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SOME PROPERTIES OF FUZZY MULTIGROUPS

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ABSTRACT

The concept of fuzzy multigroup is an algebraic structure of a fuzzy multiset that generalizes both the classical group and fuzzy group. In fact, fuzzy multigroup constitutes an application of fuzzy multiset to the elementary theory of classical group. In this paper, some notions of fuzzy multigroup such as normal series and its properties, fuzzy multigroupoid, fuzzy multimonoid, centralizer of fuzzy multigroup etc., were presented. Some theorems on these together with their proofs were formulated. Some related results on fuzzy multigroups were established.

Keywords: fuzzy multigroup, Normal series of fuzzy multigroup, fuzzy multigroupoid, fuzzy multimonoid, centralizer of fuzzy multigroup

INTRODUCTION

Classical set theory is a basic concept used to represent various situations in mathematics and some other sciences. This theory was formulated by a German Mathematician George Ferdinand Ludwig Cantor (1845-1918). Cantor defined a set as a collection into a whole of definite and well-distinguished objects (called elements) of our intuition or thought. For a set, the order of succession of its elements is ignored and also an element shall be definite and shall not be allowed to appear more than once. As such the issue of vagueness or imperfect knowledge has been a problem for a long time for philosophers, mathematicians, logicians and computer scientists, particularly in the area of artificial intelligence etc. To handle situations that involve repetition, vagueness or imperfect knowledge, tools like fuzzy sets, rough sets, soft sets, multisets etc., were suggested.

Multiset in particular is necessary as in various circumstances repetition of elements is inevitable, for example considering a graph with loops in chemical bonding, molecules of a substance, repeated roots of polynomial equations, repeated readings in volumetric analysis experiment, repeated observations in statistical samples and so on. A multiset (mset), which is a generalization of classical or standard (Cantorian) set is an unordered collection of objects with repetition allowed and it has found applications in Mathematics, computer science, economics etc. For more details on the theory and applications of multisets the reader may refer to (Blizard, 1989; Singh et al., 2007; Girish and Johns, 2012; Singh and Isah, 2016; Isah and Tella, 2015).

The concept of group theory which stems from set theory plays an important role in the past and present day mathematics and sciences. Groups appear in quantum physics, mechanics, geometry, topology, physics, chemistry etc.

Fuzzy set theory proposed by Zadeh in 1965 revolutionized the entire mathematics with its capacity to adequately tackle uncertainties. For a classical set X, a fuzzy set over X, or a fuzzy subset of X, is characterized by a membership function μ which associates values from the closed unit interval I = [0,1] to members of X. The applications of fuzzy sets are found in medical diagnosis, information analysis etc. For more details on the development and some of the applications of fuzzy set refer to (Zadeh, 1965; Miyamoto, 1997; Adam and Fatma, 2016; Alkali and Isah, 2019; Isah, 2020).

The theory of fuzzy sets has grown impressively over the years giving birth to fuzzy groups proposed in (Rosenfeld, 1971). Sincerely, the concept of fuzzy groups is an application of fuzzy set to group theory. The notion of fuzzy groups caught the attentions of algebraists and an end to its ramifications seems to be very far.

The theory of fuzzy multisets or fuzzy bags was introduced by (Yager, 1986) as an attempt to fuzzify multisets proposed by (Knuth, 1981). In terms of similarity, fuzzy multisets are the generalized fuzzy sets introduced by Zadeh. The theory of fuzzy multisets is a mathematical framework which can represent multiple occurrences of a subject item with degrees of relevance and it has been studied in relation to a variety of information systems including relational database and so on. Fuzzy multiset is a multiset of pairs, where the first part of each pair is an element of, and the second part is the degree to which the first part belongs to. An element of a fuzzy multiset can occur more than once with possibly the same or different membership values. In this paper, the theories of multigroups and fuzzy multigroups were explored and some of the properties of multigroups were extended to fuzzy multigroups.

PRELIMINARIES

Definition 1 (Zadeh, 1965) Let X be a set. A fuzzy set A of X is characterized by a membership function $\mu_A: X \longrightarrow [0,1]$.

Definition 2 (Rosefield, 1971; Morderson et al., 2005) Let X be a group. A fuzzy set A of X is called a fuzzy group (or a fuzzy subgroup) of X if

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \ \ \forall x \in X$,
(ii) $\mu_A(x^{-1}) = \mu_A(x) \ \ \forall x \in X$.

Definition 3 (Hariprakash, 2016) A semigroup X in which every element has a weight (a membership function) is called a fuzzy semigroup.

Definition 4 (Jena et al., 2011) Let X be a set. A multiset M over X is characterized by a count function $C_M: X \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers including zero. The set of all multisets of X is denoted by MS(X).

Definition 5 (yager, 1986) Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$CM_A: X \longrightarrow [0,1]$ of which the value is a multiset of the unit interval $I = [0,1]$.

Definition 6 (Nazmul et al., 2013, Ejegwa 2018) Let X be a group. A multiset M over X, is called a multigroup of X if
(i) $C_M(xy) \geq C_M(x) \wedge C_M(y)\ \forall x, y \in X$,
(ii) $C_M(x^{-1}) = C_M(x)\ \forall x \in X$.
We denote the set of all multigroups of X by MG(X).

FUZZY MULTIGROUP

Definition 7 (Shinoj et al., 2015) Let X be a group. A fuzzy multiset A of X is said to be a fuzzy multigroup of X if
(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y)\ \ \forall x \in X$,
(ii) $CM_A(x^{-1}) = CM_A(x)\ \ \forall x \in X$.
By implication, a fuzzy multiset A over X is called a fuzzy multigroup of a group X, if

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)\ \ \forall x, y \in X.$$

It follows immediately from the definition that,

$$CM_A(e) \geq CM_A(x)\ \ \forall x \in X$$

where e is the identity element of X. The set of all fuzzy multigroups of X is denoted by FMG(X).

Definition 8 (Ejegwa, 2018) Let A $\in$ FMG(X), a fuzzy submultiset B of A is called a fuzzy submultigroup of A denoted by $B \sqsubseteq A$ if B is a fuzzy multigroup. A fuzzy submultigroup B of A is a proper fuzzy submultigroup, denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 9 (Ejegwa, 2019**)** Let A be a fuzzy submultigroup of B $\in$ FMG(X). Then, A is called a normal fuzzy submultigroup of B, if $CMA(xyx^{-1}) = CMA(y), \forall\, x, y \in X$.

Definition 10 (Ejegwa and brahim, 2020**)** Let X be a group. Let B $\in$ FMG(X), then the commutator of x and y in B, is the element $[x, y] = x^{-1}y^{-1}xy \in X$, $\forall$ x, y $\in$X.

SOME PROPERTIES OF FUZZY MULTIGROUP

Theorem 1
Let B be a commutative fuzzy multigroup of a group X and e be the identity element of X. Then $CM_B([x,y]) = CM_B(e)$.
Proof
Let x, y $\in$X such that x and y commute with each other.
Now $CM_B([x,y]) = CM_B(x^{-1}y^{-1}xy) = CM_B(x^{-1}xy^{-1}y)$
$\geq CM_B(x^{-1}x) \wedge CM_B(y^{-1}y)$
$= CM_B(e) \wedge CM_B(e)$
$= CM_B(e)$
i.e., $CM_B([x,y]) \geq CM_B(e)$.
Moreover, $CM_B(e) = CM_B(x^{-1}xy^{-1}y) = CM_B((x^{-1}xy^{-1}y)e)$
$= CM_B((x^{-1}xy^{-1}y)(x^{-1}xy^{-1}y))$
$\geq CM_B(x^{-1}xy^{-1}y) \wedge CM_B(x^{-1}xy^{-1}y)$
$= CM_B(x^{-1}y^{-1}xy) \wedge CM_B(x^{-1}y^{-1}xy)$
$= CM_B(x^{-1}y^{-1}xy)$
$= CM_B([x,y])$
i.e., $CM_B(e) \geq CM_B([x,y])$
Thus, $CM_B([x,y]) = CM_B(e)$.

Theorem 2
Let B be a commutative fuzzy multigroup of a group X and e be the identity element of X. Then
$CM_B([x,y]) \geq CM_B(x) = CM_B(y)$.
Proof
Now $CM_B([x,y]) = CM_B(x^{-1}y^{-1}xy)$
$\geq CM_B(x^{-1}) \wedge CM_B(x)$
$= CM_B(x) \wedge CM_B(x)$
$= CM_B(x)$.
Also,
$CM_B([x,y]) = CM_B(x^{-1}y^{-1}xy)$
$\geq CM_B(y^{-1}) \wedge CM_B(y)$
$= CM_B(y) \wedge CM_B(y)$
$= CM_B(y)$
Thus, $CM_B([x,y]) \geq CM_B(x) = CM_B(y)$.

Definition 11 Let X be a group. A fuzzy multiset B over X is called a fuzzy multigroupoid of X if $\forall$x, y $\in$ X, $CM_B(xy) \geq CM_B(x) \wedge CM_B(y)$.

Theorem 3
A fuzzy multigroupoid B of a finite group X is a fuzzy multigroup if $CM_B(x^{-1}) = CM_B(x^{n-1})$, $\forall$x, y $\in$ X and $n \in \mathbb{N}$, the set of non-negative integers.
Proof
Since B is a fuzzy multigroupoid of X, we have $CM_B(xy) \geq CM_B(x) \wedge CM_B(y)$, $\forall$x, y $\in$ X.
Suppose $CM_B(x^{-1}) = CM_B(x^{n-1})$, $\forall$x $\in$ X and $n \in \mathbb{N}$.
We have $CM_B(x^{-1}) = CM_B(x^{n-2}x) \geq CM_B(x^{n-2}) \wedge CM_B(x)$
$\geq CM_B(x) \wedge CM_B(x) \wedge \ldots \wedge CM_B(x)$
$= CM_B(x)$.
That is, $CMB(x^{-1}) \geq CM_B(x)$.
But $CM_B(x) = CM_B((x^{-1})^{-1}) \geq CM_B(x^{-1})$
i.e., $CM_B(x) \geq CM_B(x^{-1})$
Hence, $CM_B(x^{-1}) = CM_B(x)$.

Definition 12
A fuzzy multiset B over a group X is a fuzzy semimultigroup if $CM_B(xyz) = CM_B(yxz)$, $\forall$x, y, z $\in$X.
Definition 13
A fuzzy multiset B over a group X is a fuzzy multimonoid of X if it is a fuzzy semimultigroup of X and it satisfies $CM_B(e) \geq CM_B(x)$, $\forall$x $\in$X, where e is the identity element of X.

Theorem 4
Let A be a fuzzy semimultigroup of a group X. Then, A is a fuzzy multimonoid if
$CM_A([x, y]) \geq CM_A(x)$, $\forall$x, y $\in$X.
Proof
Let x, y $\in$X such that x, y $\neq$ e, where e is identity element of X.
Since A is a fuzzy semimultigroup of X, the result follows if we show that $CM_A(e) \geq CM_A(x)$, $\forall$x $\in$X.
Suppose $CM_A([x,y]) \geq CM_A(x)$, $\forall$x, y $\in$X. By theorem 1, $CM_A(e) = CM_A([x,y])$, $\forall$x, y $\in$X.
Thus, $CM_A(e) \geq CM_A(x)$ $\forall$x $\in$X.
Therefore, A is a fuzzy multimonoid.

Definition 14
Let B $\in$FMG(X) and A be a fuzzy submultigroup of B. Then the centralizer of a fuzzy submultigroup A of B is the group:
$Z(A) = \{\, x \in X \ni CM_A(xy) = CM_A(yx)$ and $CM_A(xyz) = CM_A(yxz), \forall y, z \in X \,\}$.

Theorem 5
Let B $\in$FMG(X) and A be a fuzzy submultigroup of B. Then x $\in$Z(A) if
$CM_A(xy_1 \ldots y_n) = CM_A(y_1xy_2 \ldots y_n) = \ldots = CM_A(y_1y_2 \ldots y_nx)$ $\forall y_1, y_2, \ldots, y_n \in X$.
Proof
Let n = 1, we have $CM_A(xy_1) = CM_A(y_1x)$ $\forall y_1 \in X$.
Let n = 2, we have $CM_A(xy_1y_2) = CM_A(y_1xy_2) = CM_A(y_1y_2x)$ $\forall y_1, y_2 \in X$.
Let n = 3, we have $CM_A(xy_1y_2y_3) = CM_A(y_1xy_2y_3)$

$= CM_A(y_1xy_2y_3)$
$= CM_A(y_1y_2xy_3)$
$= CM_A(y_1y_2y_3x)\ \forall y_1, y_2, y_3 \in X$.

Thus, $x \in Z(A)$ for n = 1, 2 and 3.
Now, for n = k+1, it follows that,

$$CM_A(xy_1\ldots y_ky_{k+1}) = CM_A(y_1xy_2\ldots y_ky_{k+1}) = \ldots = CM_A(y_1y_2\ldots xy_ky_{k+1}) = CM_A(y_1y_2\ldots y_kxy_{k+1}) = CM_A(y_1y_2\ldots y_ky_{k+1}x)$$

$\forall y_1y_2\ldots y_k, y_{k+1} \in X$
i.e., $x \in Z(A)$ for n = k+1.
Hence, the result follows by induction.

Theorem 6
Let $B \in FMG(X)$ and A be a fuzzy submultigroup of B and let $T = \{x \in X : CM_A(xyx^{-1}y^{-1}) = CM_A(e)\ \forall y \in X\}$, then $T = Z(A)$.
Proof
Let $x \in T$, then for all $y, z \in X$, we have

$$CM_A((xyz(yxz)^{-1})) = CM_A(xyzz^{-1}x^{-1}y^{-1}) = CM_A(xyx^{-1}y^{-1}) = CM_A(e)$$

$\Rightarrow CM_A(xyz) = CM_A(yxz), \forall y, z \in X$ and so, $x \in Z(A)$.
Thus, $T \subseteq Z(A)$
Again, if $x \in Z(A)$ then $CM_A(xy) = CM_A(yx)$
$\Rightarrow CM_A(xyx^{-1}y^{-1}) = CM_A(e)\ \forall x, y \in X$.
So $x \in T$.
Thus $Z(A) \subseteq T$ and hence, $T = Z(A)$.

Theorem 7
Let B be a fuzzy multiset over a semigroup X and A be a fuzzy submultigroup of B. If Z(A) is nonempty, then Z(A) is a fuzzy semigroup of X. Moreover, if X is a group, then Z(A) is a normal fuzzy subgroup of X.
Proof
Let $x_1, x_2 \in Z(A)$, then for all $y, z \in X$, we have
$CM_A((x_1x_2)yz) = CM_A(y(x_1x_2)z)$ by theorem 5
and $CM_A((x_1x_2)y) = CM_A(y(x_1x_2))$
Hence, we have $x_1, x_2 \in Z(A)$.
Thus, Z(A) is a fuzzy semigroup of X
Suppose X is a group. Then Z(A) is nonempty since $e \in Z(A)$.
If $x \in Z(A)$, then

$$CM_A(x^{-1}yz) = CM_A(x^{-1}yx^{-1}xz)\ CM_A((x^{-1}yx^{-1})xz) = CM_A(xx^{-1}yx^{-1}z) = CM_A(yx^{-1}z)\ \forall y, z \in X$$

Thus, $x^{-1} \in Z(A)$.
Hence, Z(A) is a fuzzy subgroup of X.
Next, let $x \in Z(A)$ and $g \in X$, then by theorem 5
$CM_A((g^{-1}xg)yz) = CM_A(yg^{-1}xgz) = CM_A(y(g^{-1}xg)z), \forall y, z \in X$ and so, $g^{-1}xg \in Z(A)$.
Thus, Z(A) is a normal fuzzy subgroup of X.

Theorem 8
Let C be a fuzzy semimultigroup of a group X, and both A and B be fuzzy submultiset of C, then $Z(A) \cap Z(B) \subseteq Z(A \cap B)$.
Proof
Let $x \in Z(A)$ and $x \in Z(B) \Rightarrow x \in Z(A) \cap Z(B)$.
For any $y, z \in X$, we get

$$\begin{aligned} CM_{A\cap B}(xyz) &= CM_A(xyz) \wedge CM_B(xyz) \\ &= CM_A(gxyzg^{-1}) \wedge CM_B(gxyzg^{-1}), \forall g \in X \\ &= CM_A(y(gx)zg^{-1}) \wedge CM_B(y(gx)zg^{-1}) \\ &= CM_A(y(xg)zg^{-1}) \wedge CM_B(y(xg)zg^{-1}) \\ &= CM_A(y(xg)g^{-1}z) \wedge CM_B(y(xg)g^{-1}z) \\ &= CM_A(yxz) \wedge CM_B(yxz) \\ &= CM_{A\cap B}(yxz) \end{aligned}$$

Thus, $x \in Z(A \cap B)$.
Therefore $Z(A) \cap Z(B) \subseteq Z(A \cap B)$.

Definition 15
Let A be a fuzzy multigroup of a group X. Then the centre of A is defined as
$C(A) = \{x \in X: CM_A([x, y]) = CM_A(e), \forall y \in X\}$.

Definition 16
Let X be a group. Let A be a normal fuzzy submultigroup of $B \in FMG(X)$. Then, a finite chain, $1 = A_0 \leq A_1 \leq A_2 \leq A_3 \leq \ldots \leq A_n = A$ of normal fuzzy submultigroup of B, is called normal series for A and n is called the length of the series.
A finite chain $1 = A_0 \leq A_1 \leq A_2 \leq A_3 \leq \ldots \leq A_n = A$ ($0 \leq i \leq n$) of normal fuzzy submultigroup of $B \in FMG(X)$ is called subnormal series of A if A_i is normal to A_{i+1} $\forall i$.
If A is a normal fuzzy submultigroup of $B \in FMG(X)$, then a submultigroup H, which is a term in the normal series of A, is called a subnormal multigroup in A, written HS_nA or H<<A.
Moreover, taking the commutator fuzzy submultigroups of the commutator fuzzy submultigroup of C, we have a descending series of normal fuzzy submultigroups $C \geq C^1 \geq C^2 \geq C^3 \geq C^4 \geq \ldots$ which is called derived series of the fuzzy multigroup C.

CONCLUSION
The concepts of fuzzy multisets and multigroups have been studied and some notions were extended to fuzzy multigroups, some results were established. Nonetheless, in the future, other fuzzy group theoretic notions could be exploited in fuzzy multigroup setting.

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