FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 4 No. 1, March, 2020, pp 577 – 584

A NEW TWO-PARAMETER LIFETIME DISTRIBUTION WITH APPLICATION

***Usman Auwalu Hamisu, Sadiq Ahmed Abubakar and Bala Shehu**

Department of Mathematical Sciences, Bayero University, Kano- Nigeria, 3011 BUK, Kano, Nigeria

*Corresponding Author: ahu3802@gmail.com

ABSTRACT
There are several methods to combine and extend the continuous lifetime models to increase their flexibility and generality. Here we proposed a new lifetime distribution model with two parameters. Various lifetime distribution representations related to this model are derived and presented with their properties. Several Statistical measures and their properties are also studied. The method maximum likelihood estimator is discussed. Simulation studies are performed to assess the finite sample performance of the maximum likelihood estimators (MLEs) of the parameters. In the end, to show the flexibility of this distribution, an application using real data sets is presented.

Keywords: Log-logistic distribution; Hazard function; Maximum likelihood estimation; Simulation.

INTRODUCTION

The log-logistic distribution is considered as one of the most popular statistical distribution used in modeling lifetime and reliability data. It may serve as a superior alternative distribution to the most commonly used statistical distributions such as Weibull distribution, log-normal and gamma distributions. Application of the Log-logistic distribution are demonstrated in numerous applied areas such as in biostatistics and it has also been used in hydrology to model climate change by [3, 5, 7]. And the log-logistic distribution can be used to solve many practical problems especially in survival data, but it may not be effective to handle some other practical problems of interest. This is motivating us to expand the family of the log-logistic distribution by introducing a new parameter to the original distribution. A method of introducing a new parameter(s) to expand the family of distributions is not new [1, 9, 10, 14, 15, 14, 17, 22, 24] for example [16], proposed a family of distribution that can be derived based on the probability density function and hazard function. In this paper, we introduce a new lifetime distribution called two-parameter log-logistic Poisson distribution which is not considered in the literature and study its properties with application to censored life time data. The probability density function of Log-logistic distribution is given by [2] as

$$f(x;\alpha,\beta)=\frac{\beta^{\alpha}\alpha x^{\alpha-1}}{(\beta^{\alpha}+x^{\alpha})^{2}},\ x>0 \quad \textbf{(1)}$$

Where $\alpha,\beta>0$,are shape and scale parameters respectively, by taking $\beta=1$, we have one parameter log-logistics distribution; in this paper we use this one parameter distribution to introduce a new two parameter distribution by mixing it with Poisson distribution.

The Two-parameter Log-Logistics Poisson distribution (LLP)

The probability density function of the random variable X with scale parameter $\lambda>0$, and shape parameter $\alpha>0$ is give by

$$f(x;\alpha,\beta)=\frac{\lambda\alpha x^{\alpha-1}}{(e^{\lambda}-1)(1+x^{\alpha})^{2}}\mathrm{e}^{\lambda/1+\mathrm{x}^{\alpha}}\quad x>0 \qquad \textbf{(2)}.$$

The cumulative distribution and survival functions are given respectively by

$$F(x)=\frac{e^{\lambda}-\mathrm{e}^{\lambda/1+\mathrm{x}^{\alpha}}}{(e^{\lambda}-1)},\qquad x>0 \qquad \textbf{(3)}$$

$$\text{and}\quad S(t)=\frac{\mathrm{e}^{\lambda/1+\mathrm{t}^{\alpha}}-1}{(e^{\lambda}-1)}, \qquad \textbf{(4)}$$

Theorem 2.1 The probability density function of the Two-parameter Log-Logistic Poisson LLP distribution is decreasing for $1 > \alpha > 0$ and unimodel for $\alpha > 1$.

Proof. Let $K(x) = \log \lambda\alpha \left(e^{\lambda} - 1\right) + (\alpha - 1)\log x + {}^{\lambda}/_{1 + x^{\alpha}} - 2\log(1 + x^{\alpha})$. The first derivative of $K(x)$is $K'(x) = \frac{(\alpha-1)}{x} - \frac{\alpha x^{\alpha-1}}{1+x^{\alpha}}\left(2 + \frac{\lambda}{1+x^{\alpha}}\right)$. If $0 < \alpha \leq 1$, it then follows that $K'(x) < 0$, for $x > 0$. This implies that $f(x)$ is a decreasing function. Now suppose that $\alpha > 1$, then $K'(x) = \frac{U(x)}{x(1+x^{\alpha})}$, where $(x) = (1 + \alpha)x^{2\alpha} + (2 + \alpha\lambda)x^{\alpha} - (\alpha - 1)$, this shows that $K'(x) = 0$ if and only if $U(x) = 0$. This equation has a unique positive solution

$$\boldsymbol{x_o} = \left(\frac{-(2+\lambda\alpha)+\sqrt{(2+\alpha\lambda)^2+4(\alpha^2-1)}}{2(1+\alpha)}\right)^{\frac{1}{\alpha}},$$

And $U(x) > 0$ for $x < x_o$ and $U(x) < 0$ for $x > x_o$. So, $f(x)$ is a unimodal at $x = x_{o.}$ []

Hazard rate function

This is one of the important functions in application; the failure rate function of the LLP is given by

$$h(x; \alpha, \lambda) = \frac{\lambda\alpha x^{\alpha-1} e^{\lambda/_{1+x^{\alpha}}}}{(e^{\lambda/_{1+x^{\alpha}}} - 1)(1+x^{\alpha})^2}, x > 0, \qquad \textbf{(5)}$$

Theorem 2.2 The hazard rate function is decreasing function for $\alpha \leq 1$ and is a bathtub for $\alpha > 1$.

Proof.

Set $\eta(x) = -\frac{f'(x)}{f(x)} = -\frac{(\alpha-1)}{x} + \frac{\alpha x^{\alpha-1}}{1+x^{\alpha}}\left(2 + \frac{\lambda}{1+x^{\alpha}}\right)$. The first derivative of $\eta(x)$ is$\eta'(x) = \frac{V(x)}{x^2}$, where

$V(x) = \alpha - 1 + \frac{2\alpha x^{\alpha}(\alpha-x^{\alpha}-1)}{(1+x^{\alpha})^2} + \alpha\lambda\frac{x^{\alpha}(\alpha-x^{\alpha}(1+\alpha)-1)}{(1+x^{\alpha})^3}$. If $0 < \alpha \leq 1$, then the function $V(x)$ and η' are negative. So, it follows by Glaser's theorem [ref] that $h(x)$ is decreasing. Now suppose that $\alpha > 1$. The first derivative of $V(x)$ is

$$V'(x) = -\frac{2\alpha^2 x^{\alpha-1}}{(1+x^{\alpha})^2} + \frac{4\alpha^2 x^{2\alpha-1}(\alpha-x^{\alpha}-1)}{(1+x^{\alpha})^3} - \frac{\lambda\alpha^2 x^{2\alpha-1}}{(1+x^{\alpha})^3} - \frac{3x^{2\alpha-1}\alpha\lambda(\alpha-x^{\alpha}(1+\alpha)-x^{\alpha}-1)}{(1+x^{\alpha})^4}.$$

Note that $V'(x)$ is decreasing with a root at

$$x_o = \left(\frac{4(\alpha - 1) + \lambda(2 - 3\alpha) + \sqrt{\alpha^2(16 + 33\lambda) + 4\lambda(11\alpha + 1)}}{4}\right)^{\frac{1}{\alpha}}$$

So, $V(x)$ is upside-down function with $\lim_{x\to 0} V(x) = \alpha - 1 > 0$ and $\lim_{x\to\infty} V(x) = -(\alpha - 1) < 0$. For $\alpha > 1$ then η' has a root at $x = k_o$, where $V(k_o) = 0$, with $\eta' > 0$ for $x < k_o$ and $\eta' < 0$ for $x > k_o$. Since $\lim_{x\to 0} f(x) = 0$, it follows by Glaser's theorem [11] that $h(x)$is upside − down bathtub shape. []

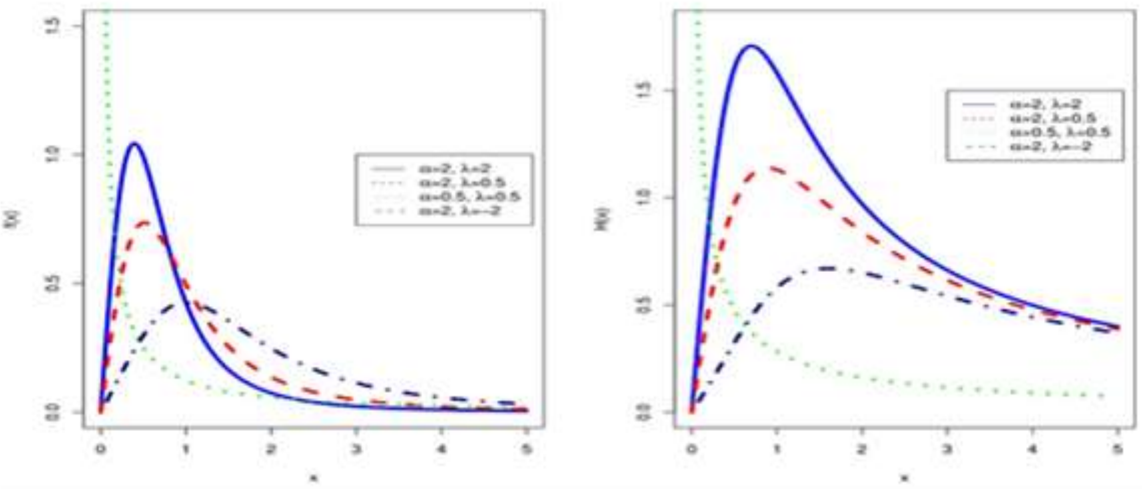

Fig.1 Plots of the probability density and hazard function respectively

Figure 1 shows the shapes of the density function and hazard function for different values of parameters. It shows that the pdf can be decreasing as indicated by the green dotted curve for α=0.5<1 or unimodal as indicated in the figure by (red-long dashed, blue solid and dark blue dashed dotted curve) for α=2>1. The hazard function can be upside-down bathtube shaped as displayed in the figure by red-long dashed curve, blue solid and dark blue dashed dotted curve for α=2>1 and decreasing as shown by the green dotted curve for α=0.5<1.

Properties

Moments

One most important and hardly emphasized property in statistical analysis is moments. Through moments we can study important features and characteristics of a distribution (e.g. mean, variance skewness, and kurtosis etc.). The k-th moment of the proposed distribution is given by

$$E[X^k] = (e^{\lambda}-1)^{-1}\lambda\sum_{r=0}^{\infty}\frac{\lambda^r\Gamma\left(1+\frac{k}{\alpha}\right)\Gamma\left(1+r-\frac{k}{\alpha}\right)}{r!\,\Gamma(r+2)}, \qquad (6)$$

It can also express by using hyper geometric function

$$\boldsymbol{E\left[X^k\right] = (e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{k}{\alpha}\right)\Gamma\left(1-\tfrac{k}{\alpha}\right)F_{1,1}\left(1-\tfrac{k}{\alpha},2,\lambda\right)}, \qquad (7)$$

Where $\boldsymbol{F_{1,1}\left(1-\frac{k}{\alpha},2,\lambda\right) = \sum_{r=0}^{\infty}\frac{\left(1-\frac{k}{\alpha}\right)_r\lambda^r}{(2)_r r!}}$

$$\boldsymbol{mean = (e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{1}{\alpha}\right)\Gamma\left(1-\tfrac{1}{\alpha}\right)F_{1,1}\left(1-\tfrac{1}{\alpha},2,\lambda\right)}, \qquad (8)$$

$$\boldsymbol{variance = (e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{2}{\alpha}\right)\Gamma\left(1-\tfrac{2}{\alpha}\right)F_{1,1}\left(1-\tfrac{2}{\alpha},2,\lambda\right) - \left((e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{1}{\alpha}\right)\Gamma\left(1-\tfrac{1}{\alpha}\right)F_{1,1}\left(1-\tfrac{1}{\alpha},2,\lambda\right)\right)^2} \quad (9)$$

$$\boldsymbol{skewness = \sigma^{-3}\left((e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{3}{\alpha}\right)\Gamma\left(1-\tfrac{3}{\alpha}\right)F_{1,1}\left(1-\tfrac{3}{\alpha},2,\lambda\right)-3\mu\sigma-\mu^3\right)} \quad (10)$$

$$\boldsymbol{kurtosis = \sigma^{-4}\left((e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{4}{\alpha}\right)\Gamma\left(1-\tfrac{4}{\alpha}\right)F_{1,1}\left(1-\tfrac{4}{\alpha},2,\lambda\right)\right.}$$
$$\boldsymbol{\left.-4\sigma^{-3}\left((e^{\lambda}-1)^{-1}\lambda\,\Gamma\left(1+\tfrac{3}{\alpha}\right)\Gamma\left(1-\tfrac{3}{\alpha}\right)F_{1,1}\left(1-\tfrac{3}{\alpha},2,\lambda\right)-3\mu\sigma-\mu^3\right)-6\mu^2\sigma-3\mu^4\right)} \quad (11)$$

Statistical inference

Maximum likelihood estimators

Let $X_1, X_2, X_3, \dots, X_n$ be a random sample coming from LLP (λ, α). The log-likelihood function (of right censored data) is given by [3];

$$l(\lambda,\alpha) = \prod_{i=1}^{n} f^{\delta_i}(x_i,\theta)[1-F(x_i,\theta)]^{1-\delta_i}$$

$$= \sum_{i=1}^{n} \delta_i[\ln(\alpha)+\ln(\lambda)] + (\alpha-1)\sum\nolimits_{i=1}^{n} \delta_i \ln(x_i) - 2\sum_{i=1}^{n} \delta_i \ln(1+x_i{}^{\alpha})$$
$$+\lambda \sum\nolimits_{i=1}^{n} \frac{\delta_i}{(1+x_i{}^{\alpha})} + \sum\nolimits_{i=1}^{n} (1-\delta_i)\ln(e^{\frac{\lambda}{(1+x^{\alpha})}}-1) - n\ln(e^{\lambda}-1).$$

Where $\delta_i = 0$, for complete observation and $\delta_i = 1$, for censored observation.

The partial derivatives of the log-likelihood function with respect to parameters α and λ are

$$\frac{\partial l}{\partial \lambda} = \frac{\sum_{i=1}^{n}\delta_i}{\lambda} + \sum\nolimits_{i=1}^{n} \frac{\delta_i}{1+x_i{}^{\alpha}} + \sum\nolimits_{i=1}^{n} \frac{(1-\delta_i)e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)} - n\frac{e^{\lambda}}{e^{\lambda}-1} \quad \textbf{(12)}$$

$$\frac{\partial l}{\partial \alpha} = \frac{\sum_{i=1}^{n}\delta_i}{\alpha} + \sum_{i=1}^{n} \delta_i \ln(x_i) - \sum\nolimits_{i=1}^{n} \frac{\delta_i x_i{}^{\alpha}\ln(x_i)(2+{}^{\lambda}/(1+x_i{}^{\alpha}))}{1+x_i{}^{\alpha}}$$

$$-\lambda \sum\nolimits_{i=1}^{n} (1-\delta_i)\frac{x_i{}^{\alpha}\ln(x_i{}^{\alpha})e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)} \quad \textbf{(13)}$$

To find the estimators, we set equation (12) and (13) to zero and solve simultaneously, but the equations are nonlinear, so it is very difficult to find the analytical solution by the way we use simulation technique to prove the existence of MLEs.

Simulation

The result of the simulation study is shown in table 1.

Table1. The averages of 10000 MLEs and simulated standard errors for LLP

			AE		SD					AE		SD	
n	CR%	(α,λ)	$\hat{\alpha}$	$\hat{\lambda}$	sd($\hat{\alpha}$)	sd($\hat{\lambda}$)	n	CR%	(α,λ)	$\hat{\alpha}$	$\hat{\lambda}$	sd($\hat{\alpha}$)	sd($\hat{\lambda}$)
50	10%	(0.5,-0.5)	0.5047	-0.5241	0.065	0.501	200	10%	(0.5,-0.5)	0.5111	-0.5021	0.0311	0.2509
		(0.5,0.5)	0.5047	0.4979	0.064	0.504			(0.5,0.5)	0.5008	0.4961	0.0307	0.2493
		(0.5,1)	0.5052	0.9993	0.063	0.519			(0.5,1)	0.501	0.999	0.0307	0.254
		(0.5,-1)	0.5049	-1.0163	0.063	0.521			(0.5,-1)	0.5004	-1.006	0.0306	0.255
	15%	(1,-0.5)	1.0121	-0.5103	0.133	0.511		15%	(1,-0.5)	1.0024	-0.4996	0.0647	0.2493
		(1,-1)	1.0096	-1.0208	0.131	0.511			(1,-1)	1.0013	-1.007	0.0632	0.2564
		(1,-1.5)	1.0109	-1.5346	0.13	0.544			(1,-1.5)	1.0024	-1.5074	0.062	0.2618
		(1,-2)	1.0106	-2.0464	0.125	0.573			(1,-2)	1.0021	-2.0155	0.0605	0.2801
	30%	(2,0.5)	2.0342	0.5046	0.297	0.568		30%	(2,0.5)	2.0086	0.4997	0.1401	0.2744
		(2,1)	2.037	1.0177	0.289	0.61			(2,1)	2.0092	1.0037	0.1394	0.2878
		(3,-4)	3.0486	-4.1432	0.383	0.82			(3,-4)	3.0098	-4.0365	0.1843	0.3808
		(4,-3)	4.0593	-3.0858	0.533	0.677			(4,-3)	4.0151	-3.0178	0.2565	0.3188
	40%	(5,-10)	5.1182	-10.908	0.654	3.025		40%	(5,-10)	5.0257	-10.1779	0.2965	1.152
		(6,-6)	6.1044	-6.3062	0.763	1.292			(6,-6)	6.0284	-6.0627	0.3641	0.5588
		(8,-6)	8.1467	-6.285	1.033	1.262			(8,-6)	8.0322	-6.0631	0.4893	0.5656

		(10,-6)	10.181	-6.2986	1.318	1.323			(10,-6)	10.0459	-6.0657	0.6143	0.559
100	10%	(0.5,-0.5)	0.5018	-0.5066	0.045	0.349	500	10%	(0.5,-0.5)	0.5004	-0.4997	0.0192	0.1553
		(0.5,0.5)	0.5251	0.4982	0.045	0.35			(0.5,0.5)	0.5006	0.4967	0.0196	0.1575
		(0.5,1)	0.5262	1.0071	0.044	0.364			(0.5,1)	0.5004	1.0009	0.0192	1.1619
		(0.5,-1)	0.5023	-1.009	0.044	0.362			(0.5,-1)	0.5003	-1.0019	0.159	0.159
	15%	(1,-0.5)	1.0041	-0.51	0.092	0.355		15%	(1,-0.5)	1.0009	-0.5006	0.0407	0.1569
		(1,-1)	1.0044	-1.0084	0.09	0.363			(1,-1)	1.0005	-1.0017	0.0402	0.1626
		(1,-1.5)	1.0042	-1.5133	0.089	0.376			(1,-1.5)	1.0004	-1.5034	0.0389	0.1666
		(1,-2)	1.0051	-2.0228	0.086	0.401			(1,-2)	1.0005	-2.0047	0.0377	0.1765
	30%	(2,0.5)	2.0175	0.5022	0.201	0.396		30%	(2,0.5)	2.0026	0.504	0.0872	0.1722
		(2,1)	2.0173	1.0108	0.199	0.416			(2,1)	2.0037	0.9989	0.0863	0.1795
		(3,-4)	3.0206	-4.068	0.259	0.55			(3,-4)	3.0022	-4.0082	0.1138	0.2365
		(4,-3)	4.0306	-3.0449	0.373	0.463			(4,-3)	4.0038	-3.0059	0.1627	0.2023
	40%	(5,-10)	5.0608	-10.387	0.427	1.725		40%	(5,-10)	5.011	-10.061	0.1852	0.6891
		(6,-6)	6.0537	-6.1394	0.519	0.82			(6,-6)	6.0092	-6.0259	0.2274	0.3416
		(8,-6)	8.0747	-6.1471	0.708	0.841			(8,-6)	8.0161	-6.0276	0.3062	0.3492
		(10,-6)	10.088	-6.1203	0.889	0.826			(10,-6)	10.0146	-60255	0.3773	0.3483

Table1.gives the approximate values of and the results for LLP distribution shown in the tables revealed that: In all cases the convergence is achieved and this highlights the numerical stability of MLE method, the differences between the average estimates and exact values are almost insignificant and these imply that, the MLE estimates presented consistently. However, the standard error of the MLEs decreases when the sample size increases.

Application

The application of the introduced distribution presented by considering two right censored data sets for illustrative purposes. The chemotherapy Plus Radiotherapy data and meloma data were taken from [13] and [4] respectively.

Table 2: MLEs of the survival chemotherapy Plus Radiotherapy

Models	λ	$\alpha 1$	$\alpha 2$	Log-likelihood	AIC
MXWEP	3.2372	0.822	0.00045	-292.4663	590.933
MXEP	3.2432	0.00056	-	-293.8144	591.629
GWD	0.5161	0.2176	0.0299	-664.853	715.446
GE	0.2154	0.00034	-	-450.099	904.199
Gamma	0.6353	0.00074	-	-296.666	597.332
Weibull	0.0108	0.6931	-	-294.659	593.317
Log-logistic	0.0012	1.167	-	-293.814	582.052
Log-Normal	5.837	1.4779	-	-289.075	582.152
Fullmodel	-55.164	0.7708	-	-286.914	577.828
Submodel	-20.822	0.808	0.2975	-286.878	579.55

Above table displayed the log-likelihood function and the Akaike Information Criterion (AIC) of our model and several known lifetime distributions. According the maximum log-likelihood function and AIC, our model and its submodel are the best and provide good fit for this data compared with remaining models as shown in the table. The generalized weibull GW distribution has a much poorer fit then the weibull or log-logistic distributions and its shows no evidence of an improved fit for this data.

Table 3: MLEs of the survival time in month for the meloma data.

Models	λ	α1	α2	Log-likelihood	AIC
MXWEP	2.18571698	1.0124993	0.02572313-	-269.1265	544.253
MXEP	2.23653557	0.0247271	-	-269.133	542.266
GWD	0.1814372	0.5541368	1.4733338	-344.5459	695.0918
WE	0.2154	2.413882	-	-269.329	542.658
GE	0.70620765	0.7062077	-	-520.9711	1045.942
Gamma	0.98253988	0.0561181	-	-269.3834	542.7668
Weibull	0.97463288	0.0608242	-	-269.3634	542.7276
Log-logistic	0.05988028	1.1011301	-	-268.7677	541.5354
Log-Normal	2.700686	1.746734	-	-270.0727	544.1454
Fullmodel	-5.7883	0.7588	-	-268.4321	540.8642
Submodel	-5.04173	0.8043	0.7773	-268.388	541.289

Above table displayed the log-likelihood function and the AIC of our model and several known lifetime distributions. According the maximum log-likelihood function and AIC, our model and its submodel are the best and provide good fit for this data compared with remaining models as shown in the table. The generalized weibull GW distribution has a much poorer fit then the weibull or log-logistic distributions and its shows no evidence of an improved fit for this data.

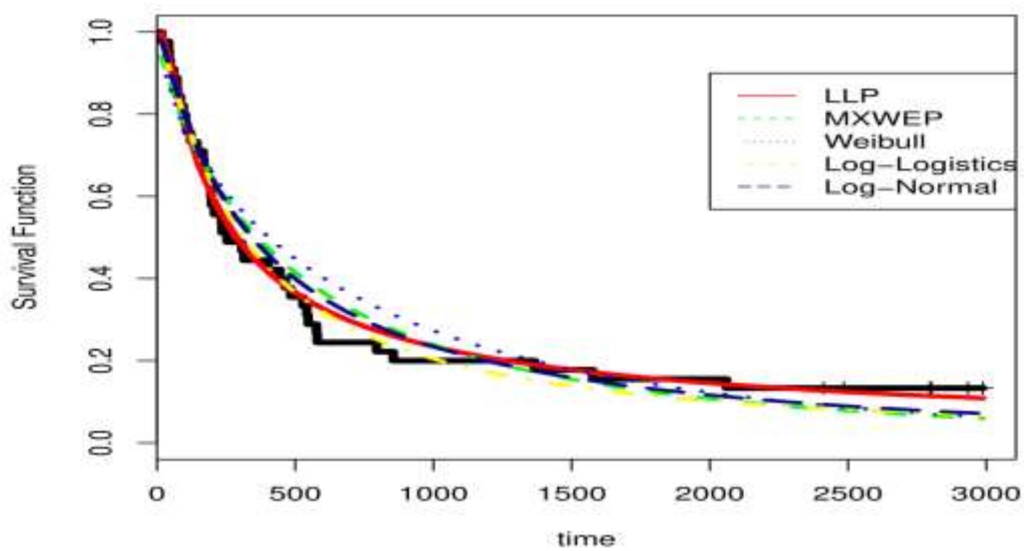

Fig. 3: Kaplan-Meiers and the plotted survival functions for the chemotherapy Plus Radiotherapy data

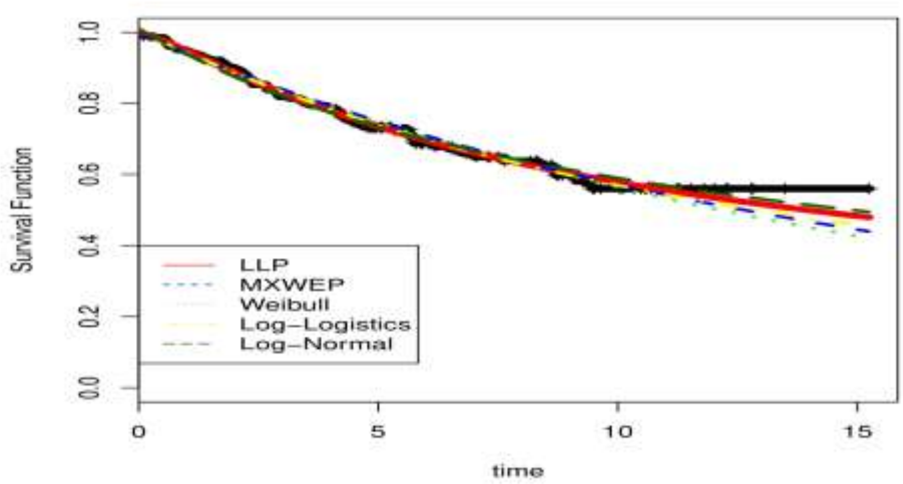

Fig. 4: Kaplan-Meiers and the plotted survival functions for the meloma data.

CONCLUSION

A new lifetime distribution is proposed which is called two parameters Log-logistic and Poisson distribution (LLP). Some statistical properties are derived and discussed for the LLP. The maximum likelihood estimation method is used to estimate the unknown parameters, simulation studies are also conducted to assess the finite sample behavior of the maximum likelihood estimators (MLEs). The LLP distribution is successfully tested using two real data sets in comparison to some other existing statistical distribution.

REFERENCES

Adamids, K. and S. Loukas (1998). A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters 79*, 35–42.

Ahmad, I. M, Sinclair, C. D., and A. Werritty (1988). Log-logistic odd frequency analysis. *Journal of Hydrology 98*, 205–255.

Ahmed, A. S. (2005). Estimation of parameters of life from progressively censored data using burr-xii model. *IEEE TRANSACTION ON RELIBILITY 54*, 159–167.

Anderson, P. K. and N. Keiding (1993). *Statistical Models Based on Counting Process*. New York: Springer-Verlag.

Ashkar, F. and S. Mahdi (2006). Fitting the log-logistic distribution by generalized moments. *Journal of Hydrology 328*, 694–703.

Brazauskas., V. (2002). Fisher information matrix for the feller-pareto distribution. *Statistics and Probability Letters 59*, 159–167.

Collett, D. (2003). *Modeling survival data in medical research*. London: Chapman and Hall.

Mahmoudi, E. A. A. J. (2011). Generalized exponential-power series distribution. *Journal of Computational Statistics and Data Analysis*.

Eugene, N., C.Lee, and F.Famoye (2002). Beta-normal distributions and its application. *Communications in Statistics - Theory and Methods 31*, 23–35.

Louzada, F. M. Roman, V. C. (2011). The complementary exponential geometric distribution: Model, properties and comparison with its counterpart. *Computational Statistics and Data Analysis 55*, 2516–2524.27.

Glaser, R. E. (1980). Bathtub and related failure rate characterization. *Journal of American Statistical Association 75*, 667–672.

Hosmer, D. and S. Lemeshow (1999). *Applied Survival Analysis*. New York: Wiley Inter Science.

Klein J. P and M. L. Moeschberger (2003). *Survival Analysis*. New York: Springer.

Kus, C. (2007). A new lifetime distribution. *Computational Statistics and Data Analysis 51*, 4497–4509.

Mahmoudi, M. T. (2011). Generalized inverse weibull-poisson distribution and its application. *Journal of Statistical Computation and Simulation*.

Marshall, A. W. and I. Olkin (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika 97*,641–652.

Morais, A.L., B.-S. W. (2011). A compound class of weibull and power series distributions. *Computational Statistics Data Analysis 55*, 1410–1425.

Rosaiah .K, Kantami, R. R. L. and C. Kumar (2006). Reliability test plans for exponentiated log-logistic distribution. *Economics Quality Control 21*, 279–289.

Singh, K. P. Bartolucci., A. A. and B. S.L. (1994). Two-step procedure for survival data. *Biometric-praximetrie 34*, 1–12.

Singh, K. P. (1989). A generalized log-logistic regression model for survival analysis: hazard function rate characteristics. *Biometric-praximetrie 29*, 63–74.

Tadikamalla, P. R. (1980). A look at the burr and related distribution. *International Statistic Review 48*, 337–344.

Tahmasbi, R. and S. Rezaei (2008). A two-parameter lifetime distribution with decreasingfailure rate. *Computational Statistics and Data Analysis 52*, 3889–3901.

Cancho, V. G. Louzada-Neto, F. G. B. (2011). The poisson exponential lifetime distribution. *Computational Statistics and Data Analysis 55*, 677–686.28.

Barreto, W. et al. (2009). A generalization of exponential-poisson distribution. *Statistics and Probability Letter 79*, 2493–2500.

Wenhao Gui, (2013). Extended log-logistic distribution and its application in minification processes. *Applied Mathematical Science 7*, 3947–3961.29.