FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 6 No. 1, March, 2022, pp 175 - 180
DOI: https://doi.org/10.33003/fjs-2022-0601-852

STUDY OF CONVEXITY IN SETS AND FUZZY SETS

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ABSTRACT

In this paper, we present an overview of both classical and fuzzy convexity, particularly, in conjunction with continuity 5 and some topological concepts, and provide proofs of some of their algebraic properties along with suitable illustrations. We have extended the work of (Ammar E. E., 1999) by proving that if μ and ν are two convex fuzzy sets, then $\mu+\nu$, $\mu-\nu$, $\mu*\nu$, and μ/ν are also convex fuzzy sets. We have also shown that the convexity of a fuzzy set implies semistrict quasi-convexity.

Keywords: *Convexity Fuzzy sets, Sets, semistrict*

INTRODUCTION

The notion of fuzzy set provides a convenient point of departure for the construction of a conceptual framework which is parallel in many respects to those used in the case of ordinary set. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of *sharply* defined the class membership rather than the presence of random variables.

The notion of convexity was extended to fuzzy sets (Zadeh L. A., 1965). This idea appears to be particularly useful in applications involving *pattern classification, optimization approximate reasoning, preference modeling, and many other related problems*. As a matter of fact, if a problem satisfies certain convexity attributes, it can be modelled easily (Syau, Lee, L., & Lixing, 2004).

A fuzzy set A is convex if and only if the sets Γ_α, defined by $\Gamma_\alpha = \{x: \mu_A(x) \geq \alpha\}$, are convex for all $\alpha \in (0,1]$. Alternately, a fuzzy set A is convex if and only if

$$\mu_A[(1-\lambda)x_1 + \lambda x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)].$$

Where Γ_α is called the $\alpha-$cut set (or $\alpha-$level set).

The study of convexity in fuzzy set-theoretic framework began with (Zadeh L. A., 1965) himself. The notions of inclusion, union, intersection, complement, relation, convexity were extended to such sets, and various properties of these notions in the context of fuzzy sets were established. In particular, a separation theorem for convex fuzzy sets was proved without requiring that the fuzzy sets be disjoint. In Zadeh's paper, an application of fuzzy set was described as a result of classifying pattern in RAND, although, the domain of the definition of the characteristic function μ_A was restricted to a subset of a set X. Zadeh (1965) generalizes the notion of linear combination of any two vectors f and g of the form $\lambda f + (1-\lambda)g,\ \lambda \in [0,1]$ to fuzzy set.

(Katsaras & Liu, 1977) studied fuzzy vector spaces and fuzzy topological vector spaces. In their work, they apply the concept of a fuzzy set to the elementary theory of vector spaces and topological vector spaces. The notion and terminology for fuzzy sets follows that of (Zadeh L. A., 1965). The topology aspect of fuzzy sets and fuzzy convexity, in spirit of vector spaces, were considered in their work.

(Lowen, 1980), studied convex fuzzy sets introduced in (Zadeh, 1965) and introduced the concept of affine fuzzy sets which are translates of fuzzy subspaces introduced by (Katsaras & Liu, 1977). One of the important result is the Representation Theorem where a useful characterization of fuzzy subspaces is given. Using this characterization, he defined the notions of dimension of a fuzzy subspace and fuzzy hyperspaces, and established that two fuzzy subspaces of the same dimension are linear translate of one another. Further, he introduced the notion of convex fuzzy cone.

(Liu, 1985), investigated some properties of convex fuzzy sets and developed some fundamental results such as: Separation theorem and theorems on shadows of convex fuzzy sets, in particular. Liu provides some suitable countable counter examples in order to show the draw backs of the theorem on shadows of convex fuzzy sets proposed by (Zadeh L. A., 1965). He added some assumptions about fuzzy topologies in other to yield several positive results on the theorem of shadows of convex fuzzy sets. In addition, a simple and direct proof of two theorems that described the relations between the fuzzy convex cone and fuzzy union subspace were formulated. Significantly, the proofs of these theorems do not appeal to (Lowen, 1980) representation theorem. He confines his findings mainly on convex fuzzy sets defined on Euclidean spaces.

(Drewniak, 1987), investigated convex and strongly convex fuzzy sets on the real line and characterized them by means of piece-wise monotonic functions. He also described the properties of the 'Support' and the 'Core' of convex and strongly convex fuzzy sets.

(Ammar & Metz, 1992), introduce a new formulation of fuzzy convexity and presented several results.

(Yang X. , 1993), introduced two weak conditions under which a fuzzy closed set is a convex fuzzy set. He also considered fuzzy sets defined on the Euclidean space and outlined its generalization to the case of fuzzy sets defined in a linear space over real or complex field.

(Yang X. , 1995), investigated some new properties of convex fuzzy sets, strictly convex fuzzy sets, and strongly convex fuzzy sets based on his critical study of several results developed in (Brown, 1971) and (Katsaras & Liu, 1977).

(Ammar E. E., 1999), based on the formulation of fuzzy convexity in (Ammar & Metz, 1992), and introduction of fuzzy line segments, introduced convex, strictly convex, quasi-convex, strictly quasi-convex, and M-convex fuzzy sets, and proved several related results.

(Syau Y. R., 2000), proposed the notion of closed and convex fuzzy sets and provided two weak conditions to show that a closed fuzzy set is a quasi-convex fuzzy set. In his work, he proved that a fuzzy set $\mu:\mathbb{R}^n \longrightarrow [0,1]$ is closed if and only if its fuzzy hypograph is a closed subset of $\mathbb{R}^n \times (0,1]$, and also gave two weak conditions that a closed fuzzy set is a convex fuzzy set.

(Cheng, Syau, & Ting, 2004), studied the concept of *semistrictly convex fuzzy sets* and proved that for the upper semicontinuous case, the class of semistrictly convex fuzzy sets lies between the convex and strictly convex classes.

(Syau, Lee, & Lixing, 2004), studied convexity and upper semicontinuity of fuzzy Sets. They investigated the interrelationships of several concepts of generalized convex fuzzy sets and established that for the upper semicontinuous case, the class of *semistrictly quasi-convex* fuzzy sets lies between the *convex* and *quasi-convex* classes. They also obtained some results on composition rules for upper semicontinuous fuzzy sets, for example, a convex combination of upper semicontinuous fuzzy sets is an upper semicontinuous fuzzy set.

In this paper, we present an overview of both classical and fuzzy convexity in conjunction with continuity and some topological concepts, and established some new results on convexity and their algebraic properties and finally provided suitable illustrations.

Preliminaries

Definition 1.

Let $f: I \longrightarrow \mathbb{R}$ be a function and $x_0 \in$ I, the function f is called *upper semicontinuous* at x_0 if $\overline{\lim}_{x\longrightarrow x_0} f(x) \le f(x_0)$ for all $x \in (x_0-\delta, x_0+\delta)$. In other words, f is upper semicontinuous at x_0 if given $\epsilon > 0$, there exists $\delta > 0$ such that $f(x) < f(x_0)+\epsilon$ for $|x-x_0| < \delta$. Similarly, f is called *lower semicontinuous* at x_0 if $\underline{\lim}_{x\longrightarrow x_0} f(x) \ge f(x_0)$ for all $x \in (x_0-\delta, x_0+\delta)$. Equivalently, f is lower semicontinuous at x_0 if for a given $\epsilon > 0$, there exists $\delta > 0$ such that $f(x_0)-\epsilon < f(x)$ for $|x-x_0| < \delta$. The function, f is called *continuous* at x_0 if and only if it is both upper and lower semicontinuous.

Definition 2.

" (Zadeh L. A., 1965)", founder of fuzzy set theory, was the first to introduce the concept of *fuzzy convexity* as follows:

As defined earlier, a fuzzy set A in X (universal set) is defined by $\mu_A: X \longrightarrow [0,1]$. That is, if $A \subseteq X$, then its generalized characteristic function μ_A is a fuzzy set. It follows that a fuzzy set $\mu:\mathbb{R}^n \longrightarrow [0,1]$ is the set of all pairs $(x, \mu(x))$. A point $x \in \mathbb{R}^n$ is called a fuzzy point if $x \in supp(\mu)$. Where $supp(\mu)$ is called the support of a fuzzy set.

A fuzzy set A is called *convex* if and only if the set Γ_α, defined by $\Gamma_\alpha = \{x: \mu_A(x) \ge \alpha\}$, are convex for all $\alpha \in (0,1]$.

Note that 0-cuts are excluded since it is always equal to $\mathbb{R}$ or $\mathbb{R}^n$. It may be observed that in view of α-cuts of a fuzzy set being crisp sets, this definition is a generalization of classical convexity.

In the same paper, he also provided an equivalent alternative definition of convexity which is more direct and easy to work with as follows:

A fuzzy set A in X is *convex* if and only if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \ge \min(\mu_A(x_1), \mu_A(x_2))$ for all $x_1, x_2 \in X$ (a universal set) and $\lambda \in [0,1]$, where μ is the membership function.

SOME INTERESTING EXAMPLES AND RESULTS

Example 1.

i. Let $f(x) = \begin{cases} \sin\frac{1}{x}, x \ne 0 \\ 1, x = 0 \end{cases}$.

Then $\overline{\lim}_{x\longrightarrow 0} \sin\frac{1}{x} = 1 = f(0)$, $\underline{\lim}_{x\longrightarrow 0} \sin\frac{1}{x} = -1 \ne f(0)$.

Hence f is upper semicontinuous at $x = 0$, but not lower semicontinuous at $x = 0$ and thereby, f is not continuous at $x = 0$.

ii. Let f be a characteristic function of rationals, that is, $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational. Then f is upper semicontinuous at every rational point and lower semicontinuous at every irrational point, and thus f is discontinuous at each point $x \in \mathbb{R}$. Analogously, the characteristic function of irrationals is upper semicontinuous at each irrational point and lower semicontinuous at each rational point, and hence discontinuous at each point of $\mathbb{R}$.

iii. Let ϕ be a *step function* defined on $I = [a, b]$, that is, ϕ assumes only one value in (x_i, x_{i+1}), a subinterval of I, for each i. Then ϕ is lower semicontinuous if and only if $\phi(x_i) \le$ minimum of the values assumed by ϕ in (x_{i-1}, x_i) and in (x_i, x_{i+1}). Analogous result holds for upper semicontinuous.

iv. Let ϕ be a function defined on a closed set. Then ϕ is lower semicontinuous if and only if the set $\{x: f(x) \le k\}$ is closed for every real number k. Analogous result holds for upper semicontinuous.

Example 2.

(a) Let the fuzzy sets μ_k, $k = 1,2$ be defined by

$$\mu_k(x) = \begin{cases} 1-(x-k)^2, & k-1 < x < k+1 \\ 0, & otherwise \end{cases}.$$

Then, μ_1 is convex and μ_2 is strictly convex.

Solution.

We show that:

$$1-(\lambda x_1 + (1-\lambda)x_2 - k)^2 \ge \lambda(1-(x_1-k)^2) + (1-\lambda)(1-(x_2-k)^2) \quad \dots (1)$$

We have $\mu_k(x) = 1-(\lambda x_1 + (1-\lambda)x_2 - k)^2$

$$= 1-(\lambda x_1 + (1-\lambda)x_2)^2 + 2k(\lambda x_1 + (1-\lambda)x_2) - k^2$$

$$= 1 - \lambda^2 {x_1}^2 - 2\lambda x_1 x_2 + 2\lambda^2 x_1 x_2 - {x_2}^2 + 2\lambda {x_2}^2 - \lambda^2 {x_2}^2 + 2k\lambda x_1 + 2kx_2 - 2k\lambda x_1 - k^2. \quad \dots (2).$$

Since $\lambda \ge \lambda^2$, we have

$\mu_k(x) \geq 1 - \lambda {x_1}^2 - {x_2}^2 + 2\lambda {x_2}^2 - \lambda {x_2}^2 + 2k\lambda x_1 + 2kx_2 - 2k\lambda x_2 - k^2$ … (3).

$= 1 - \lambda {x_1}^2 - {x_2}^2 + \lambda {x_2}^2 + 2k\lambda x_1 + 2kx_2 - 2k\lambda x_2 - k^2$

$= 1 + \lambda - \lambda {x_1}^2 - {x_2}^2 - \lambda k^2 + \lambda {x_2}^2 + 2k\lambda x_1 + 2kx_2 - \lambda - 2k\lambda x_2 - k^2 + \lambda k^2$

$= \lambda[1 - ({x_1}^2 - 2kx_1 + k^2)] + 1[1 - ({x_2}^2 - 2kx_2 + k^2)] - \lambda[1 - ({x_2}^2 - 2kx_2 + k^2)]$

$= \lambda(1 - (x_1 - k)^2) + (1 - \lambda)(1 - (x_2 - k)^2).$ … (4).

Hence the proof.

(b) Let the fuzzy sets $\mu_k,\ k = 1,2$ be defined by

$$\mu_k(x) = \begin{cases} \frac{1}{1+(x-k)^2}, & k-1 < x < k+1 \\ 0, & otherwise \end{cases}.$$

Then neither μ_1 is convex nor is μ_2 strictly convex.

We note here that μ_1 and μ_2 are quasi-convex (strictly quasi-convex), and the α-level set of μ, is convex (strictly convex) (Ammar & Metz, 1992).

(c) Let $\mu: \mathbb{R} \longrightarrow [0,1]$ be defined by

$$\mu(x) = \begin{cases} \frac{1}{4}, & x \leq 0 \\ \frac{1}{2}, & x > 0 \end{cases}.$$

Clearly, μ is not closed since $[\mu]_{\frac{1}{2}} = (0, \infty)$ which is not a closed subset of $\mathbb{R}$. Moreover, μ is not convex since it is discontinuous at $x = 0$, $\left[\lim_{x^-} \mu(x) \neq \lim_{x^+} \mu(x)\right]$.

Remark 1.

Recall that a fuzzy set is *closed* if its α-level set is a closed set for each $\alpha \in [0,1]$.

Alternatively, a fuzzy set is *closed* if and only if its fuzzy *hypergraph* is a closed subset of $\mathbb{R}^n \times (0,1)$. The example in (c) illustrates a fuzzy set which is not closed (Ammar & Metz, 1992).

Definition 3. (Ammar E. E., 1999)

For any two fuzzy sets μ and ν, using the extension principle in (Zadeh L. A., 1975), then the fuzzy sets $(\mu + \nu)$, $(\mu - \nu)$, $(\mu * \nu)$ and (μ/ν) can be defined as follows:

$$(\mu + \nu)(z) = \bigvee_{(x,y): z = x+y} \big(\mu(x) \wedge \nu(y)\big)$$ … (5).

$$(\mu - \nu)(z) = \bigvee_{(x,y): z = x-y} \big(\mu(x) \wedge \nu(y)\big)$$ … (6)

$$(\mu * \nu)(z) = \bigvee_{(x,y): z = x*y} \big(\mu(x) \wedge \nu(y)\big)$$ … (7)

$$(\mu/\nu)(z) = \bigvee_{(x, y \neq 0): z = x/y} \big(\mu(x) \wedge \nu(y)\big)$$ … (8)

Definition 4. (Syau Y. R., 2000)

A fuzzy set $\mu \in \mathcal{F}(\mathbb{R}^n)$ is called *upper semicontinuous* at a point $x \in supp(\mu)$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\mu(y) < \mu(x) + \epsilon$, for all $y \in supp(\mu)$ and

$\|y - x\| < \delta$.

If a fuzzy set μ is *upper semicontinuous* at each point of its support, then it is called *upper semicontinuous*.

By definitions given in the preceding sections, it follows that a fuzzy set μ is *upper semicontinuous* if and only if its α-level sets are convex for all $\alpha \in (0,1]$. Moreover, the class of upper semicontinuous fuzzy set is closed under addition and scalar multiplication.

Definition 5. (Wu & Cheng, 2004)

A fuzzy set $\mu: \mathbb{R}^n \longrightarrow [0,1]$ with convex support is said to be

(1) semistrictly convex if for all $x, y \in supp(\mu), \mu(x) \neq \mu(y)$, and $\lambda \in (0,1)$,
$\mu(\lambda x + (1-\lambda)y) > \lambda\mu(x) + (1-\lambda)\mu(y)$,
(2) semistrictly quasi-convex if for all $x, y \in supp(\mu), \mu(x) \neq \mu(y)$, and $\lambda \in (0,1)$,

$\mu(\lambda x + (1-\lambda)y) > min\{\mu(x), \mu(y)\}$.

Theorem 1.
If μ and ν are two convex fuzzy sets, then $\mu+\nu$, $\mu-\nu$, $\mu * \nu$, and μ/ν are also convex fuzzy sets.

Proof.
The proof of $\mu+\nu$ case was proved in (Ammar E. E., 1999), while others were not proven. Then, we shall prove the convexity of $\mu-\nu$ and μ/ν.

Proof of $(\mu-\nu)$: Let $z_i = x_i - y_i$; $x_i \in Supp(\mu), y_i \in Supp(\nu)$, $i = 1,2$; and

$x = \lambda x_1 + (1-\lambda)x_2,\ y = \lambda y_1 + (1-\lambda)y_2,\ z = \lambda z_1 + (1-\lambda)z_2, \quad \lambda \in [0,1].$ … (9).

We need to show that

$(\mu-\nu)(z) = (\mu-\nu)[\lambda z_1 + (1-\lambda)z_2] \geq [\lambda(\mu-\nu)z_1 + (1-\lambda)(\mu-\nu)z_2].$ … (10).

By definition 3, we have

$$(\mu-\nu)[\lambda z_1 + (1-\lambda)z_2] \geq \mu[\lambda x_1 + (1-\lambda)x_2] \wedge \nu[\lambda y_1 + (1-\lambda)y_2]$$

$\geq [\lambda\mu(x_1) + (1-\lambda)\mu(x_2)] \wedge [\lambda\nu(y_1) + (1-\lambda)\nu(y_2)]$ … (11)

Therefore, and ν are convex fuzzy sets.

$$\geq \lambda[\mu(x_1) \wedge \nu(y_1)] + (1-\lambda)[\mu(x_2) \wedge \nu(y_2)]$$

$$\geq \lambda(\mu-\nu)(z_1) + (1-\lambda)(\mu-\nu)(z_2). \blacksquare$$

This proves (10).

Proof of (μ/ν): Here $z = x/yy \neq 0$.

We need to show that

$(\mu/\nu)(z) = (\mu/\nu)[\lambda z_1 + (1-\lambda)z_2] \geq [\lambda(\mu/\nu)z_1 + (1-\lambda)(\mu/\nu)z_2].$ … (12).

by **definition 3** for (μ/ν), we have

$(\mu/\nu)[\lambda z_1 + (1-\lambda)z_2] \geq \mu[\lambda x_1 + (1-\lambda)x_2] \wedge \nu[\lambda y_1 + (1-\lambda)y_2]$ … (13)

$$\geq [\lambda\mu(x_1) + (1-\lambda)\mu(x_2)] \wedge [\lambda\nu(y_1) + (1-\lambda)\nu(y_2)] \because \mu \text{ and } \nu \text{ are convex fuzzy sets}$$

$\geq \lambda[\mu(x_1) \wedge \nu(y_1)] + (1-\lambda)[\mu(x_2) \wedge \nu(y_2)]$ …
(14).

$\geq \lambda(\mu/\nu)(z_1) + (1-\lambda)(\mu/\nu)(z_2). \blacksquare$

Remark 2.
It follows from the definitions of strictly convex fuzzy sets, and those given above, that any strictly convex fuzzy set is semistrictly convex (but not vice versa) and that any semistrictly convex fuzzy set is semistrictly quasi-convex (but not vice versa). Moreover, in the upper semicontinuous case, the class of semistrictly quasi-convex fuzzy sets lies between the convex and quasi-convex classes. (see (Cheng, Syau, & Ting, 2004), (Syau, Lee, L., & Lixing, 2004), &(Wu &Cheng, 2004) for more details).

The main difference between a strictly convex and semistrictly convex fuzzy sets is that the former can attain its global maximum at no more than one point whereas the latter can have a *flat* maximum. As mentioned earlier, both strict and semistrict convexity of a fuzzy set μ imply that μ is semistrictly quasi-convex. An important result in (Cheng, Syau, & Ting, 2004) shows that for a semistrictly quasi-convex fuzzy set with convex support, a local maximizer is also a global. The application of strictly, semistrictly, and hence that of semistrictly quasi-convex fuzzy sets in decision theory has become of paramount significance.

For example, let $\mu : \mathbb{R} \longrightarrow [0,1]$ be given by

$$\mu(x) = \begin{cases} \frac{2}{3}, & x = 0 \\ 1, & x \neq 0 \end{cases}.$$

It is a *semistrictly* convex fuzzy set with every nonzero point in $\mathbb{R}$ as its global maximizer. It has a flat maximum.
In the following, we include some results in this regard.

Theorem 2.
A fuzzy set $\mu : \mathbb{R}^n \longrightarrow [0,1]$ is closed if and only if μ is upper semicontinuous.

Proof.

The proof is immediate from the definitions of a fuzzy set being closed upper semicontinuous.

Theorem 3.
The convexity of a fuzzy set implies semistrict quasi-convexity.

Proof.

Let $\mu : \mathbb{R} \longrightarrow [0,1]$ be a convex fuzzy set, and let $x, y \in supp(\mu)$, $\mu(x) \neq \mu(y)$. Let $\mu(x) > \mu(y)$. We have

$\lambda\mu(x) + (1-\lambda)\mu(y) > \mu(y)$, for each $\lambda \in (0,1)$.

$\Longrightarrow \mu(\lambda x + (1-\lambda)y) \geq \lambda\mu(x) + (1-\lambda)\mu(y) > \mu(y)$, for each $\lambda \in (0,1)$.
$\Longrightarrow \mu$ is semi strictly quasi-convex. ∎

Theorem 4.
Let $\mu: \mathbb{R}^n \longrightarrow [0,1]$ be a fuzzy set with convex support. If μ is a semistrictly quasi-convex fuzzy set but not quasi-convex, then there exist distinct points $\tilde{x}, \tilde{y} \in supp(\mu)$ and $\tilde{z} \in (\tilde{x}, \tilde{y})$ such that on the closed line segment $[\tilde{x}, \tilde{y}]$, we have $0 < \mu(\tilde{z}) < \mu(\tilde{x}) = \mu(\tilde{y})$ and $\mu(x) = \mu(\tilde{x})$, for all $x \in [\tilde{x}, \tilde{z}) \cup (\tilde{z}, \tilde{y}]$.

Proof.

Let μ be semistrictly quasi-convex fuzzy set but not quasi-convex. Then, there exist distinct points $\tilde{x}, \tilde{y} \in supp(\mu)$ and $\tilde{z} \in (\tilde{x}, \tilde{y}) \subseteq supp(\mu)$ such that

$\mu(\tilde{x}) = \mu(\tilde{y})$, but $\mu(\tilde{z}) < \mu(\tilde{x}) = \mu(\tilde{y})$.

Let $z_1 \in (\tilde{x}, \tilde{z})$ and $z_2 \in (\tilde{z}, \tilde{y})$. Then, by (11) and the semistrict quasi-convexity of μ, we have $\mu(z_1) > \mu(\tilde{z})$ and $\mu(z_2) > \mu(\tilde{z})$.

Since $\tilde{z} \in (z_1, \tilde{y})$, we must have $\mu(z_1) = \mu(\tilde{y})$; otherwise, from the semistrict quasi-convexity of μ, $\mu(\tilde{z}) > \min\{\mu(z_1), \mu(\tilde{y})\}$, a contradiction. Similarly,

$\mu(z_2) = \mu(\tilde{x}) = \mu(\tilde{y})$. Hence, $\mu(x) = \mu(\tilde{x})$, for all $x \in [\tilde{x}, \tilde{z}) \cup (\tilde{z}, \tilde{y}]$. ∎

CONCLUSION
In this paper, a systematic study of convexity analysis both in classical set theory and fuzzy set theory was presented, many known results were studied, a number of examples and counter examples were provided, and a couple of new results were proven.

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