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OPTIMAL INVESTMENT POLICY AND CAPITAL MANAGEMENT IN A FINANCIAL INSTITUTION

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ABSTRACT
This research work considered an asset optimization problem where we examine how a financial institution can optimally allocate its total wealth among three assets namely; treasury, security and loan in stochastic interest rate setting and also determined how a financial institution can manage its capital. The optimal investment policy was derived through the application of stochastic optimization theory for the case of constant relative risk aversion (CRRA) utility function. Also, the Stochastic Differential Equation (SDE) for the capital adequacy ratio under Basel Accord, the SDE for the Total Risk – Weighted Assets (TRWA), the SDEs for the capital required to maintain the capital adequacy ratio under Basel II and Central Bank of Nigeria (CBN) standards were derived and solved numerically to study the capital management problem of the financial institution. Numerical examples using published data obtained from Central Bank of Nigeria (CBN) statistical bulletin and Nigeria Stock Exchange were presented to illustrate the dynamics of the optimal investment policy and how a financial institution can manage its capital. From the results, the optimal investment strategy can be achieved by shifting the financial institution investment away from the risky assets (security and loan) towards the riskless asset (treasury). It was also observed that if a financial institution observes the Basel II standard or Nigeria CBN standard of capital requirement, the financial institution would be considered to be strongly capitalized and guarantees the ability to absorb unexpected losses.

Keywords: Interest rate, capital, assets, capital adequacy, optimal policy

INTRODUCTION
Optimal assets allocation plays a vital role in banks and other financial institutions. In recent times, the volume of research done on the financial institutions has increased (Subranmanian and Yang, 2012; Nicolo *et al*., 2012; Acharya *et al*., 2011). In particular, Mukudden – Petersen and Petersen (2008) determined an optimal rate at which additional debt and equity should be raised, and strategy for the allocation of bank equity. They employed dynamic programming algorithm for stochastic optimization to verify their results. In another work by Mukudden – Petersen *et al.* (2007), they obtained an analytical solution for the associated HJB in a case where the utility functions are either of power, exponential or logarithmic type. Here, the control variates are the depository consumption, value of the depository financial institutions invested in loans, and provisions for loan losses.
Mulaudzi *et al.* (2008) studied an optimal investment strategy for banks funds in treasuries and securities in a risk and regret theoretical framework. Evidence of portfolio shifting are found in (Borio *et al*., 2001 and Lowe, 2002), where they suggested that banks may change their balance sheets in ways that can cause procyclicality. Fouche *et al.* (2006) model non – risk – based and risk – based capital adequacy. Specifically, they constructed a continuous time stochastic models for the dynamics of the leverage, equity and Tier 1 ratios and derived the Capital Adequacy Ratio (CAR). They also illustrated the relevant of their result to the banking sector by studying an optimal control problem in which an optimal assets allocation strategy is derived for the leverage ratio on a given time interval. Precisely, they determined the optimal expected terminal utility of the leverage ratio and derived the optimal assets allocation strategy that make it possible to maximize the expected terminal utility of the leverage ratio on a given time interval.
Failures spark management strategies and regulatory prescripts to mitigate risk. One of these prescriptions is the Basel Accord on capital adequacy requirements, which states that capital hold by all major international financial institutions e.g., banks should be in proportion to their perceived risks (Grant and Peter, 2014). Furthermore, governments consider it imperative to oversee and regulate financial institutions because the financial institutions play an important role in such countries' economy. Therefore, financial institutions need to manage its capital appropriately in order to satisfy the shareholders and regulator interests. Hence, financial institutions are heavily regulated. In particular, the regulation have made capital requirements as a very important component of the regulation and as well as supervision in the financial industry. From a shareholders' point of view, more utilization of capital will increase asset earnings and so will earn higher returns on equity. From the regulators' perspectives, financial institutions should increase their buffer capital in order to ensure the safety and soundness of the institutions.
The Basel committee on banking supervision (BCBS) is a body that regulates and supervises the international banking industry by imposing minimal capital requirements and other measures (Basel committee on banking supervision, 2011). In the financial sector, the global economic crisis in 2008 provided an opportunity for fundamental changes of the approach to risk and regulation in financial sector. The purpose of the Basel Accords is to ensure that capital hold by internationally active banks is enough to meet their obligations and as well absorb unexpected losses (Basel committee on banking supervision, 2011).
Under Basel I Accord, banks are to maintain total capital (calculated as the sum of Tier 1 and Tier 2 capital) equal to at least 8% of its total – risk – weighted assets (Basel committee on banking supervision, 2004). However, Basel I Accord was based on simplified calculations and classification which have led to its disappearance. As a result, the BCBS issued the Basel II Accord as the symbol of the continuous refinement

of risk and capital management. The Basel III Accord is the third global, voluntary regulatory standard on bank capital adequacy, stress testing and market liquidity risk. The reform is a set of measures introduced in response to the 2007 – 2008 financial crises. The Accord which was issued in 2010 (Debajyoti *et al*., 2013), aimed at improving the regulation, supervision and risk management within the banking sector. It also shows the continuous effort made by BCBS to improve the banking regulatory framework. It also important to note that capital adequacy ratio (CAR) for banks in Nigeria currently stands at 10% and 15% for national/regional banks and banks with international license respectively (Ugo, 2014). Therefore, many mathematical models have been formulated over the past years to explore the dynamics of asset allocation and capital management problem in financial institutions. In our contribution, we explore dynamics of a financial institution asset allocation and capital management problem in a stochastic interest rate framework by modifying the existing security and loan models in the work of Grant and Peter (2014), estimate the parameters of the models using data obtained from CBN statistical bulletin 2020 by maximum likelihood method (Jungbacker *et al*., 2011; Vaughan, 2014) and Nigeria Stock Exchange (NSE) (NSE, 2015 – 2020) and applied it to a financial institution in Nigeria.

MODEL FORMULATION

The financial market for the financial institution's assets portfolio

We assume that the financial institution can invest its wealth in a market consisting of three assets. The first asset in the financial market is a riskless treasury and its price at time t can be denoted by $S_0(t)$. It evolves according to the following stochastic differential equation:

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \qquad S_0(0) = s_0 \qquad (1)$$

The dynamics of the interest rate $r(t)$, is given by the stochastic differential equation described by:

$$dr(t) = \big(a - br(t)\big)dt + \sigma_r dW_r(t), \qquad r(0) = r_0 \qquad (3)$$

where $\sigma_r = \sqrt{k_1 r(t)}$

The second asset in the financial market is a risky security whose price is denoted by $S(t), t \geq 0$. Its dynamics can be described by the equation:

$$\frac{dS(t)}{S(t)} = \Big(r(t) + \upsilon\sigma_1 + \sigma_p\lambda_r k_1 r(t)\Big)dt + \sigma_p\sigma_r\sqrt{r(t)}dw_r(t) + \sigma_1 dw_s(t) \qquad (3)$$

From equation (3), if we assume that the risk sources σ_p of the interest rate have no effect on the price of the security then the modified security model is given by:

$$\frac{dS(t)}{S(t)} = (r(t) + \upsilon\sigma_1)dt + \sigma_1 dw_s(t) \qquad S(0) = s_0 \qquad (4)$$

where λ_1 and σ_s are constants. Let $\upsilon\sigma_1 = \lambda_s$ and $\sigma_1 = \sigma_s$ then (5) becomes

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_s)dt + \sigma_s dw_s(t) \qquad S(0) = s_0 \qquad (5)$$

The third asset is a loan to be amortized over a period $[0, T]$ whose price at time $t \geq 0$ is denoted by $L(t)$. Let us also assume that the price of the asset can be describe by a stochastic differential equation similar to (5) above then

$$\frac{dL(t)}{L(t)} = (r(t) + \lambda_l)dt + \sigma_l dw_l(t) \qquad L(0) = l_0 \qquad (6)$$

where λ_l and σ_l are constants.

The Derivation of the Financial Institution Assets Portfolio Model

Let $X(t)$ denotes the value of the financial institution assets portfolio at time $t \in [0, T], \pi_s(t)$ and $\pi_l(t)$ denote the amounts invested in the security and loan respectively. Therefore,

$$\pi_0(t) = X(t) - \pi_s(t) - \pi_l(t)$$

denotes the amount invested in the riskless asset. The assets portfolio model is given by the following SDE:

$$dX(t) = \big(X(t) - \pi_s(t) - \pi_l(t)\big)\frac{dS_0(t)}{S_0(t)} + \pi_s(t)\frac{dS(t)}{S(t)} + \pi_l(t)\frac{dL(t)}{L(t)}$$

$$= (X(t)r(t) + \pi_s(t)\lambda_s + \pi_l(t)\lambda_l)dt + \pi_s(t)\sigma_s dw_s(t) + \pi_l(t)\sigma_l dw_l(t) \qquad (7)$$

The Asset Portfolio Optimization Problem of the Financial Institution
Let the set of all admissible strategy be denoted by Π. Under the asset portfolio (7), the financial institution looks for an optimal strategy ${\pi_s}^*(t)$ and ${\pi_l}^*(t)$ which maximizes the expected utility of the terminal wealth. i.e.:

$$\underset{\pi(t)\in\Pi}{\mathrm{Max}}\, E[U(X(T))] \tag{8}$$

Based on the classical tools of stochastic optimal control, we state the optimization problem as follows:

Maximize $E[U(X(T))]$

Subject to the following constraints

$dr(t) = \big(a - br(t)\big)dt + \sigma_r dw_r(t),$

$dX(t) = (X(t)r(t) + \pi_s(t)\lambda_s + \pi_l(t)\lambda_l)dt + \pi_s(t)\sigma_s dw_s(t) + \pi_l(t)\sigma_l dw_l(t)$

$0 \le t \le T$ and $X(0) = x_0, r(0) = r_0$

The objective is to maximize the expected utility of the financial institution's portfolio at future date $T > 0$. That is, find the optimal value function

$$H(t,r,x) = \underset{\pi(t)\in\Pi}{\mathrm{Max}}\, E\big[U\big(X(T)\big)\big|r(t) = r, X(t) = x\,\big] \tag{9}$$

and the optimal strategy $\pi^*(t) = ({\pi_s}^*(t), {\pi_l}^*(t))$ such that

$$\underset{\pi^*(t)}{\mathrm{H}}(t,r,x) = H(t,r,x) \tag{10}$$

The Derivation of the Hamilton – Jacobi – Bellman Equation Associated With the Asset Portfolio Optimization Problem

The Hamilton – Jacobi – Bellman equation associated with the asset portfolio optimization problem is:

$$\underset{\pi(t)\in\Pi}{\max}\{H_t + [X(t)r(t) + \pi_s(t)\lambda_s + \pi_l(t)\lambda_l]H_x + \frac{1}{2}(\pi_s^2(t)\sigma_s^2 + \pi_l^2(t)\sigma_l^2)H_{xx}$$

$$+(\pi_s(t)\sigma_s\sigma_r + \pi_l(t)\sigma_l\sigma_r)H_{xr} + [a - br(t)]H_r + \frac{1}{2}\sigma_r^2 H_{rr}\} = 0 \tag{11}$$

$$H(T,r,x) = U(x) \tag{12}$$

where $H_t, H_x, H_r, H_{xx}, H_{rr}$ and H_{xr} denote partial derivatives of first and second orders with respect to $t, r,$ and x respectively.

The first order maximizing conditions for the optimal investment strategy $\big(\pi_s^*(t), \pi_l^*(t)\big)$ (i.e., differentiating (11) with respect to $\pi_s(t)$ and $\pi_l(t)$) gives

$$\lambda_s H_x + \pi_s(t)\sigma_s^2 H_{xx} + \sigma_s\sigma_r H_{xr} = 0 \tag{13}$$

$$\lambda_l H_x + \pi_s(t)\sigma_l^2 H_{xx} + \sigma_l\sigma_r H_{xr} = 0 \tag{14}$$

respectively. Next, we Solve (13) and (14) for $\pi_s(t)$ and $\pi_l(t)$ to obtain the optimal strategy $\big(\pi_s^*(t), \pi_l^*(t)\big)$.

From equations (13) and (14) we have

$$\pi_s^*(t) = -\frac{\lambda_s H_x}{\sigma_s^2 H_{xx}} - \frac{\sigma_r H_{xr}}{\sigma_s H_{xx}} \quad \text{and} \quad \pi_l^*(t) = -\frac{\lambda_l H_x}{\sigma_l^2 H_{xx}} - \frac{\sigma_r H_{xr}}{\sigma_l H_{xx}} \tag{15}$$

Substituting (15) into (11) gives the partial differential equation (PDE) for the value function $H(t,r,x)$.

$$H_t + xrH_x - \left(\frac{\lambda_s^2}{2\sigma_s^2} + \frac{\lambda_l^2}{2\sigma_l^2}\right)\frac{H_x^2}{H_{xx}} - \frac{\sigma_r^2 H_{xr}^2}{H_{xx}} - \left(\frac{\lambda_s\sigma_r}{\sigma_s} + \frac{\lambda_l\sigma_r}{\sigma_l}\right)\frac{H_x H_{xr}}{H_{xx}}$$

$$+(a - br)H_r + \frac{1}{2}\sigma_r^2 H_{rr} = 0 \tag{16}$$

Therefore, after substituting (15) into (11) and after simplification, we obtained that the Hamilton – Jacobi – Bellman (HJB) equation (12) is equivalent to the partial differential equation (16). The problem now is to solve (16) for the value function $H(t,r,x)$ and replace it in (15).

The Assets Portfolio Optimization Problem and its solution Under Power Utility Function.

From (16) and considering Constant Relative Risk Aversion (CRRA) utility function:

$$\mathrm{U}(x) = \frac{x^{\beta}}{\beta} \quad \beta < 1, \beta \neq 0$$

show that the value function $H(t,r,x)$ takes the following form:

$$H(t,r,x) = \frac{x^{\beta}}{\beta} f(t,r), \quad \beta < 1, \beta \neq 0 \tag{17}$$

With the boundary condition:

$$f(T,r) = 1 \text{ for all } r \tag{18}$$

From (17)

$$\left. H_t = \frac{x^{\beta}}{\beta} f_t, H_x = x^{\beta-1} f, H_r = \frac{x^{\beta}}{\beta} f_r, H_{xx} = (\beta-1)x^{\beta-2} f, H_{xr} = x^{\beta-1} f_r, H_{rr} = \frac{x^{\beta}}{\beta} f_{rr} \right\} \tag{19}$$

Where $H_t, H_x, H_r, H_{xx}, H_{xr}$ and H_{rr} are first order and second order partial derivatives of H with respect to t and r. f_t, f_r and f_{rr} represent the first order and second order partial derivatives of f with respect to t and r.

Therefore, introducing these partial derivatives in (19) into (16) and simplifying gives

$$f_t + \left[r\beta - \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(\beta-1)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(\beta-1)} \right) \right] f - \frac{\beta\sigma_r^2 f_r^2}{(\beta-1)f}$$

$$+ \left[(a - br) - \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(\beta-1)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(\beta-1)} \right) \right] f_r + \frac{1}{2}\sigma_r^2 f_{rr} = 0 \tag{20}$$

Next we conjecture $f(t,r)$ as the following:

$$f(t,r) = A(t)\exp(\phi(t)r), A(T) = 1, \Phi(T) = 0\} \tag{21}$$

From (21)

$$\left. \begin{array}{c} f_t = \left(A_1'(t) + r\phi'(t)A(t)\right)\exp(\phi(t)r) \\ f_r = \phi(t)A(t)\exp(\phi(t)r), f_{rr} = \phi^2(t)A(t)\exp(\phi(t)r) \end{array} \right\} \tag{22}$$

Hence substituting for f_t, f_r and f_{rr} in (20) and noting that $f = A(t)\exp(\phi(t)r)$ gives

$$rA(t)\exp(\phi(t)r)\left(\phi'(t) + \beta - b\phi(t)\right) + \exp(\phi(t)r)\left[A_1'(t) + \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{(\beta-1)} \right)\phi^2(t)A(t) \right.$$

$$\left. + \left[a - \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(\beta-1)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(\beta-1)} \right) \right]\phi(t)A(t) - \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(\beta-1)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(\beta-1)} \right)A(t) \right] = 0 \tag{23}$$

Next, we decompose (23) into

$$\phi'(t) + \beta - b\phi(t) = 0 \tag{24}$$

$$\left[A_1'(t) + \left(\frac{1}{2}\sigma_r^2 - \frac{\beta\sigma_r^2}{(\beta-1)} \right)\phi^2(t)A(t) + \left[a - \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(\beta-1)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(\beta-1)} \right) \right]\phi(t)A(t) \right.$$

$$\left. - \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(\beta-1)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(\beta-1)} \right)A(t) \right] = 0 \tag{25}$$

Now, solving for $\phi(t)$ in equation (24), we obtain

$$\phi(t) = \frac{\beta}{b}\left(1 - e^{-b(T-t)}\right) \tag{26}$$

Next we solve for $A(t)$ in (25). From (18), $\beta < 1$. Hence, from (25) we have

$$A_1'(t) + A(t)\left[\left(\frac{1}{2}\sigma_r^2 + \frac{\beta\sigma_r^2}{(1-\beta)}\right)\phi^2(t) + \left[a + \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(1-\beta)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(1-\beta)}\right)\right]\phi(t)\right.$$

$$\left. + \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(1-\beta)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(1-\beta)}\right)\right] = 0 \tag{27}$$

Let

$$p(t) = \left(\frac{1}{2}\sigma_r^2 + \frac{\beta\sigma_r^2}{(1-\beta)}\right)\phi^2(t) + \left[a + \left(\frac{\beta\lambda_s\sigma_r}{\sigma_s(1-\beta)} + \frac{\beta\lambda_l\sigma_r}{\sigma_l(1-\beta)}\right)\right]\phi(t)$$

$$+ \left(\frac{\beta\lambda_s^2}{2\sigma_s^2(1-\beta)} + \frac{\beta\lambda_l^2}{2\sigma_l^2(1-\beta)}\right)$$

then from equation (27), we have the following

$$\frac{dA(t)}{dt} + p(t)A(t) = 0 \tag{28}$$

Solving equation (28) and imposing the boundary condition $A(T) = 1$ gives

$$A(t) = \exp\left(P(T) - P(t)\right) \tag{29}$$

Hence,

$$f(t,r) = A(t)\exp(\phi(t)r)$$

$$= \exp\left(\left(P(T) - P(t)\right) + \frac{\beta}{b}\left(1 - e^{-b(T-t)}\right)r\right) \tag{30}$$

and

$$H(t,r,x) = \frac{x^\beta}{\beta}\exp\left(\left(P(T) - P(t)\right) + \frac{\beta}{b}\left(1 - e^{-b(T-t)}\right)r\right) \tag{31}$$

Theorem 1

Given (15), (19) and (22), the optimal proportion of wealth invested in security, loan and treasury are:

$$\pi_{sp}^*(t) = \left(\frac{\lambda_s}{\sigma_s^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_s b(1-\beta)}\right)$$

$$\pi_{lp}^*(t) = \left(\frac{\lambda_l}{\sigma_l^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_l b(1-\beta)}\right)$$

$$\pi_{0p}^*(t) = 1 - \left[\left(\frac{\lambda_s}{\sigma_s^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_s b(1-\beta)}\right)\right] - \left[\left(\frac{\lambda_l}{\sigma_l^2(1-\beta)}\right) + \left(\frac{\beta\beta_1\sigma_r}{\sigma_l b(1-\beta)}\right)\right]$$

where $\beta_1 = \left(1 - e^{-b(T-t)}\right)$

Derivation of the Capital Adequacy Ratio Model
Total Risk – Weighted Assets (TRWA) model equation
The dynamics of the total risk – weighted assets at time t, can be described by the stochastic differential equation:

$$dY_{rw}(t) = 0 \times \left(X(t) - \pi_s(t) - \pi_l(t)\right)\frac{dS_0(t)}{S_0(t)} + 0.2 \times \pi_s(t)\frac{dS(t)}{S(t)} + 0.5 \times \pi_l(t)\frac{dL(t)}{L(t)} \tag{32}$$

where, 0, 0.2 and 0.5 are the risk weights associated with the treasury, security and loan under Basel II Accord respectively. Therefore,

$$dY_{rw}(t) = [0.2\pi_s(t)(r(t)+\lambda_s) + 0.5\pi_l(t)(r(t)+\lambda_l)]dt + 0.2\pi_s(t)\sigma_s dw_s(t)$$
$$+0.5\pi_l(t)\sigma_l dw_l(t) \quad (33)$$

Capital Adequacy Ratio Model Equation

The Basel Accord and central bank of Nigeria lay down regulations seeking to provide incentives for greater awareness of differences in risk through more risk sensitive minimum capital requirements based on numerical formula. The Capital Adequacy Ratio (CAR) also known as capital to risk weighted assets ratio is the measure of the amount of a financial institution's capital relative to the amount of its credit exposures. An international standard has been adopted that requires a financial institution e.g. bank to comply with minimum capital requirements. The purpose of maintaining minimum capital adequacy ratios is to guarantee that banks are prepared to absorb a reasonable level of loses before becoming insolvent. Hence, it promotes protection of depositors, the stability and effectiveness of the financial system. The capital adequacy ratio dynamics can be described by:

$$CAR = \frac{K(t)}{Y_{rw}(t)} \quad (34)$$

where, $K(t)$ is the total capital and $Y_{rw}(t)$ is the total risk – weighted assets capital of the financial institution respectively.

Let $CAR = Z(t)$, then from (34)

$$Z(t) = \frac{K(t)}{Y_{rw}(t)} \quad (35)$$

Proposition 1 (SDE for capital adequacy ratio)

Let the dynamics of the total capital of the financial institution be

$$dK(t) = k(t)dt$$

and the total risk – weighted assets $Y_{rw}(t)$ be described by (33). The dynamics of the Basel II capital adequacy ratio $Z(t)$ satisfies the following stochastic differential equation:

$$dZ(t) = f\big(Y_{rw}(t)\big)dK(t) + K(t)df\big(Y_{rw}(t)\big)$$
$$= \frac{kdt}{Y_{rw}(t)} + \left\{\frac{1}{Y_{rw}^3(t)}([0.2\pi_s(t)\sigma_s]^2 + [0.5\pi_l(t)\sigma_l]^2)\right.$$
$$\left. -\frac{1}{Y_{rw}^2(t)}0.2\pi_s(t)(r(t)+\lambda_s) - 0.5\pi_l(t)(r(t)+\lambda_l)\frac{1}{Y_{rw}^2(t)}\right\}dt$$
$$-\frac{1}{Y_{rw}^2(t)}\{+0.2\pi_s(t)\sigma_s dw_s(t) + 0.5\pi_l(t)\sigma_l dw_l(t)\} \quad (36)$$

Proof:

Let $f\big(Y_{rw}(t)\big) = \frac{1}{Y_{rw}(t)}$, $dK(t) = k(t)dt$, then

$$Z(t) = K(t)f\big(Y_{rw}(t)\big)$$

$$dZ(t) = d\big[K(t)f\big(Y_{rw}(t)\big)\big] \quad (37)$$

Applying Ito product rule to the RHS (right hand side) of (37) yields

$$dZ(t) = f\big(Y_{rw}(t)\big)dK(t) + K(t)df\big(Y_{rw}(t)\big) \quad (38)$$

From Ito Lemma,

$$df\big(Y_{rw}(t)\big) = f'(t)dt + f'\big(Y_{rw}(t)\big)dY_{rw}(t) + \frac{1}{2}f''(Y_{rw}(t))[dY_{rw}(t)]^2$$

$$= -\frac{dY_{rw}(t)}{Y_{rw}^2(t)} + \frac{[dY_{rw}(t)]^2}{Y_{rw}^3(t)} \quad (39)$$

From (33)

$$[dY_{rw}(t)]^2 = \left[\left(0.2\pi_s(t)(r(t)+\lambda_s) + 0.5\pi_l(t)(r(t)+\lambda_l)\right)dt + 0.2\pi_s(t)\sigma_s dw_s(t) + 0.5\pi_l(t)\sigma_l dw_l(t)\right]^2$$

Note that

$dt.dt = dt.dW_t = dW_t.dt = 0, dW_t.dW_t = dt$. Therefore,

$$[dY_{rw}(t)]^2 = ([0.2\pi_s(t)\sigma_s]^2 + [0.5\pi_l(t)\sigma_l]^2)dt$$

Hence,

$$df\left(Y_{rw}(t)\right) = -\frac{dY_{rw}(t)}{Y_{rw}^2(t)} + \frac{[dY_{rw}(t)]^2}{Y_{rw}^3(t)}$$

$$df\left(Y_{rw}(t)\right) = \left\{\frac{1}{Y_{rw}^3(t)}([0.2\pi_s(t)\sigma_s]^2 + [0.5\pi_l(t)\sigma_l]^2)\right.$$

$$\left. -\frac{1}{Y_{rw}^2(t)}0.2\pi_s(t)(r(t)+\lambda_s) \; -0.5\pi_l(t)(r(t)+\lambda_l)\frac{1}{Y_{rw}^2(t)}\right\}dt$$

$$-\frac{1}{Y_{rw}^2(t)}\{+0.2\pi_s(t)\sigma_s dw_s(t) + 0.5\pi_l(t)\sigma_l dw_l(t)\}$$

Now, returning back to (37) we have

$$dZ(t) = \; f\left(Y_{rw}(t)\right)dK(t) + K(t)df\left(Y_{rw}(t)\right)$$

$$= \frac{kdt}{Y_{rw}(t)} + \left\{\frac{1}{Y_{rw}^3(t)}([0.2\pi_s(t)\sigma_s]^2 + [0.5\pi_l(t)\sigma_l]^2)\right.$$

$$\left. -\frac{1}{Y_{rw}^2(t)}0.2\pi_s(t)(r(t)+\lambda_s) -0.5\pi_l(t)(r(t)+\lambda_l)\frac{1}{Y_{rw}^2(t)}\right\}dt$$

$$-\frac{1}{Y_{rw}^2(t)}\{0.2\pi_s(t)\sigma_s dw_s(t) + 0.5\pi_l(t)\sigma_l dw_l(t)\} \quad (40)$$

Proposition 2 (The SDE for the Capital Required to Maintain Total Capital Ratio at 8%)

Given that the capital adequacy ratio is:

$$\text{CAR} = Z(t) = \frac{K(t)}{Y_{rw}(t)}$$

Then the dynamics of the capital required to maintain the total capital ratio at 8% according to Basel II accord is:

$$dK(t) \; = \{0.016\pi_s(t)(r(t)+\lambda_s) + 0.04\pi_l(t)(r(t)+\lambda_l)\}dt + 0.016\pi_s(t)\sigma_s dw_s(t)$$

$$+0.04\pi_l(t)\sigma_l dw_l(t) \quad (41)$$

Proof:

From Total Capital Ratio $= Z(t) = \frac{K(t)}{Y_{rw}(t)}$

we obtain

$$\frac{K(t)}{Y_{rw}(t)} = 0.08$$

$$K(t) = 0.08Y_{rw}(t)$$

Therefore, the dynamics of the capital required to maintain total capital ratio at 8% is:

$$dK(t) = 0.08dY_{rw}(t)$$

$$= 0.08\left(0.2 \times \pi_s(t)\frac{dS(t)}{S(t)} + 0.5 \times \pi_l(t)\frac{dL(t)}{L(t)}\right)$$

$$= \{0.016\pi_s(t)(r(t) + \lambda_s) + 0.04\pi_l(t)(r(t) + \lambda_l)\}dt + 0.016\pi_s(t)\sigma_s dw_s(t)$$

$$+0.04\pi_l(t)\sigma_l dw_l(t) \quad (42)$$

Proposition 3 (The SDE for the Capital Required to Maintain Total Capital Ratio at 15%)

Given that the capital adequacy ratio is:

$$\text{CAR} = Z(t) = \frac{K(t)}{Y_{rw}(t)}$$

Then the dynamics of the capital required to maintain the total capital ratio at 15% according to the Central Bank of Nigeria is:

$$dK(t) = \{0.03\pi_s(t)(r(t) + \lambda_s) + 0.075\pi_l(t)(r(t) + \lambda_l)\}dt + 0.03\pi_s(t)\sigma_s dw_s(t)$$

$$+0.075\pi_l(t)\sigma_l dw_l(t) \quad (43)$$

Proof:

From Total Capital Ratio:

$$Z(t) = \frac{K(t)}{Y_{rw}(t)}$$

we obtain

$$\frac{K(t)}{Y_{rw}(t)} = 0.15$$

$$K(t) = 0.15Y_{rw}(t)$$

Therefore, the dynamics of the capital required to maintain total capital ratio at 15% is:

$$dK(t) = 0.15dY_{rw}(t)$$

$$= 0.15\left(0.2 \times \pi_s(t)\frac{dS(t)}{S(t)} + 0.5 \times \pi_l(t)\frac{dL(t)}{L(t)}\right)$$

$$dK(t) = \{0.03\pi_s(t)(r(t) + \lambda_s) + 0.075\pi_l(t)(r(t) + \lambda_l)\}dt + 0.03\pi_s(t)\sigma_s dw_s(t)$$

$$+0.075\pi_l(t)\sigma_l dw_l(t) \quad (44)$$

Numerical Examples

Here, we present the numerical simulation for the evolution of the optimal investment strategy, TRWA, the capital required to maintain the CAR at 8% and 15% and CAR. We take the investment period $T = 10$ years, $\beta = 0.5, k = 0.1, K = 1$, $Z = 0.08$ and $Z = 0.15$ from BCBS and CBN capital adequacy requirements, and assumed that $Y_{rw} = 1.4, \lambda_l = 0.0031$, $\sigma_l = 0.0874$. The remaining parameters $b = 2.5148, \lambda_s = 0.0022, \sigma_s = 0.0854, \sigma_r = 0.3535, r = 0.1493$ are estimated from data obtained from CBN statistical bulletin and Nigeria Stock Exchange Fact Book.

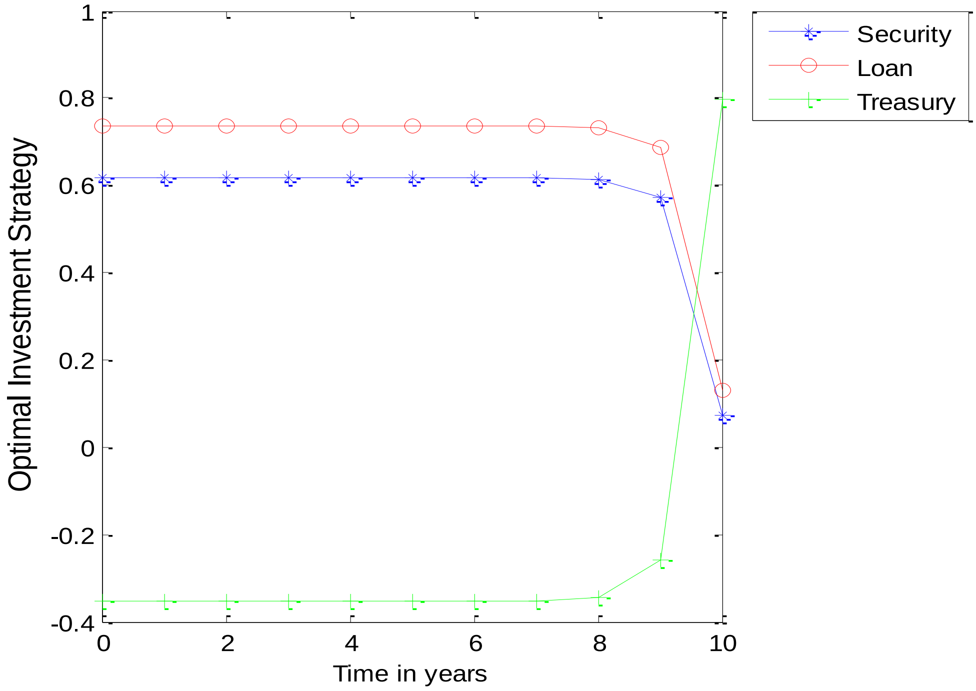

Fig. 1: The effect of time on the optimal investment strategy

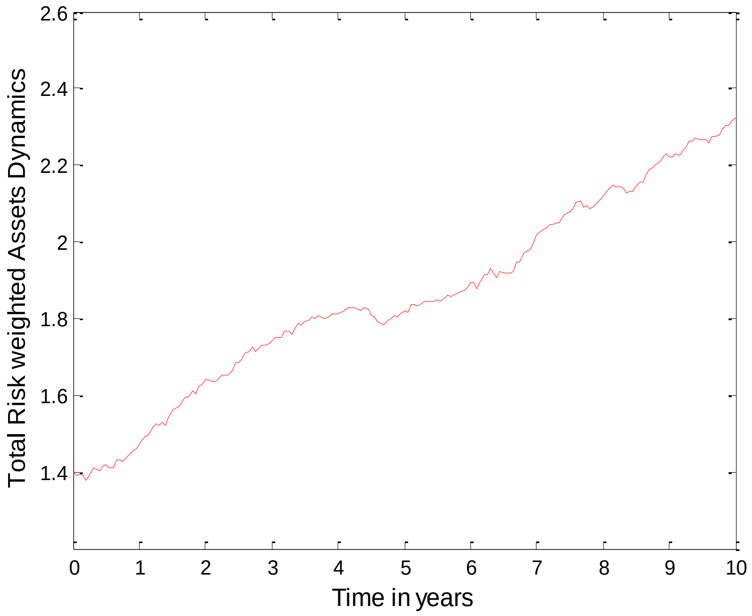

Fig. 2: A Simulation of the total risk – weighted assets, $Y_{rw}(t)$

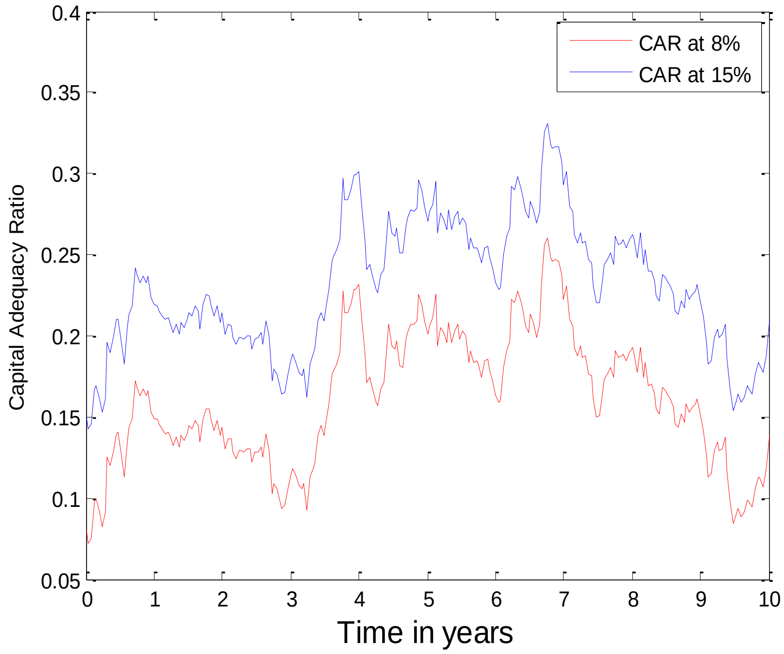

Fig. 3: A Simulation of the behavior of the capital adequacy ratio $Z(t)$ at 8% and 15%

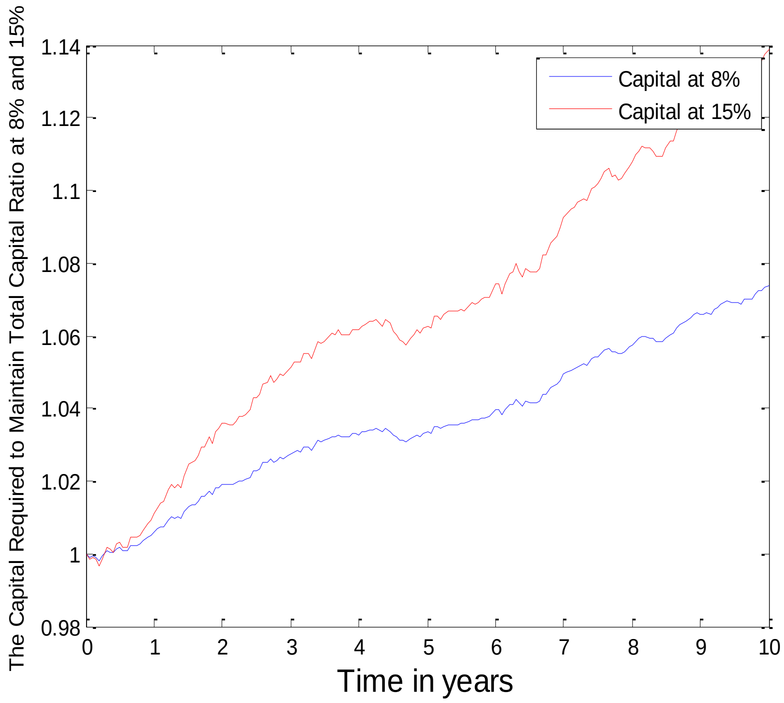

Fig. 4: A Simulation of the capital, $K(t)$, required to maintain the capital adequacy ratio at 8% and 15%

Figure 1 illustrates the trends of how the optimal proportion of the wealth invested in the three assets change with time. From figure 1, there is a positive relationship between optimal investment in the treasury and time. That is, as time increases so also the optimal investment in the treasury. However, the optimal proportion invested in the security almost remains unchanged and the optimal proportion invested loan decreases with time. Figure 1 also shows that the optimal proportion invested in the treasury is negative at the beginning of the investment horizon which indicates that the investor takes a short position in the treasury. But toward the end of the investment period, the investor invests more in the treasury to reach the optimal investment strategy.

Figure 2 illustrates how the evolution of the risk weighted – asset is affected by the stochastic variables characterizing the economy. By Basel II standard and Nigeria CBN, the financial institution is considered to be strongly capitalized and guaranteed the ability to absorb unexpected losses as shown in Figure 3. Therefore, as shown in figure 3, the higher the CAR the more resilient the financial institution but this also have its own down side as shown in figure 4. From figure 4, we observed that more capital is needed to maintain the capital adequacy ratio at 15% than 8%. Therefore, the higher the percentage of the capital adequacy ratio, the more capital needed to maintain the prescribed capital adequacy ratio by the financial institution. This would tie up capital needed for investment by the investor. Therefore, prescribed capital adequacy ratio should be kept in a range such that the financial institution is well capitalized and guarantee that the financial institution can absorb reasonable unexpected losses, and also relieve fund for investor for investment which is important to the shareholders and the economy.

CONCLUSION

Allocating optimally a financial institution's resources among competing investments is very important. In this research work, we have considered asset optimization problem of a financial institution where the interest rate is driven by stochastic interest rate model. The volatilities of the security and loan are assumed to be constant. Here, the investor objective is to maximize the utility of the terminal wealth. The financial market consists of three assets namely; security, loan and treasury. We derived the optimal investment strategy under the CRRA utility function, obtained the explicit solution of the resulted Hamilton – Jacobi – Bellman equation for the optimal asset allocation problem. We also derived an explicit stochastic differential equation (SDE) for the capital adequacy ratio (CAR) which is the ratio of the financial institution total capital to the total risk – weighted assets under Basel II Accord. Furthermore, we derived the SDE for the total risk – weighted assets (TRWA) and SDE for the capital required to maintain the capital adequacy ratio under Basel II and Nigeria CBN standards and solved the derived SDEs numerically by Euler – Maruyama method. We also estimated some of the parameters of the models using maximum likelihood method and apply it to financial institution in Nigeria.

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