n+3}+2442f_{n+2}+48f_{n+1}+14f_n\right)+3330hy'_{n+3}\\
&\frac{478}{111}y_{n+4}-\frac{441}{74}y_{n+3}+\frac{367}{222}y_{n+1}=h^2\left(\frac{3849}{1480}f_{n+3}+\frac{69}{40}f_{n+2}+\frac{207}{1480}f_{n+1}-\frac{9}{1480}f_n\right)+hy'_{n+4}\\
&480y_{n+4}-720y_{n+3}+240y_{n+1}=h^2\left(37f_{n+4}+432f_{n+3}+222f_{n+2}+32f_{n+1}-3f_n\right)
\end{aligned}\right\}\quad(8)$$

$\varphi_i(x)$ of Shifted Chebyshev Polynomial type

The choice of $\varphi_i(x)$ in (3) is of Shifted Chebyshev polynomial with the choice of x_n and x_{n+4} as interpolation points, while the points $x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} are selected as collocation points.

With this choices, a continuous scheme in the form of (4) is derived, such that the scheme and its first derivative are evaluated at { $x_{n+1}, x_{n+2}, x_{n+3}$ } and { $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ } respectivelyto generate a block of discrete schemes as presented in (9)

$$\left.\begin{aligned}
&\frac{1}{2}y_{n+4}-y_{n+2}+\frac{1}{2}y_n=h^2\left(\frac{1}{30}f_{n+4}+\frac{8}{15}f_{n+3}+\frac{13}{15}f_{n+2}+\frac{8}{15}f_{n+1}+\frac{1}{30}f_n\right)\\
&\frac{3}{4}y_{n+4}-y_{n+3}+\frac{1}{4}y_n=h^2\left(\frac{9}{160}f_{n+4}+\frac{83}{120}f_{n+3}+\frac{37}{80}f_{n+2}+\frac{11}{40}f_{n+1}+\frac{7}{480}f_n\right)\\
&\frac{1}{4}y_{n+4}-y_{n+1}+\frac{3}{4}y_n=h^2\left(\frac{7}{480}f_{n+4}+\frac{11}{40}f_{n+3}+\frac{37}{80}f_{n+2}+\frac{83}{120}f_{n+1}+\frac{9}{160}f_n\right)\\
&\frac{1}{4}y_{n+4}-\frac{1}{4}y_n=h^2\left(\frac{16}{45}f_{n+3}+\frac{4}{15}f_{n+2}+\frac{16}{15}f_{n+1}+\frac{14}{45}f_n\right)+hy'_n\\
&\frac{1}{4}y_{n+4}-\frac{1}{4}y_n=h^2\left(\frac{19}{720}f_{n+4}+\frac{5}{24}f_{n+3}+\frac{19}{30}f_{n+2}+\frac{61}{360}f_{n+1}+\frac{3}{80}f_n\right)+hy'_{n+1}\\
&\frac{1}{4}y_{n+4}-\frac{1}{4}y_n=h^2\left(\frac{1}{90}f_{n+4}+\frac{14}{45}f_{n+3}-\frac{14}{45}f_{n+1}-\frac{1}{90}f_n\right)+hy'_{n+2}\\
&\frac{1}{4}y_{n+4}-\frac{1}{4}y_n=h^2\left(\frac{3}{80}f_{n+4}-\frac{61}{360}f_{n+3}-\frac{19}{30}f_{n+2}-\frac{5}{24}f_{n+1}-\frac{19}{720}f_n\right)+hy'_{n+3}\\
&\frac{1}{4}y_n-\frac{1}{4}y_{n+4}=h^2\left(\frac{14}{45}f_{n+4}+\frac{16}{15}f_{n+3}+\frac{4}{15}f_{n+2}+\frac{16}{45}f_{n+1}\right)-hy'_{n+4}
\end{aligned}\right\}\quad(9)$$

Furthermore, the choices of interpolation and collocation points are reviewed for this choice of polynomial. A continuous polynomial in the form of (4) is again derived with x_n, x_{n+2} and x_{n+3} as interpolation points, while $x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} are selected collocation points.

Similarly, a simultaneous numerical integrator is derived when the continuous formulation, its first and second derivative are evaluated at { x_n, x_{n+4} }, { $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ } and { x_n } respectively as given in (10)

$$
\left.\begin{array}{l}
y_{n+4}-\frac{32}{17}y_{n+3}+\frac{14}{17}y_{n+2}+\frac{1}{17}y_{n}=h^{2}\left(\frac{4}{51}f_{n+4}+\frac{44}{51}f_{n+3}+\frac{8}{51}f_{n+2}+\frac{4}{51}f_{n+1}\right)\\
\frac{19}{17}y_{n+3}-\frac{37}{17}y_{n+2}+y_{n+1}+\frac{1}{17}y_{n}=h^{2}\left(-\frac{1}{204}f_{n+4}+\frac{23}{204}f_{n+3}+\frac{185}{204}f_{n+2}+\frac{11}{68}f_{n+1}\right)\\
\frac{480}{17}y_{n+3}-\frac{720}{17}y_{n+2}+\frac{240}{17}y_{n}=h^{2}\left(-\frac{3}{17}f_{n+4}+\frac{52}{17}f_{n+3}+\frac{402}{17}f_{n+2}+\frac{252}{17}f_{n+1}+f_{n}\right)\\
\frac{424}{51}y_{n+3}-\frac{441}{34}y_{n+2}+\frac{475}{102}y_{n}=h^{2}\left(-\frac{3}{85}f_{n+4}+\frac{69}{85}f_{n+3}+\frac{606}{85}f_{n+2}+\frac{303}{85}f_{n+1}\right)-hy'_{n}\\
\frac{26}{17}y_{n+3}-\frac{61}{34}y_{n+2}+\frac{9}{34}y_{n}=h^{2}\left(\frac{1}{6120}f_{n+4}+\frac{73}{680}f_{n+3}+\frac{1007}{680}f_{n+2}+\frac{4319}{6120}f_{n+1}\right)+hy'_{n+1}\\
\frac{40}{51}y_{n+3}-\frac{23}{34}y_{n+2}-\frac{11}{102}y_{n}=h^{2}\left(-\frac{8}{765}f_{n+4}+\frac{11}{85}f_{n+3}+\frac{19}{85}f_{n+2}-\frac{127}{765}f_{n+1}\right)+hy'_{n+2}\\
\frac{62}{51}y_{n+3}-\frac{45}{34}y_{n+2}+\frac{11}{102}y_{n}=h^{2}\left(\frac{9}{680}f_{n+4}-\frac{207}{680}f_{n+3}-\frac{33}{680}f_{n+2}+\frac{111}{680}f_{n+1}\right)+hy'_{n+3}\\
\frac{8}{17}y_{n+3}-\frac{7}{34}y_{n+2}-\frac{9}{34}y_{n}=hy'_{n+4}-h^{2}\left(\frac{253}{765}f_{n+4}+\frac{109}{85}f_{n+3}+\frac{26}{85}f_{n+2}+\frac{287}{765}f_{n+1}\right)
\end{array}\right\} \quad (10)
$$

$\varphi_i(x)$ of Shifted Legendre Polynomial type

Again, $\varphi_i(x)$ in (3) is taken to be of Shifted Legendre polynomial, where the points $x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} are selected interpolation points with x_{n+1}, x_{n+2} and x_{n+3} as collocation points. The continuous formulation of this approach in the form of (4), its first and second derivatives are also evaluated at { x_n }, { $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ } and { x_{n+4} } respectively as in (11).

$$
\left.\begin{array}{l}
y_{n+4}+16y_{n+3}-34y_{n+2}+16y_{n+1}+y_{n}=h^{2}\left(\frac{8}{3}f_{n+3}+\frac{44}{3}f_{n+2}+\frac{8}{3}f_{n+1}\right)\\
\frac{127}{30}y_{n+4}+\frac{1012}{15}y_{n+3}-\frac{4399}{30}y_{n+2}+\frac{1124}{15}y_{n+1}=h^{2}\left(\frac{508}{45}f_{n+3}+\frac{2767}{45}f_{n+2}+\frac{80}{9}f_{n+1}\right)+hy'_{n}\\
\frac{4}{15}y_{n+4}+\frac{27}{10}y_{n+3}-\frac{36}{5}y_{n+2}+\frac{127}{30}y_{n+1}=h^{2}\left(\frac{3}{5}f_{n+3}+\frac{12}{5}f_{n+2}\right)-hy'_{n+1}\\
\frac{7}{30}y_{n+4}+\frac{32}{15}y_{n+3}-\frac{119}{30}y_{n+2}+\frac{8}{15}y_{n+1}=h^{2}\left(\frac{23}{45}f_{n+3}+\frac{77}{45}f_{n+2}+\frac{1}{9}f_{n+1}\right)+hy'_{n+2}\\
\frac{4}{15}y_{n+4}+\frac{1}{30}y_{n+3}-\frac{28}{15}y_{n+2}+\frac{47}{30}y_{n+1}=h^{2}\left(\frac{32}{45}f_{n+3}+\frac{68}{45}f_{n+2}+\frac{1}{9}f_{n+1}\right)-hy'_{n+3}\\
\frac{127}{30}y_{n+4}-\frac{36}{5}y_{n+3}+\frac{27}{10}y_{n+2}+\frac{4}{15}y_{n+1}=h^{2}\left(\frac{12}{5}f_{n+3}+\frac{3}{5}f_{n+2}\right)+hy'_{n+4}\\
12y_{n+4}+204y_{n+3}-444y_{n+2}+228y_{n+1}=h^{2}\left(33f_{n+3}+185f_{n+2}+23f_{n+1}-f_{n}\right)\\
12y_{n+4}-36y_{n+3}+36y_{n+2}-12y_{n+1}=h^{2}\left(f_{n+4}+9f_{n+3}-9f_{n+2}-f_{n+1}\right)
\end{array}\right\} \quad (11)
$$

METHOD ANALYSIS

Order and Error Constant

Following [35], the local truncation error associated with the method is defined by

$$L[y(x);h]=\sum\left(\alpha_j y(x_n+jh)-h^2\beta_j f(x_n+jh)\right) \tag{12}$$

where $y(x)$ is assumed to have continuous derivative of sufficiently high order.

Thus expanding (12) by Taylor series, the co-efficient of the expansion is generalized recursively as

$$c_o=\alpha_o+\ldots+\alpha_k \qquad c_1=\sum_{j=0}^{k} j\alpha_j \qquad c_2=\frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j-(\beta_0+\ldots+\beta_k)$$

$$c_p=\frac{1}{p!}\sum_{j=1}^{k} j^p\alpha_j-\frac{1}{(p-2)!}\sum_{j=1}^{k} j^{(p-2)}\beta_j, \qquad p=3,4,\ldots,q \tag{13}$$

The method will be of order $(q-2)$ with error constant as the value of c_q if $c_o=c_1=\ldots=c_{q-1}=0$ and $c_q\neq 0$

Hence, the discrete schemes of (7) are of non-uniform order $[6,5,5,5,5,5,5,5]^T$ and error constants $\left[-\frac{2}{945},\frac{1}{480},\frac{4}{315},\frac{1}{480},-\frac{61}{14},\frac{2}{7},-\frac{61}{14},\frac{4}{315}\right]^T$ respectively.

Similarly, the order and error constant of each of the discrete schemes of (8) is $[5,5,5,5,5,5,5,5]^T$ and $\left[\frac{17}{4440},\frac{1}{8880},\frac{5093}{372960},\frac{57}{14},\frac{1173}{28},\frac{143}{14},-\frac{131}{41440},-\frac{6}{3}\right]^T$ respectively.

As well, the discrete schemes of (9) is respectively of non-uniform order $[6,5,5,5,5,5,5,5]^T$ and of error constants $\left[-\frac{1}{945},-\frac{1}{480},\frac{1}{480},\frac{4}{315},-\frac{61}{10080},\frac{1}{630},-\frac{61}{10080},\frac{4}{315}\right]^T$.

Again, (10) is found to be of uniform order $[5,5,5,5,5,5,5,5]^T$ with respective error constants $\left[-\frac{1}{255},-\frac{1}{4080},\frac{1}{17},\frac{11}{2380},-\frac{491}{173060},\frac{23}{7140},-\frac{67}{19040},\frac{293}{21420}\right]^T$.

The discrete schemes of (11) is also of non-uniform order $[6,5,5,5,5,5,5,5]^T$ and of error constant $\left[\frac{23}{3780},-\frac{1}{350},-\frac{1}{350},\frac{17}{6300},\frac{1}{350},-\frac{1}{350},-\frac{1}{20},-\frac{1}{20}\right]^T$ respectively.

Consistency and Zero Stability

Since the derived block (7), (8), (9), (10) and (11) are of order greater than one, they are sufficiently consistent.

The roots of the first characteristic polynomial of the derived blocks were found to have all their roots within the unit circle with the spurious roots being simple, hence the Zero stability of the derived methods (7), (8), (9), (10) and (11).

Following the determination of the consistence and stability properties of the derived methods, we deduced that the derived blocks are convergent.

RESULTS AND DISCUSSION

Solution to test problems are approximated using each of the derived system of discrete schemes. The uniform performances of the various approaches (7), (8), (9), (10) and (11) are investigated. Comparative observation of the numerical results are discussed.

Problem 1

$$y'' = 2y + 1, \quad y(0) = 1, \quad y'(0) = 0, \quad 0 \le x \le 1, \quad h = 0.1$$

Exact Solution: $y(x) = \frac{3}{4}\left(e^{\sqrt{2}x} + e^{-\sqrt{2}x}\right) - \frac{1}{2}$

The maximum error encountered within the interval of solution using each of the set of derived methods (7), (8), (9), (10) and (11) are tabulated below.
Computations are performed iteratively on Maple 18 software.

Table 1: Numerical Result for Problem 1, Comparing Maximum Error for Derived Methods

S/N0	Method	Max Error
1	Block (7)	2.873689207650E-07
2	Block (8)	2.873689207655E-07
3	Block (9)	2.873689207654E-07
4	Block (10)	2.873689207836E-07
5	Block (11)	2.873689207894E-07

Table 1 clearly reveals that all varied choices of interpolation polynomial as well as variants of interpolation and collocation points lead to considerably the same result.
It is apparent that, regardless of the choice of orthogonal polynomial chosen as an interpolating function for deriving a *k*-step method, even though the derived discrete schemes comprising the various block methods varies, exceptionally the same results will be obtained when implemented in solving a particular problem.
It is also easily seen from (7) that uniform block of discrete schemes were obtained when the choices of interpolating function varied, but with the same choices of interpolation and collocation points.

Problem 2

$$y'' = 2\cos(x) + x, \quad y(0) = 0, \quad y'(0) = 1, \quad 0 \le x \le 1, \quad h = 0.1,\ 0.01,\ 0.001$$

Exact Solution: $y(x) = \frac{x^3}{6} - 2\cos(x) + 2 + x$

The maximum error encountered for each of the varied step sizes is presented in Table 2

Table 2: Maximum Error for Variable Step Size Mode of Solution to Problem 2

s/no	step size *h*	maximum error
1	0.1	1.3595×10^{-08}
2	0.01	9.7297×10^{-15}
3	0.001	9.5110×10^{-21}

Table 2 above shows conformity of the derived methods with necessary convergent criteria, that is, vanishing error as step size vanishes. Thus, it is concluded that the derived methods are consistent and convergent solvers.

CONCLUSION

Conclusively, it is established that whenever interpolation and collocation approach is employed as a derivation technique, the choices of interpolation or collocation points as well as

basis or interpolating function does not affect the final outcome in terms of performance of derived methods, when a *k*-step solution algorithm is derived.
This idea is extended to multi step methods for higher order initial and boundary value problems of ordinary differential equations, it is however noteworthy to state that what may increase the performance of derived linear multistep methods over existing methods in terms of accuracy is if a higher step number method is derived.
Further research points towards considering newer ideas of computation, especially for problems in the form of system of differential equation. Such methods may include multi-grid method and quantum computational methods.

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