*FUDMA Journal of Sciences (FJS)*
*ISSN online: 2616-1370*
*ISSN print: 2645 - 2944*
*Vol. 5 No. 4, December 2021, pp 44 - 48*
*DOI: https://doi.org/10.33003/fjs-2021-0504-779*

# ON THE STUDY OF CHAINS SOFT SETS, SOFT ORDERED SETS AND SOFT SEMILATTICES.

**Yusuf, A. O., Nworah, C. and Isyaku, M.**

Department of Mathematical Sciences, Federal University Dutsin-Ma, Katsina, Nigeria

Authors Email: oyusuf@fudutsinma.edu.ng, nchinenye@fudutsinma.edu.ng

**ABSTRACT**
In this paper, we study the concept of chains soft sets and set-valued function of chains soft sets. The definitions of chains soft sets or linear order or total order soft set are given. The notions of binary relation of comparability of the elements of set-valued functions are also discussed. Linearization's of a partial order soft set are also defined. The definition and some algebraic structure of soft semilattice are also given.

**Keywords**: set-valued, notions, partial order, Linearization

**Chains Soft Sets**
The set value functions $F(x)$ and $G(y)$ are comparable in a preorder when either $F(x) \leq G(y)$ or $F(y) \leq F(x)$, and otherwise are incomparable. The binary relation of comparability may be seen to be reflexive and symmetric but not in general transitive.

**Definition 2.1**:
A chain soft sets or linear order or total order soft set symbolically denoted as $F(x) \leq G(y)$ or $F(y) \leq F(x)$, is a partial order soft set in which all pairs of elements that is the set-valued functions are comparable.

**Definition 2.2:**
A pre-order soft set $(F, A)$ of a pre-order $(Y, \leq')$ augments a pre-order soft set $(G, B)$ of pre-order $(X, \leq)$ when $Y = X$ and $F(x) \leq G(y)$ implies $F(x) \leq' G(y)$. Hence a chain soft set $(F, A)$ can be described as a partial order soft set with no proper augment that is a partial order soft set. (But a chain soft set can always be augmented to a clique.)

**Definition 2.3:**
A linearization of a partial order soft set $(F, A)$ is a chain soft set augmenting $(F, A)$ i.e., a maximal antisymmentric augment of $(F, A)$ .

**Theorem 2.1:**
Every partial order soft set $(F, A)$ over partial order $(X, \leq)$ in which $F(x)$ and $F(y)$ are incomparable has an augment in which they are comparable.

**Proof**:
Form $\leq'$ by adding to $\leq$ all pairs $(F(x'), (y'))$ for which $F(x') \leq F(x)$ and $F(y) \leq F(y')$. The result contains $(F(x), F(y))$ since $F(x) \leq F(x)$ and $F(y) \leq F(y)$. It remains reflexive since nothing is removed. It is transitive because for any triple $F(x') \leq F(y') \leq F(z)$ where $F(x') \leq F(y')$ is one of the added pairs, we have $F(y) \leq F(y') \leq F(z)$, whence $(F(x'), F(z))$ will also have been added, and similarly for $F(z) \leq F(x') \leq' F(y')$. It is antisymmentric because if $F(x') \leq' F(y') \leq' F(x')$ then $F(y) \leq F(y') \leq' F(x') \leq F(x)$ contradicting incomparability of $F(x)$ and $F(y)$.
Hence it is a partial order soft set extending $(X, \leq)$ and containing $(F(x), F(y))$.

**Lemma 2.1**:
Every soft lattice-theoretical term denoted by $\varphi$ is equivalent to a normal term, with the equivalence holding in every distributive lattice and hence being an equation of the theory of distributing lattices.

**Proof:**

Given any soft lattice $(F, A)$ where $F(x)$, $F(y)$ and $F(z)$ are set -value function of the soft lattice, then we begin by "pushing down the $'\wedge'$s." Choose any subterm of $\varphi$ of the form $F(x) \wedge (F(y) \vee F(z))$ such that $F(x)$ does not contain subterms of that form (otherwise we would choose the latter subterm ), and rewrite it as $\big(F(x) \wedge F(y)\big) \vee \big(F(x) \wedge F(z)\big)$.

This transformation is justified by the distributivity law in the sense that, for any distributive soft lattice $(F, A)$ over a lattice L and any assignment of element of L to variables of $\varphi$, the transformation leaves unchanged the value of every subterm.

Now define an inversion of $\varphi$ to be a pair consisting of an occurrence of a $\wedge$ above an occurrence of $\vee$ in the parse tree of $\varphi$, not necessarily immediately above. A rewriting step of the above kind eliminates the inversion at the top of the rewritten subterm, and creates no new inversions. (Although $x$ is duplicated, it contains no inversion, and although the $\wedge$ is duplicated no inversion is associated with that $\wedge$ is duplicated.) Hence this transformation can be repeated only as many times as there were inversion in $\varphi$ at the outset. The result of so transforming $\varphi$ exhaustively is a term of the form $\psi(\mu_1, \ldots, \mu_k)$ where the operations of ψ are all ⋁ and those of the $\mu_i's$ are all ⋀.

Duplicate variables within each meet may now be eliminated by associativity and commutativity laws to bring the duplicates together, and then eliminating them using idempotence of ⋀. The same technique permits duplicate meets to be eliminated. We now have a subset of $2^{\vee}$ which we must make as order filter. If there exists a meet $\mu_i$ and a variable $p$ of $\varphi$ such that $\mu_i \wedge p \neq \mu_j$ for any $i \leq j$, then expand the set of $\mu's$ by taking $\mu_{k+1} = \mu_i \wedge p$.

This step is justified by $F(x) = F(x) \vee (F(x) \wedge F(y))$ with $F(x) = \mu_i$ and $F(y) = p$, and we say that $\mu_i$ *subsumes* $\mu_i \wedge p$. Iterate until the set is closed under conjugation with variables of $\varphi$. The result is now a normal term equivalent in every distributive soft lattice to the one we started with.

**Soft ordered sets**

A partial ordered set, or more briefly just ordered set, is a system $\wp = (P, \leq)$ where $P$ is a nonempty set and $\leq$ is a binary relation on $P$ satisfying, for all $x, y, z \in P$,

(i) $x \leq x$ (reflexivity)

(iii) $if x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity)

Throughout this section, $\wp$ is a partial ordered set and A is any nonempty set. R will refer to an arbitrary binary relation between elements of A and elements of $\wp$. i.e., $R \subseteq A \times \wp$. A set- value function $F: A \longrightarrow \phi(\wp)$ can be defined as $F(x) = \{y \in \wp / xRy\}$. the pair $(F, A)$ is a soft set over $\wp$.

**Definition 3.1:**

Let $(F, A)$ be a soft set over $\wp$. Then $(F, A)$ is said to be a soft partial order set if $F(x)$ is a partial ordered subset of $\wp$ for all $x \in A$.

The sets of all soft partial ordered set is given as $\mathfrak{H}p(\wp)$.

More generally, if $(F, A)$ is a soft ordered set and $A \subseteq B$, then the restriction of $\subseteq$ to $A$ is a soft partial order set, leading to a new soft ordered set $(G, B)$.

The set $\mathbb{R}$ of real numbers as a parameters, with its natural order is an example of a rather special type of soft partial ordered set, namely a totally soft ordered set, or soft chain.

$SC$ is a soft chain if for every set-valued function $F(x), F(y) \in SC$, either $F(x) \subseteq F(y)$ or $F(y) \subseteq F(x)$ for $x \leq y$ or $y \leq x$. At the opposite extreme we have soft antichains, soft ordered sets in which $\subseteq$ concides with the equality relation =.

We say that a set-valued function $F(x)$ covered by another set-valued function $F(y)$ in $\wp$, written $F(x) \prec F(y)$, if $F(x) \subseteq F(y)$ for $x \leq y$ and there is no $z \in A$ with $F(x) \subset F(z) \subset F(y)$. It is clear that the covering relation determines the soft partial order in a finite ordered set $\wp$. In fact, the order $\subseteq$ is the smallest reflexive, transitive relation containing $\prec$ . We can use this to define a Hasse diagram for a finite ordered set $\wp$; the elements of $A$ are represented by points in the plane, and a line is drawn from $x$ up to $y$ prescription is not precise, but it is close enough for government purposes.

The natural maps associated with the category of soft ordered sets are the soft order preserving maps, those satisfying the condition $F(x) \subseteq F(y)$ implies$f(F(x)) \subseteq f(F(y))$. We say that $(F, A)$ is isomorphic to $(G, B)$, written $(F, A) \cong (G, B)$, if there is a map $f: A \longrightarrow B$ which is one-to-one, onto, and $f$ and $f^{-1}$ are soft order preserving, i.e., $F(x) \subseteq F(y)$ iff $f(F(x)) \subseteq f(F(y))$.

**Theorem 3.1:**

Let $(F, A)$ be a soft ordered set and let $\phi: A \longrightarrow \beta(\wp)$ be defined by $\phi(x) = \{y <\in \wp: y \leq x\}$. Then $\wp$ is isomorphic to the range of $\phi$ order by $\subseteq$.

**Proof:**

If $x \leq y$, then $y \leq z$ implies $x \leq z$ by transitivity law, and hence $\phi(x) \subseteq \phi(y)$. Since $x \in \phi(x)$ by reflexivity law, $\phi(x) \subseteq \phi(y)$ implies $x \leq y$. Thus $x \leq y$ iff $\phi(x) \subseteq \phi(y)$. That $\phi$ is one-to-one then follows by antisymmetry.

**Definition 3.1:**

The soft ordered set $(F, A)$ has a maximal ( or greatest) element if there exist $x \in A$ such that $F(x) \subseteq F(y)$ for all $y \in A$. An element $x \in A$ is maximal if there is no element $y \in A$ with $F(y) \supseteq F(x)$. Clearly there concept are different. Minimum and minimal elements are defined dually.

**Lemma 3.1:**

Given any soft ordered set $(F, A)$, the following are equivalent.

(1) Every nonempty subset $B \subseteq A$ contain an element minimal in $B$.

(2) $(F, A)$ contains no infinite descending chain,$a_0 > a_1 > a_2 > \cdots > a_n$

(3) If $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n$ in $(F, A)$, then there exists $k$ such that $a_n = a_k$ for all $n \geq k$.

**Proof:**

The equivalence of (2) and (3) is clear, and likewise that (1) implies (2). There is, however, a subtlety in the (2) implies (1). Suppose $(F, A)$ fails (1) and that $B \subseteq A$ has no minimal element. In order to find an infinite descending chain in $B$, rather than just arbitrarity long finite chains, we must use the Axiom of choice. One way to do this is as follows:

Let $f$ be a choice function on the subsets of $B$, i.e., $f$ assigns to each nonempty subset $C \subseteq B$ an element $f(c) \in C$.

Let $a_0 = f(B)$, and for each $i \in w$ define $a_{i+1} = f(\{b \in B: b < a_i\})$; argument of $f$ in this expression is nonempty because $B$ has no minimal elemet. The sequence so defined is an infinite descending chain, and hence $(F, A)$ fails (2).

The conditions described by the preceding lemma are called the descending chain condition (DCC). The dual notion is called the ascending chain condition (ACC).

**Lemma 3.2**:

Let $(F, A)$ be a soft ordered set satisfying the descending chain condition (DCC). If $\emptyset(x)$ is a statement such that;

(1) $\emptyset(x)$ holds for all minimal elements of $A$, and

(2) whenever $\emptyset(y)$ holds for all $y < x$, then $\emptyset(x)$ is true for every element of $A$.

**Proof:** Note that (1) is in fact a special case of (2). It is included in the statement of the lemma in practice minimal elements usually require a separate argument. The proof is immediate. The contrapositive of (2) states that the set $F = \{x \in A : \emptyset(x) \text{ is false}\}$ has no minimal element. Since $(F, A)$ satisfies the DCC, $F$ must therefore be empty.

We now turn our attention more specifically to the structure of soft ordered sets.

**Definition 3.2:** The width of a soft ordered set $(F, A)$ denoted by $F_A$ is defined by $\omega(F_A) = Sup\{|A|: A \text{ is a soft antichain in } F_A.\}$ where $|A|$ denotes the cardinality of $A$.

A second invariant is the soft chain covering number $c(F_A)$, defined to be the least cardinal $\gamma$ such that $A$ is the union of $\gamma$ chains in $F_A$. Because no soft chain can contain more than one element of a given soft antichain, we must have $|A| \leq |I|$ whenever $A$ is an antichain in $F_A$ and $A = \cup_{i \in I} C_i$ is a chain covering. Therefore $\omega(F_A) \subseteq c(F_A)$ for any soft ordered set $F_A$. The following result, due to R. P. Dilworth, says in particular that if $F_A$ is finite, then $\omega(F_A) = c(F_A)$.

**Theorem 3.2**:

If $\omega(F_A)$ is finite, then $\omega(F_A) = c(F_A)$.

**Proof**: In the finite case. We need to show $\omega(F_A) \subseteq c(F_A)$, which is done by induction on $|P|$. Let $\omega(F_A) = k$, and let $G_B$ be a maximal soft chain in $F_A$. If $F_A$ is a soft chain, $\omega(F_A) = c(F_A) = I$, so assume $G_B \neq F_B$. Because $G_B$ can contain at most one element of any maximal antichain, the width $\omega(F_A - G_B)$ is either $k$ or $k-1$, and both possibilities can occur. If $\omega(F_A - G_B) = k$, and let $A = \{a_1, \dots, a_k\}$ be a maximal antichain in $F_A - G_B$. As $|P| = k$, it is also a maximal antichain in $F_A$. Set

$$L = \{x \in P : x \leq a_i \text{ for some } i\}$$

$$U = \{x \in P : x \leq a_j \text{ for some } j\}.$$

Since every element of $P$ is comparable with some element of $A$, we have $P = L \cup U$, while $A = L \cap U$. Moreover, the maximality of $G_B$ insures that the largest element of $G_B$ does not belong to $L$ ( remember $A \subseteq A - G_B$), so $|L| < |A|$. dually, $|U| < |A|$also. Hence $L$ is a union of $k$ chains, as a union of chains. By renumbering, if necessary, we may assume that $a_i \in D_i \cap E_i$ for $1 \leq i \leq k$, so that $G_{B_i} = D_i \cup E_i$ is a soft chain. Thus $A = L \cap U = G_{B_i} \cup \dots \cup G_{B_k}$ is a union of $k$ chain.

**Theorem 3.3:**

Let $F_A$ be a soft ordered set. Then

(1) $d(F_A)$ is the smallest cardinal $\gamma$ such that $F_A$ can be embedded into the direct product of $\gamma$ chain.
(2)$d(F_A) \subseteq c(F_A)$.

**Proof:**

First suppose $\subseteq$ is the intersection of total soft order $\subseteq_i$ $(i \in I)$ on $A$. If we let $G_{B_i}$ be the soft chain $(A, \subseteq_i)$, then it easy to see that the natural map $\phi: A \longrightarrow \Pi_{i \in I} G_{B_i}$, with $(\phi(x))i = x$ for all $x \in A$, satisfies $x \leq y$ iff $\phi(x) \subseteq \phi(y)$. Hence $\phi$ is an embedded.

Conversely, assume $\phi: A \longrightarrow \Pi_{i \in I} G_{B_i}$ is an embedded of $A$ into a direct product of chains. We want to show that this leads to a representation of $\subseteq$ as the intersection of $|I|$ total soft orders. Define

$$xR_i y \text{ if } \begin{cases} x \leq y \\ or \\ \phi(x) \subseteq \phi(y) \end{cases}$$

We notice that $R_i$ is a partial order extending $\subseteq$. To see that $\subseteq$ is the intersection of the $\subseteq_i$ $s$, suppose $x \nsubseteq y$. Since $\phi$ is an embedding, then $\phi(x)_i \nsubseteq \phi(y)_i$ for some $i$. Thus $\phi(x)_i \supset \phi(y)_i$ implying $yR_i x$ and $y \subseteq_i x$ or equivalently $x \subseteq_i y$ $(as\ x \neq y)$, as desired.

Thus the order on $F_A$ is the intersection of $k$ total soft orders if and only if $F_A$ can be embedded into the direct product of $k$ chains, yielding (1).

For (2), assume $F_A = \cup_{j \in J} G_{B_j}$ with each $G_{B_j}$ a soft chain. Then, for each $j \in J$, the soft ordered set $\vartheta(G_{B_j})$ of order ideals of $G_{B_j}$ is also a soft chain. Define a map $\phi: A \longrightarrow \Pi_{i \in I} \vartheta(G_{B_i})$ by $(\phi(x))_j = \left\{y \in G_{B_j}: y \subseteq x\right\}$. (Note $\theta \in \vartheta(G_{B_i})$, and $(\phi(x))_j = \theta$ is certainly possible.) The $\phi$ is clearly soft order-preserving. On the other hand, if $x \nsubseteq y$ in $A$ and $x \in G_{B_j}$, then $x \in (\phi(x))_j$ and $x \notin (\phi(y))_j$, so $(\phi(x))_j \nsubseteq (\phi(y))_j$ and $\phi(x) \nsubseteq \phi(y)$. Thus $A$ can be embedded into a direct product of $|J|$ soft chain. Using (1), this shows $d(A) \subseteq c(A)$.

**SOFT SEMILATTICES**

A semi lattice is an algebra $S = (\mathcal{S}, *)$ satisfying, for all $x, y, z \in \mathcal{S}$,

(i) $x * x = x$,

(ii) $x * y = y * x$,

(iii) $x * (y * z) = (x * y) * z$

In other words, a semilattice is an idempotent commutative semigroup. The symbol $*$ can be replaced by any binary operation symbol, and in fact we will most often use one of $\wedge$, $\vee$, $+ or \cdot$, depending on the setting. The most natural example of a semilattice is $(\beta(x), \text{n})$, or more generally any collection of subsets of $X$ closed under intersection. In this section, we give the definition of soft semilattice. Throughout this section, $\mathcal{S}$ is a semilattices and $A$ is any nonempty set . $R$ will refer to an arbitrary binary relation between elements of $A$ and elements of $\mathcal{S}$. That is, $R \subseteq A \times \mathcal{S}$. A set-valued function $F: A \longrightarrow \mathcal{P}(\mathcal{S})$ can be defined as $F(x) = \{y \in \mathcal{S}/xRy\}$. The pair $(F, A)$ is a soft set over $\mathcal{S}$.

**Definition 4.1**:

Let $(F, A)$ be a soft set over $\mathcal{S}$. Then $(F, A)$ is said to be a soft semilattice over $\mathcal{S}$ if $F(x)$ is sub-semilattice of $\mathcal{S}$, for all $x \in A$.

**Theorem 4.1**:

In a soft semilattices $(F, A)$, define $F(x) \subseteq F(y)$ if and only if $F(x) \vee F(y) = F(x)$. Then $((F, A), \subseteq)$ is an ordered soft set in which every pair of element has a greatest lower bound. Conversely, given an ordered soft set $((G, B), \subseteq)$ with that property, define $G(x) \vee G(y) = g.l.b\left(G(x), G(y)\right)$. Then $(A, \vee)$ is a soft semilattice.

**Proof**:

Let $(F, A)$ be a soft semilattice, and define $\subseteq$ as above. First we check that $\subseteq$ is a partial order.

(1) $(F(x) \vee F(x)$ implies $F(x) \subseteq F(x)$, $\forall x \in A$, and $F(x) = \{z \in \mathcal{S}: zRx\}$.

(2) if $F(x) \subseteq F(y)$ and $F(y) \subseteq F(x)$, then $F(x) = F(x) \vee F(y) = F(y) \vee F(x) = F(y)$.

(3) if $F(x) \subseteq F(y) \subseteq F(z)$, then $F(x) \vee F(z) = \big(F(x) \vee F(y)\big) \vee F(z) = F(x) \vee F(y) = F(x)$, so $F(x) \subseteq F(z)$.

Since $\big(F(x) \vee F(y)\big) \vee F(z) = F(x) \vee \big(F(x) \vee F(y)\big) = \big(F(x) \vee F(x)\big) \vee F(y) = F(x) \vee F(y)$, we have $F(x) \vee F(y) \subseteq F(x)$; similarly, $F(x) \vee F(y) \subseteq F(y)$. Thus $F(x) \vee F(y)$ is a lower bound for $\{F(x), F(y)\}$. To see that it is the greatest lower bound, suppose $F(z) \subseteq F(x)$ and $F(z) \subseteq F(y)$.Then $F(z) \vee \big(F(x) \vee F(y)\big) = \big(F(z) \vee F(x)\big) \vee F(y) = F(z) \vee F(y) = F(z)$, so $F(z) \subseteq F(x) \vee F(y)$,as desired. The proof of the converse is likewise a direct application of the definition.

**CONCLUSION**

In this paper we have introduced the concept of chains soft sets and the concept of set-valued function. The notion of soft semi lattices and its algebraic structures are also discussed.

**REFERENCES**


A. M. Ibrahim, M. K. Dauda and D. Singh. Composition of Soft Set Relations and Construction of Transitive Closure, *Journal of Mathematical Theory and Modeling*, Vol. 2, No. 7, pp. 98 – 107, 2012.

A. M. Ibrahim and A. O. Yusuf. Development of Soft Set Theory, *American International Journal of Contemporary Research*, Vol. 2 No. 9, 2012.

A. Kharal and B. Ahmad. Mappings on Soft Classes, *Information Sciences, INS-D-08-1231 by ESS,* pp. 1 – 11, 2010.

A. O. Atagun and A. Sezgin. Soft Substructures of Rings, Fields and Modules, *Computers and Mathematics with Applications*, Vol. 61, pp. 592 – 601, 2011.

A. Sezgin and A. O. Atagun. On operation of Soft Sets, *Computers and Mathematics with Applications* Vol. 61, 1457 – 1467, 2011.

D. A. Molodtsov. Soft Set Theory- First Results, *Computers and Mathematics with Applications* 37 (4/5), 19 – 31, 1999.

A. C. Davis, A characterization of complete lattices, *pacific J. Math. 5(1955) 311-319*.

A. Tarski, A lattice- theoretical fixpoint theorem and its applications, *pacific J. Math., (1955) 285- 309.*