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BAYESIAN ESTIMATION OF FOUR PARAMETERS ADDITIVE CHEN-WEIBULL DISTRIBUTION

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ABSTRACT

Models with bathtub-shaped failure rate function have been widely accepted in the field of reliability and medicine and are particularly useful in reliability related decision making and cost analysis. In this study, the additive Chen-Weibull (ACW) distribution with increasing and bathtub-shaped failure rates function is studied using Bayesian and non-Bayesian approach using two real data set. The Bayes estimator were obtained by assuming non-informative prior (Half-Cauchy) under square error loss function (SELF), the Laplace Approximation and Monte Carlo Markov Chain (MCMC) techniques conducted in R were used to approximate the posterior distribution of ACW model. In addition, the maximum product of spacing method (MPS) of estimation is also considered using mpsedist function in BMT package in R with good set of initial values of parameters. We compared the performance of the two difference estimation methods by using Kolmogorov-Smirnov test. And the result showed that MPSEs method outperformed Bayesian approach

Keywords: Bayes estimators, prior distribution, square error loss function, Chen Distribution, Weibull distribution

INTRODUCTION

The techniques of proposing, generalized, modified or extended classes of distributions have pulled theoretical and applied statisticians due to their flexibility in modeling data in practice. The generalization, modifications and extensions of probability distribution makes it richer and more flexible for modeling data in practice. One way of generalizing, modifying and extension of probability distribution is by adding a new parameter to the base line model. Weibull distribution which was named after Swedish mathematician (Weibull, 1938), who described it in detail as it is a widely used life time distribution not only in reliability but also in many other fields. However, the failure rate function of the Weibull distribution can only be increasing, decreasing, or constant. But it fails to handle the life time data with bathtub-shaped failure rate function. Therefore, many generalizations, extensions, and modifications of the Weibull distribution have been developed to meet the requirement. Example, (Xie and Lai, 1996) proposed an additive Weibull (addW) distribution by combining two Weibull distribution together with cumulative distribution function (CDF)

$$F(x) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}}, x \geq 0;\ \alpha, \beta \geq 0, \theta > 1, 0 < \gamma < 1. \quad (1)$$

(Sarhan and Zaindin, 2000) Introduce a new three (3) parameter distribution called modified Weibull distribution by generalizing exponential, Rayleigh, linear failure and Weibull distributions. , it is observed that the MWD can have constant, increasing and decreasing hazard rate functions which are desirable for data analysis purposes. (Chen, 2000) Proposed a new two (2) parameter distribution with bathtub shaped or increasing hazard rate function called Chen distribution with positive parameter α and β. The cumulative distribution function (CDF) of Chen distribution is given by

$$F(x) = 1 - e^{\lambda\left(1 - e^{x^{\beta}}\right)}, x \geq 0;\ \lambda, \beta > 0. \quad (2)$$

Neetu et al., (2012) Proposed a new five (5) parameter distribution called Beta generalized Weibull distribution The new distribution generalize BGE, BW, GW,GR, BE, GE, Weibull, Rayleigh and exponential distribution.. The new distribution can have monotone, Bathtub-shaped, unimodal failure rate for different parametric combinations. The new distribution showed that its density can be expressed as a mixture of GW densities. Bo He et al., (2016) Proposed a new five (5) parameter distribution called Additive modify Weibull distribution. The new distribution can have increasing, decreasing and bathtub-shaped hazard rate functions. Elgarhy et al., (2017) Proposed a new four parameter distribution called Exponentiated Weibull-Exponential distribution (EWED). Eisa et al., (2018) introduced a new four parameter distribution called extended Exponentiated Weibull (EEW) distribution. The new distribution can have a decreasing, increasing, decreasing-increasing-decreasing (DID), upside down bathtub (unimodal) and bathtub-shaped failure rate function. Mohammed et al., (2019) proposed a new five parameter generalization of the extended Weibull distribution called generalized extended Exponentiated Weibull (GExEW) distribution by adding one shape parameter to the base line distribution (ExEW). (Tien and Radim, 2020) Proposed and studied a new life time distribution called additive Chen-Weibull distribution with four (shape and scale) parameters. The new distribution is a continuous life time distribution with increasing and bathtub-shaped failure rates proposed by combining the Chen and Weibull distribution in a series system with two independent components. This situation is of particular interest in applications, where the failure times of a system with two or more failure modes must be modeled. One component, representing the first failure mode follows the Chen distribution and the other follows the Weibull distribution. Each component has a potential failure time associated with each failure mode. The ACW model is demonstrated to be better than many existing life time distribution when fitting to two real reliability data sets. The CDF of the additive Chen-Weibull (ACW) distribution with four parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)'$ is defined by

$$F(x) = 1 - e^{\lambda(1-e^{x\gamma}) - (\alpha x)^{\beta}},$$

Where $x \geq 0, \alpha, \beta, \gamma > 0, \lambda \geq 0$

The probability density function (PDF) is defined by

$$f(x) = (\lambda\gamma x^{\gamma-1} e^{x\gamma} \alpha\beta(\alpha x)^{\beta-1}) e^{\lambda(1-e^{x\gamma}) - (\alpha x)^{\beta}}, \quad (3)$$

Where $x \geq 0$.

And the failure rate and reliability functions are respectively

$$h(x) = (\lambda\gamma x^{\gamma-1} e^{x\gamma} + \alpha\beta(\alpha x)^{\beta-1} \quad (4)$$

And

$$R(x) = e^{\lambda(1-e^{x\gamma}) - (\alpha x)^{\beta}} \quad (5)$$

As shown in the literature, the Bayesian approach is a widely used for the estimation of unknown parameters of any proposed probability distribution. Example, (Chris and Noor, 2012) studied Bayesian analysis of the survival function and failure rate of Weibull distribution with censored data. The Bayes estimator are obtained under three different loss function. (Ahmad and Ahmad, 2013) considered the Bayesian approach for the estimation of the scale parameter of two parameter Weibull distribution with known shape. They obtained Bayes' estimator of Weibull distribution by using Jeffery's and extension of Jeffery's prior under linear exponential loss function and symmetric loss function. (Al Omari, 2016) Studied Bayesian using MCMC of Gompertz distribution based on interval censored data with three loss functions. Vikas et al., (2017) discussed classical and Bayesian methods of estimation for power Lindley distribution with application to waiting time data. Kamran et al., (2019) studied a three-parameter Frechet distribution with medical application. The Bayesian estimators were obtained using LINEX and general entropy loss function by considering Gama and non-informative priors through Lindley's approximation. (Tien and Radim, 2020) Used Bayesian assuming gamma prior under square error loss function (SELF) and maximum likelihood estimate (MLE) for the estimation of the four unknown parameters of ACW model.

The main objectives of this article is to provide complete Bayesian analysis by considering half-Cauchy distribution prior under square error loos function (SELF) for the estimation of the four unknown parameters of the ACW model. The rest of the article is organized as follows. We derived the maximum product of spacing method of estimation (MPSE) for estimating the four unknown parameters of the ACW model. Bayes estimators by assuming half-Cauchy prior under square error loss function (SELF) has been provided. Application to two real reliability data sets and the comparison of the two estimation methods using Kolmogorov-Smirnov test statistics are also presented. And finally the conclusion is also provided.

MATERIAL AND METHODS

Estimation Using Maximum Product of Spacing Method

We discussed the maximum product of spacing method for estimating the parameters of any probability distribution. In this section, the method of maximum product of spacing (MPS) introduced by (Chen and Amin, 1979) will be used to estimate the unknown parameters of additive Chen-Weibull (ACW) distribution. Let $X_{1,} X_{2,} \ldots . X_n$ be i.i.d random variables from the additive chen-weibull distribution and let $X_{(1),} X_{(2)}, \ldots X_{(n)}$ be the corresponding order statistics.

The cumulative distribution function (CDF) of additive Chen-Weibull (ACW) distribution with four parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)^T$ is defined by

$$F(x) = 1 - e^{\lambda(1-e^{x\gamma}) - (\alpha x)^{\beta}}, \quad x \geq 0, \alpha, \beta, \gamma > 0, \lambda \geq 0 \quad (6)$$

Then, we define spacing as

$$D_i = F(X_i) = 1 - e^{\lambda(1-e^{x_i\gamma}) - (\alpha x)^{\beta}} \quad (7)$$

$$D_{n+i} = 1 - F(X_i) = 1 - 1 - e^{\lambda(1-e^{x_i\gamma}) - (\alpha x)^{\beta}}$$

And the general term of spacing is given by

$$D_i = F(X_i) - F(X_{(i-1)})$$

Such that $\sum_i^n D_i = 1$

In method of product spacing, we choose $\boldsymbol{\theta}$ such that it maximizes the product of spacing or in other words it maximizes the geometric mean of spacing i.e.

$$M = \prod_{i=1}^{n+1} D_i^{\frac{1}{n+1}} \quad (8)$$

We defined the term **S** which is obtained by taking log on both side of the equation (8) i.e Set $S = logM$ we get

$$S = \frac{1}{(n+1)} \sum_{1=1}^{n+1} logD_i$$

$$S = \frac{1}{(n+1)} \sum_{i=1}^{n+1} \log(1 - e^{\lambda(1-e^{x_i\gamma}) - (\alpha x_i)^{\beta}} - 1 - e^{\lambda(1-e^{x_{(i-1)}\gamma}) - (\alpha x_{(i-1)})^{\beta}})$$

$$S = \frac{1}{(n+1)} \sum_{i+1}^{n+1} \lambda(1 - e^{x_i\gamma}) - (\alpha x_i)^{\beta} - \lambda(1 - e^{(x_{i-1})^{\gamma}})\alpha(x_{i-1})^{\beta} \quad (9)$$

To obtain the normal equations for the unknown parameters, we differentiate partially equation (9) with respect to the four (4) parameters (α, β, γ and λ) and equate them to zero. The estimators for α, β, γ and λ can be obtained by

$$\frac{ds}{d\alpha} = -\frac{1}{(n+1)}\sum_{i=1}^{n+1}(x_i)^{\beta} + \lambda - \lambda e^{(x_{i-1})^{\gamma}}(x_{i-1})^{\beta} = 0$$

$$\frac{ds}{d\beta} = -\frac{1}{(n+1)}\sum_{i=1}^{n+1}(\alpha x_i)^{\beta} - \lambda - \lambda e^{(x_{i-1})^{\gamma}}\alpha(x_{i-1})^{\beta-1} = 0$$

$$\frac{ds}{d\gamma} = -\frac{1}{(n+1)}\sum_{i=1}^{n+1} x_i \lambda e^{x_i\gamma}\alpha(x_{i-1})^{\beta}\alpha(x_{i-1})e^{(x_{i-1})^{\gamma}} = 0$$

$$\frac{ds}{d\lambda} = -\frac{1}{(n+1)}\sum_{i=1}^{n+1} 1 - e^{x_i\gamma} + \alpha(x_{i-1})^{\beta}(1e^{(x_{i-1})})^{\gamma} = 0$$

The above expressions cannot be solve analytically, therefore, the iterative procedure techniques (conjugate-gradient algorithm solution) will be used in order to obtain the estimate of the parameters of ACW distributions.

Bayesian estimation under half-Cauchy prior

The Bayesian analysis of concerned reliability model begins with the specification of likelihood function. For this let us assume that, $Y: y_{1,}y_2, \ldots, y_n$ be the observed lifetimes from additive chen-weibull (ACW) distribution with probability density function (PDF)

$$f(x) = (\lambda\gamma x^{\gamma-1}e^{x\gamma} + \alpha\beta(\alpha x)^{\beta-1})e^{\lambda(1-e^{x\gamma})-(\alpha x)^{\beta}}, \qquad x \geq 0.$$

The corresponding likelihood function can be defined as

$$L(Y|\alpha,\beta,\gamma,\lambda) = \left[\prod_{i=1}^{n}\left(\lambda\gamma x_i^{\gamma-1}e^{x_i^{\gamma}} + \alpha\beta(\alpha x_i)^{\beta-1}\right)\right]\exp\left[-\sum_{i=1}^{n}(\lambda(1-e_i^{x_i^{\gamma}}\right) + (\alpha x_i)^{\beta-1})] \tag{10}$$

The next step in Bayesian statistics is to choose a prior distribution that express uncertainty about the parameters of the model before the data is observed. We consider an independent and non-informative prior distribution for the four unknown parameters $(\alpha, \beta, \gamma, \lambda)$. Both the positive parameters are assumed to be half-Cauchy distributed according to their hyperparameters, and are denoted by

$\alpha \sim$ Half-Cauchy (35), $\beta \sim$ half-Cauchy (35), $\gamma \sim$ half-Cauchy (35) and $\lambda \sim$ half-Cauchy (35)

The joint prior distribution is defined as

$$P(\alpha,\beta,\gamma,\lambda) = P((\alpha^2+(35)^2)(\beta^2+(35)^2)(\gamma^2+(35)^2)(\lambda^2+(35)^2)) \tag{11}$$

by Bayes' rules, the joint posterior distribution can be obtained as

$$P(\alpha,\beta,\gamma,\lambda|Y) \propto L(Y|\alpha,\beta,\gamma,\lambda)P(\alpha,\beta,\gamma,\lambda) \tag{12}$$

Taking the log of the prior densities, the logarithm of the unnormalized joint posterior density is calculated according to the Bayes' rule as:

$$\log P(\alpha,\beta,\gamma,\lambda|Y) \propto \log L\,(Y|\alpha,\beta,\gamma,\lambda) + \log p\, P((\alpha^2+(35)^2)(\beta^2+(35)^2)(\gamma^2+(35)^2)(\lambda^2+(55)^2)) \tag{13}$$

To get the correct posterior inference for the positive parameters in the situation that involves optimization of the log-posterior, itself a difficult numerical problem. The package Laplaces Demon favours unconstrained parameterization by making the log-transformation of the positive parameter (Romana and Athar, 2016).

Estimation under Square Error Loss Function

The Bayesian estimates of the four parameters of additive Chen-Weibull (ACW) distribution assuming independent Half-Cauchy prior under square error loss function (SELF) is given by

$$\hat{\alpha} = \int \alpha P(\gamma,\alpha,\lambda,\beta|Y)d\theta \qquad \hat{\lambda} = \int \lambda p((\gamma,\alpha,\lambda,\beta|Y)\,d\theta$$

$$\hat{\beta} = \int \beta p((\gamma,\alpha,\lambda,\beta|Y)\,d\theta \qquad \hat{\gamma} = \int \gamma p((\gamma,\alpha,\lambda,\beta|Y)\,d\theta$$

As we can see from the above expressions that, the marginal posterior densities of the four parameters cannot be obtained in closed-form, therefore the Bayes estimator cannot be analytically computed through the posterior means. Therefore, the numerical approximation method (Laplace approximation) and simulation technique Monte Carlo Markov Chain (MCMC) will be used to approximate the posterior densities of the parameters.

The independent Metropolis-Hasting algorithm is a general MCMC algorithm introduced by (Hastings, 1970) will be used to simulate a random sample from the posterior distribution. The implementation of independent Metropolis-Hasting algorithm and Laplace approximation are given below. Let as assume a target distribution $\boldsymbol{p}(\boldsymbol{\theta}|\boldsymbol{y})$ from which we wish to generate a sample of size T. the metropolis-Hastings algorithm can be described by the following iterative steps; where $\boldsymbol{\theta}^{(t)}$ is the vector of generated values in t iteration of the algorithm

.

Algorithm 1

Given the marginal posterior distribution $P((\gamma|\alpha,\lambda,\beta,Y)$, $P((\alpha|\gamma,\lambda,\beta,Y)$, $P((\beta|\gamma,\alpha,\lambda,Y)$, $P((\lambda|\gamma,\alpha,\beta,Y)$ and sample size N:

Step 1: select a starting value of the chain $\gamma^{(0)}, \alpha^{(0)}, \beta^{(0)}, \lambda^{(0)}$.

Step 2: set $m = 1$.

Step 3: Using the M-H, generate $\gamma^{(m)}$ from $P((\gamma|\alpha^{(m-1)}, \lambda^{(m-1)}, \beta^{(m-1)}Y)$.

Step 4: Using the M-H, generate $\alpha^{(m)}$ from $P(\alpha|\gamma^{(m)}, \lambda^{(m-1)}, \beta^{(m-1)}, Y)$.

Step 5: Using the M-H generate $\alpha^{(m)}$ from $P(\beta|\gamma^{(m)}, \alpha^{(m)}, \lambda^{(m-1)}, Y)$.

Step 6: Using the M-H generate $\lambda^{(m)}$ from $P((\lambda|\gamma^{(m)},\alpha^{(m)},\beta^{(m)},Y)$.
Step 7: set $m = m + 1$.
Step 8: Repeat step 2 to 7until $m = N$ to obtain the samples of γ,α,λ and β
With size$N$, respectively.

Data

A two real data sets will be used for illustrations purposes that is; Aarset and Meeaker-escobar data with a random sample of 50 lifetime's devices and the failure and running times of 30 devices respectively. Two failure modes were observed for this data.

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RESULT AND DISCUSSIONS

In this section, we have given an application of Bayesian and maximum product of spacing method of estimation (MPSE) of the additive Chen-Weibull (ACW) distribution to two real reliability data sets to illustrate the applicability of ACW distribution by the two different estimation methods. The Kolmogorov-Smirnov test is also provided for the comparison between the two estimation methods

Meeker-Escobar data

Table 1. Meeker-Escobar data

2	10	13	23	23	28	30	65	80	88
106	143	147	173	181	212	245	247	261	266
275	293	300	300	300	300	300	300	300	300

Table 1 represent the Meeker-Escobar data with a failure and running times of 30 devices. Two failure modes were observed for this dataset. Many authors in the literature used this data for illustration purpose and the most recent studies are given by (Tien and Radim, 2020), Almaliki et al., (2013), Bo He et al., (2016) and Mohammed et al., (2019).

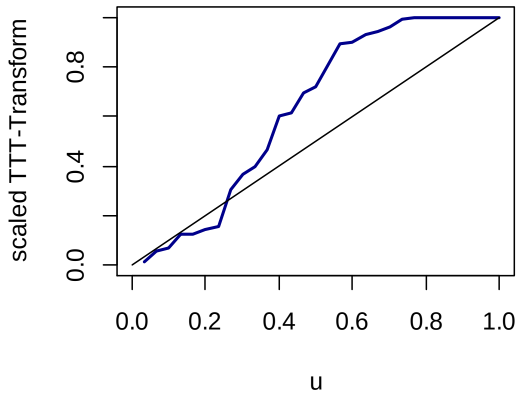


Figure 1. The empirical scaled TTT-transform plot for Meeker-Escobar data
Source: Author's computation aided by R package V 3.6.3

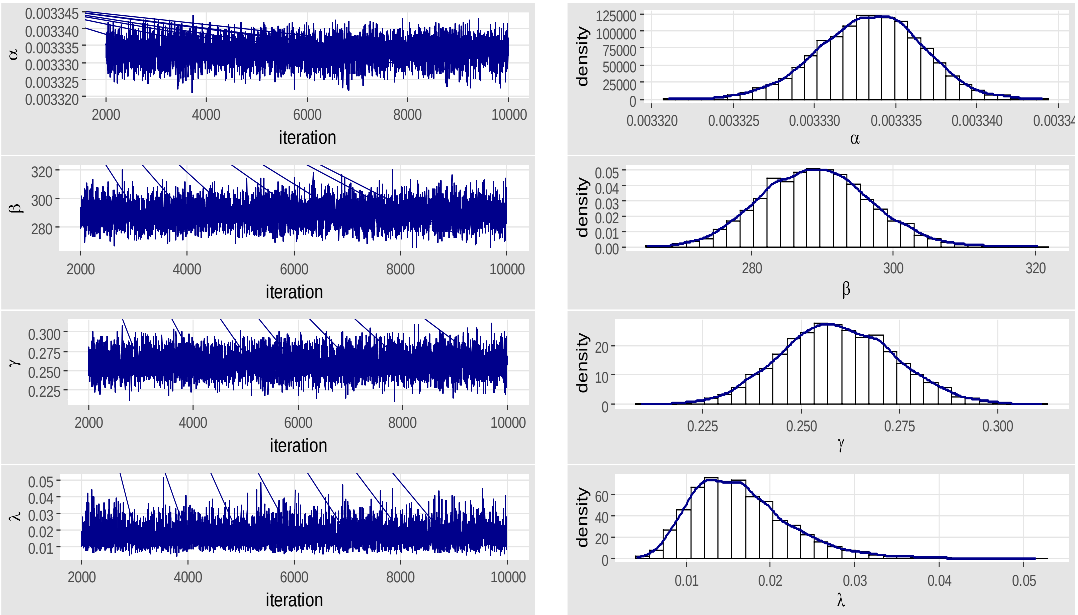
Fig. 1. Provides the empirical scaled TTT-transform plot for Meeker-Escobar data sets. From this plot we can observed that the Meeker-Escobar data have a bathtub-shaped failure rate function.
The independent Metropolis-Hasting algorithm (algorithm 1) in MCMC is used to simulate a random sample from each of the marginal posterior density of four unknown parameters to approximate the posterior distribution of ACW model.

Table 2. Bayes estimates for the parameters of ACW model to Meeker-Escobar Data.

Parameter	Bayes	SD	Bayes 95%CI
γ	0.25942	0.0144	[0.2322, 0.2877]
α	0.00333	0.0000	[0.0033, 0.0033]
λ	0.01660	0.0588	[0.0079, 0.0306]
β	289.109	7.6153	[274.98, 304.49]

Source: Author's computation aided by R package V 3.6.3

Table 2 Shows Bayes estimates and Bayes 95% CI, standard deviation for $(\alpha,\beta,\gamma,$ and$\lambda)$. Additionally, the asymptotic approximation method (Laplace approximation) is also used to simulate a random sample from the each of the marginal posterior density using sampling important resampling and approximate the posterior densities of the four parameters of the ACW model. The estimates of the four parameters ($\gamma, \alpha, \beta,$ and λ) by asymptotic approximation method are respectively computed as, the Bayes estimates are 0.25959, 0.00333, 0.01729 and 289.698. The standard deviation of the four parameters are given as 0.0224, 0.0000, 0.0098 and 12.319 and the Bayes 95% C.Is are given as [0.2108, 0.3008], [0.0033, 0.0033], [0.0056, 0.0472] and [269.44, 319.97]. Therefore, it has been observed throughout that the Monte Carlo Markov Chain (MCMC) technic particularly (algorithm 1) summarizes the posterior more precisely in terms of the lower standards deviations of the parameters as compared to that of asymptotic approximation.



(A) **(B)**

Figure 2. (A) Trace plots and (B) plots of the marginal posterior densities of the parameters for the posterior distribution of additive Chen-Weibull model using the IM

Source: Author's computation aided by R package V 3.6.3

Fig. 2 (A) trace plot and (B) the marginal posterior density estimates of the parameters obtained by MCMC algorithm (algorithm 1). The trace plots of each parameter showed that IM algorithm converges quickly to the same target distribution. The marginal posterior densities of the four parameters are distributed approximately symmetrically around the central values which means that they provide good Bayesian estimates under square error loss function.

Table 3: MPSE of ACW model Using Meeker-Escobar data

Parameter	MPSEs
γ	0.15168
α	0.00333
λ	0.01518
β	13.4583

Source: Author's computation aided by R package V 3.6.3

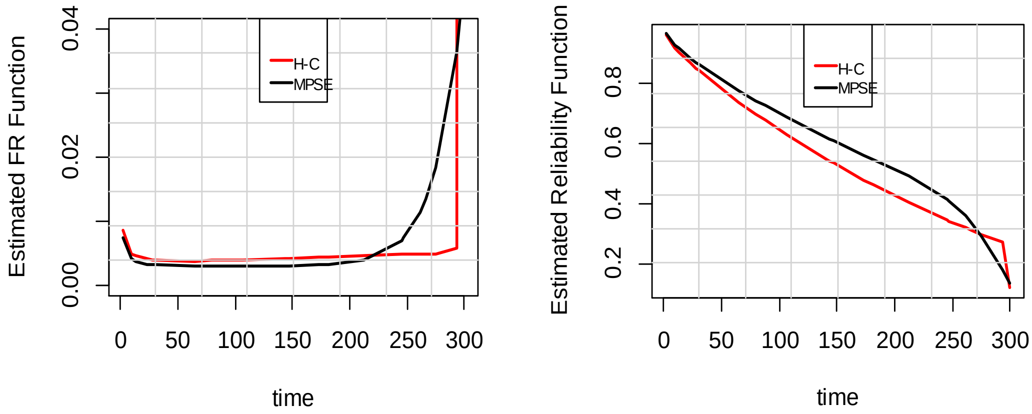
Table 3 gives the maximum product of spacing (MPS) point estimate of the four unknown parameters of additive Chen-Weibull (ACW) model of meeker-Escobar data using mpedist function in BMT package in R with good set of initial values of the parameters.

Table 4. K-S and its p-value when fitting to ACWTo Meeker-Escobar data

Method of Estimation	K-S(p-value)
Bayesian	0.14084 (0.5912)
MPSEs	0.13729 (0.6238)

Source: Author's computation aided by R package V 3.6.3

Table 4 represents the K-S statistic and its p-value for the comparison of the two different estimation methods (Bayesian and MPSEs) when fitting ACW model to Meeker-Escobar dataset. The result from K-S statistic showed that MPSE with KS = 0.13729 (p-value = 0.6238) perform better than the Bayesian estimate.

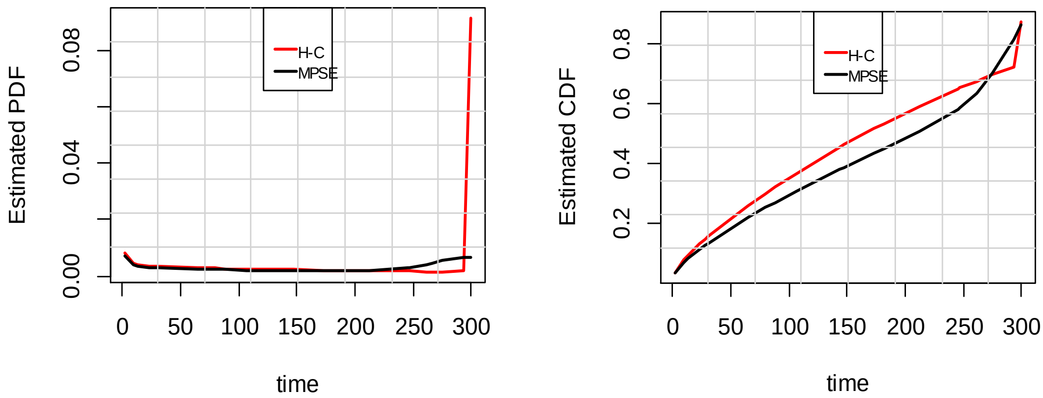


(A) **(B)**

Figure 3. The estimated (A) failure rate function and (B) reliability function obtained by fitting ACW distribution using Bayesian and MPS method of estimation to meeker-Escobar data

Source: Author’s computation aided by R package V 3.6.3

Fig 3 showed a visual comparison of reliability/survival function and failure/hazard rate functions plots, we can see from (A) that, the FR of Bayesian estimate started to increase at around $x = 300,$ which shows a low and long constant FR at mid time region. the FR of MPSE started to increase at nearly $x = 220$, which showed low and short constant FR at mid time region.

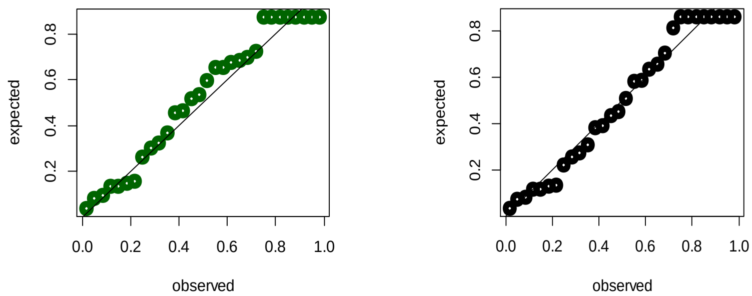


(A) **(B)**

Figure 4. The estimated (A) PDF and (B) CDF of the ACW model using Bayesian and MPS method of estimation to meeker-Escobar data.

Source: Author’s computation aided by R package V 3.6.

Fig. 4 showed (A) the probability density function (PDF) and (B) the cumulative distribution function (CDF) of the two different estimation method have different shape. From the results obtained we can conclude that, the MPSEs provides a better fit of the four unknown parameters of ACW model for Meeker-Escobar data.



(A) **(B)**

Figure. 5. Comparison of theP-P plots of (A) Bayesian (B) MPS estimate when fitting ACW model with meeker-Escobar data

Source: Author’s computation aided by R package V 3.6.

Fig. 5 showed p-p plot of the two different estimation methods for fitting the ACW model to Meeker-Escobar data. P-p plot also gives more information about the appropriateness of each estimation methods for fitting ACW model Meeker-Escobar data. From the results obtained we can conclude that, the MPSEs provides a better fit of the four unknown parameters of ACW model for Meeker-Escobar data.

Aarset data

Table 5: Aarset data

0.1	0.2	1.0	1.0	1.0	1.0	1.0	2.0	3.0	6.0
7.0	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 5 represent the Aarset data with a random sample of 50 lifetime's devices. Many authors in the literature used this data for illustration purpose and the most recent studies are given by Govind et al., (1993), Lai et al., (2003), (Sarhan and Apolo, 2013), Almaliki et al., (2013), Bo He et al., (2016), Hongtao et al., (2016) and Mohammed et al., (2019).

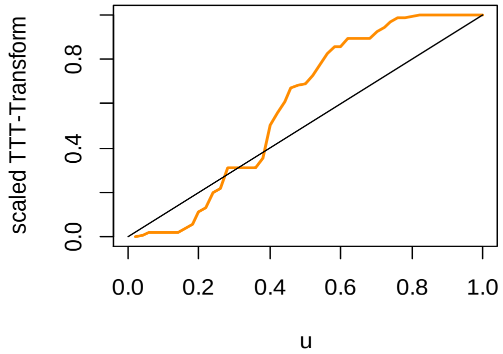


Figure. 6. The empirical scaled TTT-transform plot for Aarset data
Source: Author's computation aided by R package V 3.6.3

Fig. 6. Provides the empirical scaled TTT-transform plot for Aarset datasets. From this plot we can observed that the Aarset data have a bathtub-shaped failure rate function.

The independent Metropolis-Hasting algorithm (algorithm 1) in MCMC is also used to simulate a random sample from each of the marginal posterior density of the four unknown parameters to approximate the posterior density of ACW model.

Table 6. Bayes estimates for the parameters when fitting ACW model to Aarset

Parameters	**Bayes**	**SD**	**Bayes 95%C.I**
γ	0.2834	0.0147	[0.2553, 0.3123]
α	0.0118	0.0000	[0.0117, 0.0119]
λ	0.0420	0.0094	[0.0264, 0.0629]
β	80.151	13.103	[57.284, 108.92]

Source: Author's computation aided by R package V 3.6.3

Table 6. Gives the Bayes estimates, Bayes 95% CIs, and standard deviation for $(\alpha, \beta, \gamma,$ and $\lambda)$. Additionally, the asymptotic approximation method (Laplace approximation) is also used to simulate a random sample from the each of the marginal posterior density using sampling important resampling and approximate the posterior densities of the four parameters of the ACW model. The estimates of the four parameters by Laplace approximation of $(\gamma, \alpha, \beta,$ and $\lambda)$ are respectively given as, the Bayes estimates are 0.2812, 0.0118, 0.0434 and 73.000. The standard deviation are 0.0254, 0.0001, 0.0170 and 22.8416. The Bayes 95% C.Is are given as [0.2309, 0.3301], [0.0117, 0.0119], [0.0187, 0.0849] and [37.8081, 124.67]. Therefore, it has been observed throughout in this study that, the MCMC algorithm (independent-Metropolis) summarizes the posterior more precisely in terms of the lower standards deviations of the parameters as compared to that of Laplace approximation

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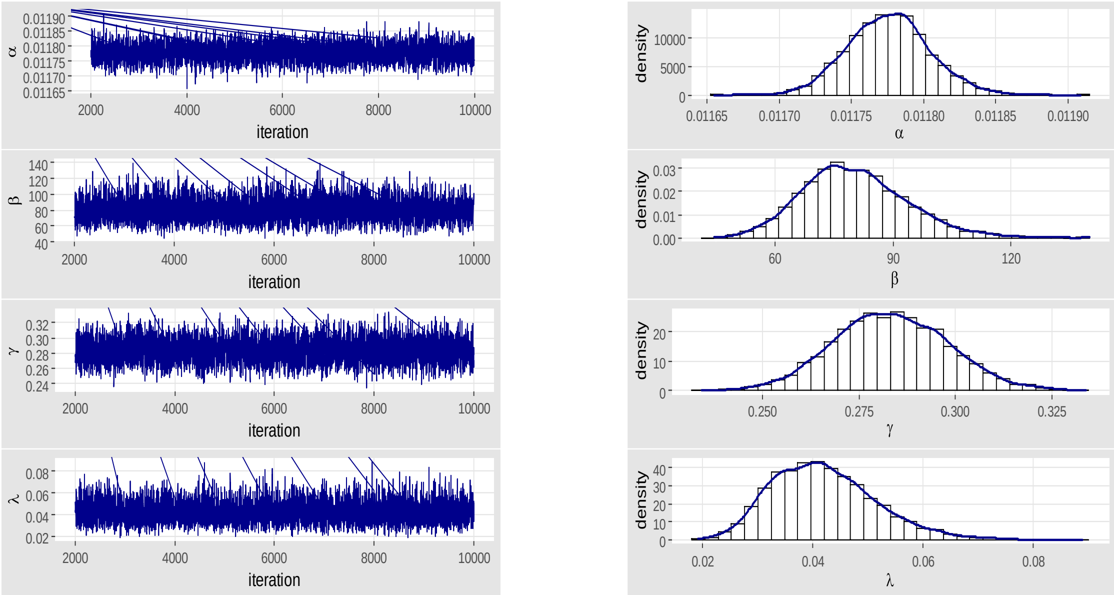


(A) **(B)**

Figure. 7 (A) Trace plots and (B) plots of the marginal posterior densities of the parameters for the posterior distribution of additive Chen-Weibull model using the IM
Source: Author's computation aided by R package V 3.6.3
Fig. 7 shows the trace plots and density estimates of the parameters obtained by MCMC algorithm. The trace plots of each parameter showed that the IM algorithm converge quickly to the same target distribution. The densities are distributed approximately symmetrically around the central values which means that they provide good Bayes estimates under square error loss function.

Table 7: MPSE of ACW model Using Aarset data

Parameter	MPSEs
γ	0.2759
α	0.0118
λ	0.0423
β	38.231

Source: Author's computation aided by R package V 3.6.3

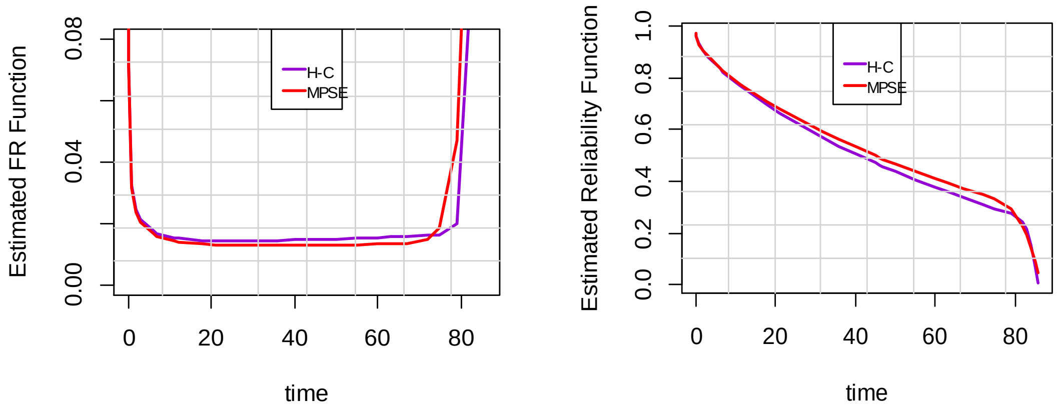
Table 7 gives the maximum product of spacing (MPS) point estimate of the four unknown parameters of additive Chen-Weibull (ACW) model of Aarset data using mpdist function in BMT package in R with good set of initial values of the parameters.

Table 8. K-S and its p-value when fitting to ACWTo Meeker-Escobar data

Method of Estimation	K-S (p-value)
Bayesian	0.089235(0.8208)
MPSE	0.069895(0.9675)

Source: Author's computation aided by R package V 3.6.3

Table 8 represents the K-S statistic and its p-value for the comparison of the two different estimation methods (Bayesian and MPSEs) when fitting ACW model to Aarset data. The result from K-S statistic showed that MPSE with KS = 0.069895 (p-value=0.9675) Perform better than the Bayesian estimate assuming half-Cauchy prior with KS = 0.089235 (p-value = 0.8208). In addition, the MPSE also perform better than the Bayesian estimate assuming gamma prior with KS = 0.079682 (p-value = 0.9087) and MLE with KS = 0.070375(p-value=0.9654) studied by (Tien and Radim 2020).

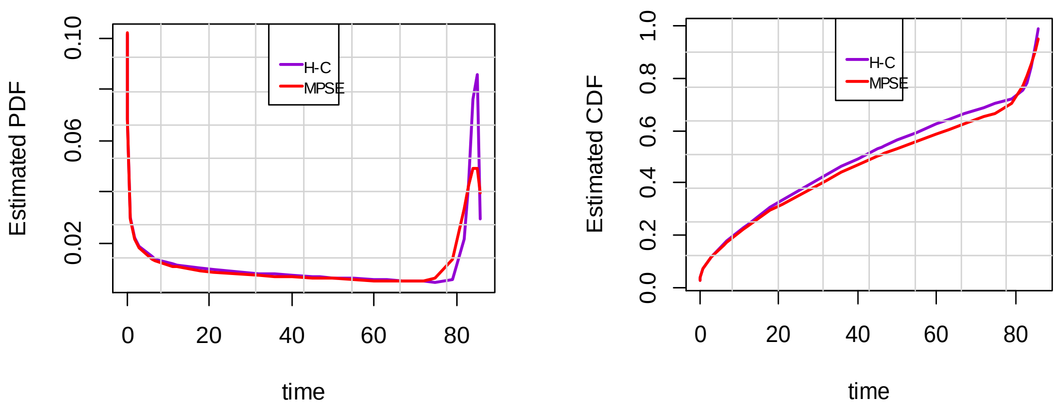


(A) **(B)**

Figure. 8. The estimated (A) failure rate function and (B) reliability function.

Source: Author's computation aided by R package V 3.6.3

Fig 8. Showed a visual comparison of reliability/survival function (R) and failure/hazard rate functions obtained by fitting ACW distribution using Bayesian and MPS method of estimation to Aarset data, it has been cleared from these plots that, the FR of both Bayesian and MPS estimate predict relatively low and long constant FR at mid time region.

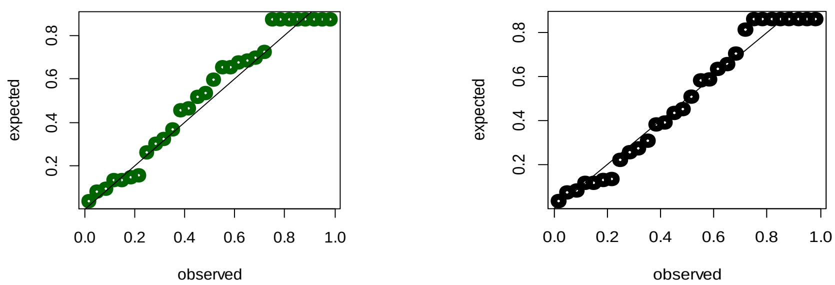


(A) **(B)**

Figure. 9. The estimated (A) PDF and (B) CDF of the ACW model

Source: Author's computation aided by R package V 3.6.3

Fig. 9 showed the plots of (A) probability density function (PDF) with corresponding (B) cumulative distribution function (CDF) plots obtained by fitting ACW distribution using Bayesian and MPS method of estimation to Aarset data. We can observed from these plots that, the two different estimation methods have almost similar shape as there is only slight difference between the two estimation methods. From the results obtained we can conclude that, the MPSEs provides a better fit than the Bayesian assuming two different priors (half-Cauchy and gamma) and MLE studied by (Tien and Radim 2020) of the four unknown parameters of ACW model to Aarset data.



(A) **(B)**

Figure. 10. Comparison of the P-P plots of (A) Bayesian (B) MPS estimate when fitting ACW model with Aarset data

Source: Author's computation aided by R package V 3.6.3

Fig. Fig. 10 showed p-p plot of the two different estimation methods for fitting the ACW model to Aarset data. It is cleared that, the p-p plot also provides more information about the appropriateness of each estimation methods for fitting ACW model to Aarset data.

CONCLUSION

In this study, the additive Chen-Weibull (ACW) model is used to analyze the lifetime data using Bayesian assuming half-Cauchy prior and maximum product of spacing method of estimation (MPSE). A two real data sets were used for illustration purposes. In Bayesian paradigm, the analytic approximation and MCMC techniques were implemented using the function LaplaceApproximation and LaplacesDemon Respectively. Therefore, it has been observed throughout that the simulation technique, particularly independent-Metropolis algorithm summarizes the posterior more precisely in terms of the lower standard deviations of the parameters as compared to that of Laplace approximation. From the results obtained we can conclude that, the MPSEs provides a better fit than the Bayesian estimate.

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