



IMPLICIT FOUR-STEP APPROACH WITH APPLICATION TO NON-LINEAR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

A unique and efficient implicit four-step approach with application to nonlinear third order ordinary differential equations is considered in this article. In the derivation of this method Collocation and Interpolation techniques were engaged and power series approximate solution was used as the interpolating polynomial. The third derivative of the power series was collocated at the entire grid points, while the interpolation was done at the first three points. Appropriate study of the basic properties of the method was done. The results generated when the new block method was applied on nonlinear third order ordinary differential equations are better in terms of accuracy than the existing methods.

Keywords: Implicit Four-step, Non-linear Third Order, Interpolation, Collocation, Ordinary Differential Equations, and Power series.

INTRODUCTION

The numerical solution of nonlinear third order initial value problems (IVPs) of ordinary differential equations (ODEs) directly using a unique implicit four-step linear multistep block method is studied in this research. These ODEs which are frequently met in our everyday lives are of the form

$$y^{iii}(x) = f(x, y, y^i, y^{ii}), y(x_0) = y_0, y^i(x_0) = y_1, y^{ii}(x_0) = y_2 \quad (1)$$

Equation (1) arises in diverse fields of applied mathematics, amongst which are elasticity, fluid mechanics, and quantum mechanics as well as in control system, engineering and physics. The existence and uniqueness of the solution for these equations have been discussed extensively in Adeniran & Omotoye (2016) and Wend (1969). In general, finding the exact solutions of these equations is not easy. For instance, the application problem in fluid mechanics named Fluid flow does not have exact solution, hence it is important to get the numerical solutions [3, 7, 9]. For a long time, different numerical methods have been developed in order to approximate the solution of equation (1). Among these methods are block method, linear multistep method, hybrid method, Taylor series and Rung-kutta method, see Henrici (1962), Kayode et al., (2018), Adoghe et al., (2016), Adeniran & Omotoye (2016), Abdelrahim et al., (2019), Ukpebor (2019), Ogunware et al., (2018), and Yao et al., (2011).

This article is motivated to derive a Four-step approach with an application to nonlinear third order ordinary differential equations via power series as the basic function. This work is motivated by the success story of block methods for solving ordinary differential equations directly without reducing it to system of first order ordinary differential equation. The advantages of the method lie in the fact that it is economical, saves time and computationally reliable.

MATERIALS AND METHOD

In this section, the procedure for derivation of the proposed method for solving (1) is presented. Let the exact solution $y(x)$ to approximate (1) be of the form

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j \quad (2)$$

with the third derivative given as

$$y'''(x) = \sum_{j=3}^{c+i-1} j(j-1)(j-2)a_j x^{j-3} \quad (3)$$

In this case, c is the number of collocation points and i is the number of interpolation points. (2) is called interpolation equation while (3) is called collocation equation.

Applying the conditions (2) and (3) at some strategic points give the following equations

$$h^7 a_7 + h^6 a_6 + h^5 a_5 + h^4 a_4 + h^3 a_3 + h^2 a_2 + h a_1 + a_0 = y_{n+1} \tag{4}$$

$$128h^7 a_7 + 64h^6 a_6 + 32h^5 a_5 + 16h^4 a_4 + 8h^3 a_3 + 4h^2 a_2 + 2h a_1 + a_0 = y_{n+2} \tag{5}$$

$$2187h^7 a_7 + 729h^6 a_6 + 243h^5 a_5 + 81h^4 a_4 + 27h^3 a_3 + 9h^2 a_2 + 3h a_1 + a_0 = y_{n+3} \tag{6}$$

$$6a_3 = f_n \tag{7}$$

$$210h^4 a_7 + 120h^3 a_6 + 60h^2 a_5 + 24h a_4 + 6a_3 = f_{n+1} \tag{8}$$

$$3360h^4 a_7 + 960h^3 a_6 + 240h^2 a_5 + 48h a_4 + 6a_3 = f_{n+2} \tag{9}$$

$$17010h^4 a_7 + 3240h^3 a_6 + 540h^2 a_5 + 72h a_4 + 6a_3 = f_{n+3} \tag{10}$$

$$53760h^4 a_7 + 7680h^3 a_6 + 960h^2 a_5 + 96h a_4 + 6a_3 = f_{n+4} \tag{11}$$

Remark 2.1: It should be noted that equations (4) to (11) are generated by evaluating the approximate solution (2) and its third derivative (3) at the suitable points.

Combining (4-11) and solve with Computer Aided Software such as Maple 18 to obtained the values of $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$ as follows. see Ogunware et al., (2015) and Omole & Ukpebor (2020) for more details

$$a_0 = -\frac{1}{240} h^3 f_n - \frac{29}{60} h^3 f_{n+1} - \frac{21}{40} h^3 f_{n+2} + \frac{1}{60} h^3 f_{n+3} - \frac{1}{240} h^3 f_{n+4} + 3y_{n+1} - 3y_{n+2} + y_{n+3}$$

$$a_1 = \frac{1}{10080 h} \left(677h^3 f_n + 10480h^3 f_{n+1} + 7254h^3 f_{n+2} + 64h^3 f_{n+3} + 5h^3 f_{n+4} - 25200y_{n+1} + 40320y_{n+2} - 15120y_{n+3} \right)$$

$$a_2 = -\frac{1}{720} \frac{1}{h^2} \left(118h^3 f_n + 477h^3 f_{n+1} + 96h^3 f_{n+2} + 35h^3 f_{n+3} - 6h^3 f_{n+4} - 360y_{n+1} + 720y_{n+2} - 360y_{n+3} \right)$$

$$a_3 = \frac{1}{6} f_n$$

$$a_4 = -\frac{1}{288} \frac{25f_n - 48f_{n+1} + 36f_{n+2} - 16f_{n+3} + 3f_{n+4}}{h}$$

$$a_5 = \frac{1}{1440} \frac{35f_n - 104f_{n+1} + 114f_{n+2} - 56f_{n+3} + 11f_{n+4}}{h^2}$$

$$a_6 = -\frac{1}{1440} \frac{5f_n - 18f_{n+1} + 24f_{n+2} - 14f_{n+3} + 3f_{n+4}}{h^3}$$

$$a_7 = \frac{1}{5040} \frac{f_n - 4f_{n+1} + 6f_{n+2} - 4f_{n+3} + f_{n+4}}{h^4}$$

Substituting ($a_0 - a_7$) into (2) gives a Four-step implicit continuous coefficient of the form:

$$y(t) = \alpha_1(t)y_{n+1} + \alpha_2(t)y_{n+2} + \alpha_3(t)y_{n+3} + h^2(\beta_0(t)+\beta_1(t)+\beta_2(t)+\beta_3(t)+\beta_4(t)) \tag{12}$$

where $\alpha_1(t), \alpha_2(t)...$ and $\beta_0(t), \beta_1(t), \dots, \beta_4(t)$ are continuous coefficients. See Kayode et al., (2018), Adoghe et al., (2016), Adeniran & Omotoye (2016), Abdelrahim et al., (2019), Ukpebor (2019) for more reference

The continuous method (12) is used to the discrete schemes below. That is, evaluating (12) at $t=4$ and $t=0$ and evaluate the first and second derivatives of (12) at all points gives the following discrete schemes. For more details, please see Yao et al., (2011), Wend (1969) and Ogunware et al. (2015).

$$\frac{1}{240} h^3 f_n - \frac{1}{60} h^3 f_{n+1} + \frac{21}{40} h^3 f_{n+2} + \frac{29}{60} h^3 f_{n+3} + \frac{1}{240} h^3 f_{n+4} - 3y_{n+2} + 3y_{n+3} + y_{n+1} = y_{n+4} \tag{13}$$

$$-\frac{1}{240} h^3 f_n - \frac{29}{60} h^3 f_{n+1} - \frac{21}{40} h^3 f_{n+2} + \frac{1}{60} h^3 f_{n+3} - \frac{1}{240} h^3 f_{n+4} + 3y_{n+1} - 3y_{n+2} + y_{n+3} = y_n \tag{14}$$

With the first derivatives as follows

$$\frac{1}{10080 h} \left(677h^3 f_n + 10480h^3 f_{n+1} + 7254h^3 f_{n+2} + 64h^3 f_{n+3} + 5h^3 f_{n+4} - 25200y_{n+1} + 40320y_{n+2} - 15120y_{n+3} \right) = y'_n \tag{15}$$

$$-\frac{1}{5040} \frac{1}{h} \left(29h^3 f_n - 452h^3 f_{n+1} - 1296h^3 f_{n+2} + 52h^3 f_{n+3} - 13h^3 f_{n+4} + 7560y_{n+1} - 10080y_{n+2} + 2520y_{n+3} \right) = y'_{n+1} \tag{16}$$

$$\frac{1}{10080} \frac{1}{h} \left(5h^3 f_n - 104h^3 f_{n+1} - 1482h^3 f_{n+2} - 104h^3 f_{n+3} + 5h^3 f_{n+4} - 5040y_{n+1} + 5040y_{n+3} \right) = y'_{n+2} \tag{17}$$

$$\frac{1}{5040} \frac{1}{h} \left(13h^3 f_n - 52h^3 f_{n+1} + 1296h^3 f_{n+2} + 452h^3 f_{n+3} - 29h^3 f_{n+4} + 2520y_{n+1} - 10080y_{n+2} + 7560y_{n+3} \right) = y'_{n+3} \tag{18}$$

$$\frac{1}{10080} \frac{1}{h} \left(5h^3 f_n + 64h^3 f_{n+1} + 7254h^3 f_{n+2} + 10480h^3 f_{n+3} + 677h^3 f_{n+4} + 15120y_{n+1} - 40320y_{n+2} + 25200y_{n+3} \right) = y'_{n+4} \tag{19}$$

With the following second derivatives

$$-\frac{1}{360} \frac{1}{h^2} \left(118h^3 f_n + 477h^3 f_{n+1} + 96h^3 f_{n+2} + 35h^3 f_{n+3} - 6h^3 f_{n+4} - 360y_{n+1} + 720y_{n+2} - 360y_{n+3} \right) = y''_n \tag{20}$$

$$\frac{1}{720} \frac{1}{h^2} \left(15h^3 f_n - 308h^3 f_{n+1} - 456h^3 f_{n+2} + 36h^3 f_{n+3} - 7h^3 f_{n+4} + 720y_{n+1} - 1440y_{n+2} + 720y_{n+3} \right) = y''_{n+1} \tag{21}$$

$$-\frac{1}{360} \frac{2h^3 f_n - 19h^3 f_{n+1} + 19h^3 f_{n+3} - 2h^3 f_{n+4} - 360y_{n+1} + 720y_{n+2} - 360y_{n+3}}{h^2} = y''_{n+2}$$

$$\frac{1}{720} \frac{1}{h^2} \left(7h^3 f_n - 36h^3 f_{n+1} + 456h^3 f_{n+2} + 308h^3 f_{n+3} - 15h^3 f_{n+4} + 720y_{n+1} - 1440y_{n+2} + 720y_{n+3} \right) = y''_{n+3} \tag{22}$$

$$-\frac{1}{360} \frac{1}{h^2} \left(6h^3 f_n - 35h^3 f_{n+1} - 96h^3 f_{n+2} - 477h^3 f_{n+3} - 118h^3 f_{n+4} - 360y_{n+1} + 720y_{n+2} - 360y_{n+3} \right) = y''_{n+4} \tag{23}$$

Combining equations (13-23) and solve simultaneously gives the block formula below which will be used to solve (1) directly with developing separate starting values

$$y_{n+1} = \frac{113}{1120} h^3 f_n + \frac{107}{1008} h^3 f_{n+1} + \frac{43}{1680} h^3 f_{n+3} - \frac{47}{10080} h^3 f_{n+4} - \frac{103}{1680} h^3 f_{n+2} + \frac{1}{2} h^2 y''_n + h y'_n + y_n \tag{24}$$

$$y_{n+2} = \frac{331}{630} h^3 f_n + \frac{332}{315} h^3 f_{n+1} - \frac{8}{2} h^3 f_{n+2} + \frac{52}{315} h^3 f_{n+3} - \frac{19}{630} h^3 f_{n+4} + 2h^2 y''_n + 2h y'_n + y_n \tag{25}$$

$$y_{n+3} = \frac{1431}{1120} h^3 f_n + \frac{1863}{560} h^3 f_{n+1} - \frac{243}{560} h^3 f_{n+2} + \frac{45}{112} h^3 f_{n+3} - \frac{81}{1120} h^3 f_{n+4} + \frac{9}{2} h^2 y''_n + y_n + 3h y'_n \tag{26}$$

$$y_{n+4} = \frac{248}{105}h^3f_n + \frac{2176}{315}h^3f_{n+1} + \frac{32}{105}h^3f_{n+2} + \frac{128}{105}h^3f_{n+3} - \frac{8}{63}h^3f_{n+4} + 8h^2y''_n + 4hy'_n + y_n \tag{27}$$

With first derivatives

$$y'_{n+1} = \frac{367}{1440}h^2f_n + \frac{3}{8}h^2f_{n+1} - \frac{47}{240}h^2f_{n+2} + \frac{29}{360}h^2f_{n+3} - \frac{7}{480}h^2f_{n+4} + hy''_n + y'_n \tag{28}$$

$$y'_{n+2} = \frac{53}{90}h^2f_n + \frac{8}{5}h^2f_{n+1} - \frac{1}{3}h^2f_{n+2} + \frac{8}{45}h^2f_{n+3} - \frac{1}{30}h^2f_{n+4} + 2hy''_n + y'_n \tag{29}$$

$$y'_{n+3} = \frac{147}{160}h^2f_n + \frac{117}{40}h^2f_{n+1} + \frac{27}{80}h^2f_{n+2} + \frac{3}{8}h^2f_{n+3} - \frac{9}{160}h^2f_{n+4} + 3hy''_n + y'_n \tag{30}$$

$$y'_{n+4} = \frac{56}{45}h^2f_n + \frac{64}{15}h^2f_{n+1} + \frac{16}{15}h^2f_{n+2} + \frac{64}{45}h^2f_{n+3} + 4hy''_n + y'_n \tag{31}$$

With the second derivatives

$$y''_{n+1} = \frac{251}{720}hf_n + \frac{323}{360}hf_{n+1} - \frac{11}{30}hf_{n+2} + \frac{53}{360}hf_{n+3} - \frac{19}{720}hf_{n+4} + y''_n \tag{32}$$

$$y''_{n+2} = \frac{29}{90}hf_n + \frac{62}{45}hf_{n+1} + \frac{4}{15}hf_{n+2} + \frac{2}{45}hf_{n+3} - \frac{1}{90}hf_{n+4} + y''_n \tag{33}$$

$$y''_{n+3} = \frac{27}{80}hf_n + \frac{51}{40}hf_{n+1} + \frac{9}{10}hf_{n+2} + \frac{21}{40}hf_{n+3} - \frac{3}{80}hf_{n+4} + y''_n \tag{34}$$

$$y''_{n+4} = \frac{14}{45}hf_n + \frac{64}{45}hf_{n+1} + \frac{8}{15}hf_{n+2} + \frac{64}{45}hf_{n+3} + \frac{14}{45}hf_{n+4} + y''_n \tag{35}$$

ANALYSIS OF THE BLOCK METHODS

Order and error Constants of the Block Methods

According to Adeniran & Omotoye (2016), Abdelrahim et al., (2019) and Ukpebor (2019), the order of the new block method (24) – (27) is obtained by using the Taylor series and it is found it has uniformly order five, with an error constants vector

$$C_{p+3} = \left[\frac{139}{40320}, \frac{1}{45}, \frac{243}{4480}, \frac{32}{45} \right]^T \tag{36}$$

Consistency

Definition 3.1: The Four-step block method (24-27) is said to be consistent if it has an order more than or equal to one i.e.

$P \geq 1$. Therefore, the method is consistent (Abdelrahim et al., (2019) and Lambert 1973).

Zero Stability

Definition 3.2: The hybrid block method (24-27) said to be zero stable if the first characteristic polynomial $\pi(r)$ having

roots such that $|r_z| \leq 1$ and if $|r_z| = 1$, then the multiplicity of r_z must not greater than six as discussed in Wend (1969) and Ogunware et al. (2015).

In order to find the zero-stability of Four-step block method (24-27), we only consider the first characteristic polynomial of the method as follows

$$\Pi(r) = r \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = r^3(r-1) \tag{37}$$

which implies $r = 0, 0, 0, 1$. Hence the method is zero-stable since $|r_z| \leq 1$.

Convergence

Theorem (3.1): Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method (22-24) are consistent and zero stable, it implies the method is convergent for all point (as reported in Kayode et al., (2018), Adoghe et al., (2016) and Ukpebor (2019)).

Implementation of the Block Methods

In this section, we implement our derived method (24) – (27) and its first and second derivatives (28) – (31) and (32) – (35) respectively with the aid of MATLAB coding to solve third order nonlinear problems in order to show the level of accuracy and efficiency of the method.

Numerical Examples

The method is specifically developed to examine third order nonlinear problems to test the accuracy of the proposed methods and our results are compared with the results obtained using existing methods.

The following problems are taken as test problems:

Examples

1. $y''' = y'(2xy'' + y')$

$y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0, h = 0.01$

Exact solution: $y(x) = 1 + \frac{1}{2} \ln\left(\frac{2+x}{2-x}\right)$

Source: Adoghe et al., (2016)

2. Application Problem (Fluid flow)

7. $2y''' + yy'' = 0$

$y(0) = 0, y'(0) = 0, y''(0) = 1, h = 0.1$

Source: Adeniran & Omotoye (2016)

Table1: Showing the result of Problem 1

x	y-exact solution	y-computed solution	Error K=4,P=5	Error in Adoghe et al (2016) K=4, P=7
0.1	1.0500417292784914	1.0500417292688820	9.6094e-012	1.93182e-08
0.2	1.1003353477310756	1.1003353470238757	7.0720e-010	5.61699e-07
0.3	1.1511404359364668	1.1511404292437832	6.6927e-009	3.77772e-06
0.4	1.2027325540540821	1.2027325226369256	3.1417e-008	1.34474e-05
0.5	1.2554128118829952	1.2554127068075998	1.0508e-007	3.26773e-05
0.6	1.3095196042031119	1.3095193190398202	2.8516e-007	5.82875e-05
0.7	1.3654437542713964	1.3654430765839469	6.7769e-007	7.16807e-05
0.8	1.4236489301936019	1.4236474629330653	1.4673e-006	2.57657e-05
0.9	1.4847002785940520	1.4846972958267568	2.9828e-006	1.71002e-04
1.0	1.5559952073071430	1.5559890182352183	6.1891e-006	6.71370e-04

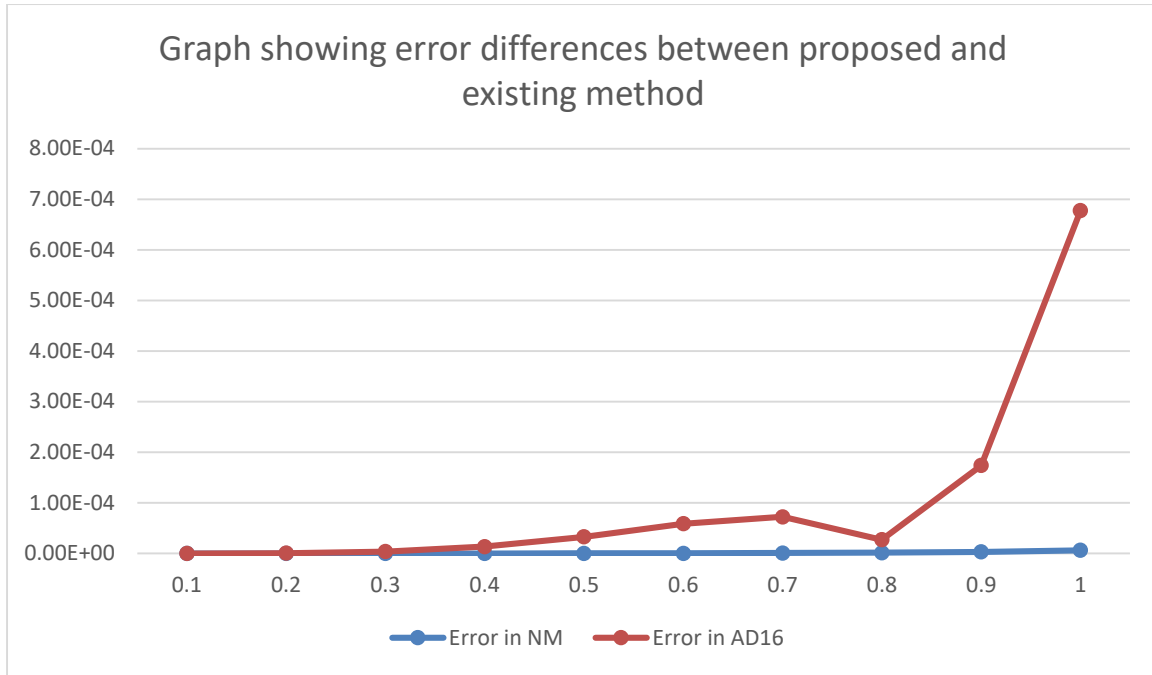


Figure 1: Graph showing the error differences between the proposed method namely ‘NM’ and the existing method namely ‘AD16’.

Table 2: Showing the result of Problem 2

x	y-computed solution	y-computed solution in Adeniran & Omotoye (2016)
0.1	0.004999916668877	0.00499997916667
0.2	0.019997333981535	0.01999866667262
0.3	0.044979767219860	0.004449984812841
0.4	0.079914841165289	0.079991467273443
0.5	0.124740629167025	0.124967453567222
0.6	0.179356502477224	0.179902834981102
0.7	0.243614844746559	0.244755061293070
0.8	0.317314056925792	0.319454487433778
0.9	0.400193307358953	0.4038948458877279
1.0	0.491929477903173	0.4979224398235544

Remark 4.1: it should be noted that problem 2 is an application problem and does not have an exact solution. Hence the comparison of the computed solution of the proposed method with similar work in the literature.

DISCUSSION OF RESULTS

In this section, the tables of results will be extensively discussed. Table 1 shows the exact solution, computed solution and the error in the method for problem 1. The comparison of error in new method with another error in the literature is also made. Specifically, Adoghe et al., (2016) who proposed a linear multistep method of order 5. As it could be seen in Table 1, the four-step block method of order 5 proposed in this work is better in terms of accuracy than that of Adoghe et al (2016). On the other hand, Table 2 shows the computation of an application problem in Fluid Mechanics namely Thin Flow. The problem was solved by Adeniran and Omotoye (2016) using $h=0.1$. The results show that the proposed method is more accurate when compared with other method in the literature. The method is therefore computationally reliable and recommended for general use.

CONCLUSION

In this article, the derivation of the new block method for solving third order nonlinear ordinary differential equations directly is studied. The method is of order $p=5$ which shows that it is consistent. The positive aspect of the method over the existing numerical methods is its ability to solve problem with exact solution and without exact solution and performance in terms of accuracy and convergence in the literature. The comparison of errors in the new method with other existing method is shown in Figure 1. The new method gives minimal error and also solves a notable real life problem namely Thin Flow which has application in fluid mechanics.

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