FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 6 No. 3, June, 2022, pp 177 - 190
DOI: https://doi.org/10.33003/fjs-2022-0603-775

NOTE ON THE HISTORY OF (SQUARE) MATRIX AND DETERMINANT

[*1,3,9]Babarinsa Olayiwola, [6]Jerome Adepoju, [3]Azfi Zaidi Mohammad Sofi, [3]Asrul Mohd Hery, [1]Sunday Imoni, [1]Adamu Wakili, [4]Benson Onojhojobi, [1]Sheidu Momoh, [1]Rotimi Kehinde, [1]Daniel Lanlege, [4]Shobanke Dolapo, [1]Helen Olaronke Edogbanya, [1]Oyem Anselm, [4]Isaac Adeniyi, [4]Chinenye Ezenweke, [1]Sani Umaru, [1]Emmanuel Sabastine, [1]Kefas Bitrus,[1]Veronica Cyril-Okeme, [4]Eunice Upahi, [1]Emmanuel Akaligwo, [1]Ozioko Luke Arinze, [4]Simon Barguma, [1]Friday Edibo, [1]Mayowa Atteh, [1]Alloy Idoko, [5]Enoch Opeyemi, [6]Adesanmi Alao Mogbademu, [7]Olaitan Moses Ojo, [8]Akinola Eluwole, [9]Akeem B. Disu, [10]Adeeko Tajudeen, [11]Jeraldine Chibuzor Anukwu, [10]Samson Damilola Oluwafemi, [12]Kunle Ogunleye, [13]Shekh Abdullah-al-Musa Ahmed,[14]Zaato Solomon Gbene, [12]Koffa Durojaiye Jude, [15]Ojonubah James Omaiye, [2,16]Edna Manga, [2,16]Musa Josiah Marut, [17]Niri Martha Choji, [18]Emeka Helen Oluyemisi, [2,19]Adamu Abubakar Umar, [20]Mansur Hassan, [21]Oluwaseyi Jaiyeoba, [22]Sanusi A. Jubril, [23]Mamuda Mamman, [24]Amiru Sule, [25]Godwin Amechi Okeke, [17]Garba Gambo Zaku, [11]Osagie Abel Uyimwen, [14]Samsidine Aidara

[1]Department of Mathematics, Federal University Lokoja, P.M.B 1154, Kogi, Nigeria
[2]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia
[3]Faculty of Bioengineering and Technology, Universiti Malaysia Kelantan, 17600 Jeli, Malaysia
[4]Department of Statistics, Federal University Lokoja, P.M.B 1154, Kogi, Nigeria
[5]Department of Mathematics, Federal University Oye-Ekiti, P.M.B 373, Oye, Ekiti, Nigeria
[6]Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria
[7]Pompea College of Business, University of New Haven, 300 Boston Post Road, Connecticut, U.S.A
[8]Deparment of Geophysics, Federal University Oye-Ekiti, P.M.B 373, Oye, Ekiti, Nigeria
[9]Department of Mathematics, National Open University of Nigeria, Abuja, Nigeria
[10]Department of Physics, University of Abuja, P.M.B 117, Abuja, Nigeria
[11]Department of Geosciences, University of Lagos, Akoka, Lagos, Nigeria
[12]Department of Physics, Federal University Lokoja, P.M.B 1154, Kogi, Nigeria
[13]Computer Sciences and Engineering Department, Bangladesh University of Business and Technology, Rupnagar Mirpur-2, Dhaka-1216, Bangladesh
[14]Faculty of Entrepreneurship and Business, Universiti Malaysia Kelantan, 16100 Kota Bharu, Malaysia
[15]Department of Mathematics, Federal College of Education Okene, P.M.B 1026, Kogi, Nigeria
[16]Department of Mathematics, University of Jos, P.M.B 2084, Jos, Nigeria
[17]Department of Mathematics, Plateau State University, Bokkos, Jos, Nigeria
[18]Department of Mathematical and Physical Sciences, Afe Babalola University, Ado, Ekiti, Nigeria
[19]Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University, P.M.B 1045, Zaira-Kaduna, Nigeria
[20]Department of Mathematics, Yusuf Maitama Sule University, P.M.B 3320, Kano, Nigeria
[21]Department of Computer Graphics Technology, Purdue University, West Lafayette, Indiana USA
[22]Department of Statistics, Kano University of Science and Technology, Wudil, Kano, Nigeria
[23]Department of Mathematics, Niger State College of Education, Minna, Niger, Nigeria
[24]Department of Mathematics, Federal University Gusau, Gusau, Zamfara, Nigeria
[25]Department of Mathematics, School of Physical Sciences, Federal University of Technology, Owerri, Imo, Nigeria

*Corresponding authors' email: **olayiwola.babarinsa@fulokoja.edu.ng;** Phone**:** +2348060032554

ABSTRACT

This paper reviews the theory of matrices and determinants. Matrix and determinant are nowadays considered inseparable to some extent, but the determinant was discovered over two centuries before the term matrix was coined. Our review associate determinant with the matrix as part of linear systems but not with polynomials. Thus, the paper first gives the background on matrix with vast applications in all fields of study and then reviews the history of determinants which is based on its major contributors in chronological order from the sixteenth century to the twenty-first century.

Keywords: Matrix, determinant, linear systems, history of mathematics

INTRODUCTION

Matrices and determinants have been the aesthetic aspect of mathematics that anyone who studies it, not only found interesting but also able to teach the topics or pass the knowledge to others without loss of generalities. Many mathematicians have made contributions to the history of matrices and determinants (Burton, 2003; Eves, 1969). The modern theory of determinants was put forward by German mathematicians in Karl Theodor Wilhelm Weierstrass and Leopold Kronecker's lectures, but the notes were published after their death (Kronecker, 1903). The earliest contributors to determinants associate it with polynomials. Hence, they defined the term determinant without any reference to the existence of a square matrix. If the square matrix is an essential element of a determinant, then determinants would have been used more than a century after the death of Cramer. Instead, polynomial was considered which brings about the dual meaning of determinant. The early history of determinants focused on a system of $n+1$ linear equations in n unknowns to eliminate the unknowns, linear transformations, and the solution of a system of n linear equations in n unknowns (Miller, 1930). These systems were mostly represented in a rectangular form, known as a matrix. A matrix is a rectangular array of entries or elements (numbers, expressions, or symbols) in rows and columns. A matrix is referred to be a square matrix of size $n \times n$ if the number of rows (n) and columns (m) are equal otherwise, it is a non-square matrix of size ($n \times m$). Square matrices (which include Arrowhead matrix, Hadamard matrix, Sylvester matrix, Walsh matrix, Bézout matrix, Hessian matrix, Symplectic matrix, Bernoulli matrix, Hourglass matrix, Adjacency matrix, Edmonds matrix, Hat matrix, and Supnick matrix) are more interesting due to their unique properties than the counterpart non-square matrices (Babarinsa, 2018). In this write-up, basic matrix terminologies will not be defined except where necessary for further discussion and reference. Readers are expected to consult matrix-related books or articles. It is often impossible for an article (if not a book) to cover all the forms of square matrices (sparse or dense), except by devoting the work to a specific square matrix. Hence, we focus our review on the theory of determinants on a finite "square" matrix.

MATRICES

History of matrices

The word "matrix" was not coined for over four millennia, yet the history of matrices can be traced to ancient times. According to Debnath (2013a), a lot of evidence from mathematics history suggests the discovery of matrices may have started with the existence of magic squares. About 4000 years ago, magic squares were engraved on stones, metals, or paintings. If M_n is an $n \times n$ magic square which contains each entry $1, 2, 3, \dots, n^2$ exactly one with the same sum of each row, each column, and each of the two diagonals, then the common sum is the weight denoted as $wt(M_n)$ and defined by

$$wt(M_n) = \frac{1}{n}\sum_{k=1}^{n^2} k = \frac{n(n^2+1)}{2}$$

Magic squares (not Sagrada Familia magic square or Parker square) are esoteric in ancient times and were used in India, China, and Japan (Yoke, 1991). The study of systems of simultaneous linear equations starts from the origin of mathematical matrices which can be traced to Babylon and were recorded in a tablet dating 300 BC (Shafarevich and Remizov, 2013). Chinese came much closer to matrices methods to solve simultaneous linear equations than the Babylonians through Han Dynasty (200 BC – 100 BC). The author wrote the linear equations in columns rather than rows in modern methods, which is documented in "*Nine Chapters of the Mathematical Art"*. Chapter eight of the book was dedicated to *Fangcheng* - rectangular array, popularly known today as a matrix. Fangcheng's problems are displayed in two dimensions on the counting board (Hart, 2011; Shen, Crossley, Lun, and Liu, 1999). However, the concept of the matrix did not resurface and garner further attention until the end of the 17th century. In 1850, an English mathematician James Joseph Sylvester coined the term "matrix". Matrix is a Latin word for "womb", derived from *mater*—mother, and is defined as an oblong arrangement of terms (Sylvester and Baker, 2012). Sylvester further explains, "*I previously described a "Matrix" as a rectangular array of terms from which several determinant systems could emerge as though from a single parent*" (Sylvester, 1867). He coined the word womb to treat a matrix as a generator of determinants (Tucker, 1993). Sometimes the understanding of a whole field of science is suddenly advanced by the discovery of an idea (Pickover, 2011).

Contributors to matrix theory

In 1841, British mathematician Arthur Cayley used the letter A (uppercase) to represent matrix and lowercase for its elements (Debnath, 2013b). He released the initial article on the inverse of a matrix and focused more on the power of square matrices and matrix polynomials. Then, He provided definitions for addition, multiplication, scalar multiplication, and inverse in matrix algebra. In 1844, the combination of a row matrix and a column matrix was first proposed by German mathematician Hermann Günther Grassmann (1809-1877). Almost a century apart, an American mathematical physicist, Josiah Willard Gibbs (1839-1903), published a treatise on vector analysis to represent general matrices, called dyadic. Vector analysis got more improvement when an English physicist Paul Adrien Dirac (1902-1984) introduced the term "bra" (row) vector and "ket" (column) vector. The result from scalar multiplication of "bra-ket" or "ket-bra" form a simple matrix (Tucker, 1993).

In 1855, Cayley successfully established that there is a strong connection between matrices and linear transformations in his memoir on the "*theory of linear transformations*". In 1858, he published *"A memoir on the theory of matrices"* and discussed geometric transformation with abstract matrix operations. The problem with Cayley's writing is that he did not have a fixed notation for matrices. MacDuffee (1934) and Wedderburn (1934) used double vertical lines for matrices in their leading English books on matrices. These lines are now recommended for (matrix) norms. A British mathematician named Cuthbert Edmund Cullis (1875-1955) was the first to represent matrices using modern bracket (or parenthesis) notation in his 1913 treatise "*Matrices and Determiniods*" (Dossey, Otto, Spence, and Eynden, 2001). These days, entire rows or columns in a matrix are indicated by an asterisk. Later, Cayley developed matrix algebra alongside some matrix terminologies and introduced two vertical lines for a determinant on the side of the array (matrix). He used 0 for the zero matrix and 1 for the identity matrix. Though, Bell (2014) attributed Cayley as the founder view of the history of a matrix which is misleading since his paper in 1858 "*A memoir on the theory of matrices*" was not known due to where he published it. The same work of Cayley was done by a French mathematician, Edmond Nicolas Laguerre (1834-1886), in 1867 but his paper was not known too. Nevertheless, the paper of a German mathematician, Ferdinand Georg Frobenius (1849-1917), was not only known on the theory of matrix due to the world-leading journal (Crelle's journal) of the time he published it but also his paper is more substantial than those by Laguerre and Cayley (Hawkins, 1974).

Jan de Witt (1625-1672), a Dutch mathematician and statesman in 1660, never thought of the term "symmetric matrix" in his book "*Elements of Curves"* but showed how to transform a Canonical form of a conic given equation in arrays (Descartes, 1886). Later. a German mathematician David Hilbert (1862-1943) coined the Latin word "*spectrum*" for the set of eigenvalues (latent roots) of a matrix or operator. Eigenvalues and eigenvectors of a matrix are important aspects of engineering. Cayley-Hamilton sole theorem points out in a memoir that a square matrix is a root of its characteristic polynomial. However, in 1878, Frobenius proved the Cayley-Hamilton theorem. He then introduced the concept of the rank of a matrix from the results on Jordan canonical and orthogonal matrices. Frobenius did not use the term matrix, his paper deals with coefficients of forms and bilinear forms. Aitken (1956) and Weyl (1922) discussed the trace "*spur*' of a matrix is equal to the product of its eigenvalues, and the determinant of the matrix is equal to the sum of its eigenvalues. A German mathematician Ferdinand Gotthold Max Eisenstein (1823 - 1852) showed that matrix products are non-commutative which conformed to be non-abelian and he introduced the algebraic notation for products, inverses, and powers of linear substitutions (Hawkins, 1974).

Richard Dedekind (1831–1916), a German mathematician, in his study of algebraic numbers first discovered the set of $n \times n$ square matrices form an abstract mathematical system called a ring. In 1800, Carl Friedrich Gauss developed the method known as Gaussian elimination in his *Disquisitio de Elementis Elliptic is Palladis* (Bernardes and Roque, 2018; Grcar, 2011). He used the method to solve the normal equations associated with the method of least squares. However, some authors consider the Gaussian elimination was already known to a Chinese mathematician, He Chang Tsang, around 200 BC - 263 AD as the author of the method (Degos, 2015). Gauss-Jordan elimination (with reference to Wilhelm Jordan but not Camille Jordan) was considered as part of the development of geodesy (Athloen and McLaughlin, 1987). Russell and Whitehead (1913), in the article "*Principia Mathematica*", proposed the context of the axiom of reducibility for "matrix". The notion of the truth table in mathematical logic in connection with matrix was established in a 1946 paper titled "*Introduction to Logic"*(Tarski, 1946). AlanTuring introduced the LU decomposition of a matrix in 1948 while Roger Penrose developed the theory of generalized inverse matrices (Kyrchei, 2015; Rao and Mitra, 1972).

Applications of matrices

Matrix notations and computations have had a profound influence on all branches of mathematics: linear algebra, number theory, differential equation, numerical analysis, abstract algebra, modeling, operations research, and graph theory (Bôcher and Duval, 1922). Applications of matrices have spread like a wildfire in almost all fields of education such as engineering, computer science, statistics, economics, chemistry, physics, biology, geology, accounting, business, and industry to mention but a few (Jaffe, 1984). One of the great qualities of a matrix is the ability to create code in a situation where you need to send a private message to an ally (Babarinsa, Arif, and Kamarulhaili, 2019; Kippenhahn, 1999). The concept of matrix militarization was back to Julius Ceaser in 49 BC (Churchhouse, 2002). In the early 20th century, militaries of the world began to take advantage of the great ability of the matrix to create code in enigma machine during World War I by German engineer Arthur Scherbius (Kruh and Deavours, 2002), work of ballistic tables by Mauro Picone in World War I (Benzi, a), and to plane vibrations analysis – flutter - during World War II by female mathematician Olga Taussky Todd (Channell, 1977).

Nowadays, not only militaries of the world cannot survive without the application of matrices but also individuals and groups for handling large amounts of data. We depends on matrices in designing computer game graphics (Eberly, 2001; Lengyel, 2012), cyberspace internet (D'Andrea, Ferri, and Grifoni, 2010; Vaishnav, Choucri, and Clark, 2013),space communication (Tarokh, Seshadri, and Calderbank, 1998; Tirkkonen and Hottinen, 2002), facial recognition (Mangal, Malik, and Aggarwal, 2020; Rohil and Kaushik, 2014), PageRank algorithm for Google search engine (Djungu and Manneback, 2020), analyzing relationships (Henry and Fekete, 2007), choreographers plotting complicated dance steps (Raptis, Kirovski, and Hoppe, 2011), network analysis (Hawe, Webster, and Shiell, 2004), sound analysis (Sueur, Aubin, and Simonis, 2008), health and safety (Kariuki and Löwe, 2007; Lenhart and Travis, 1986), quantum theory (Mehra and Rechenberg, 1982), Markov chains (Bylina and Bylina, 2009; Searle, 2000), seismic survey (Berkhout, 2008; MacBeth and Li, 1996), chemical analysis (Gutman, 1977), decision making (Feng and Zhou, 2014; Saaty, 2003), population growth (Kendall, 1949; Lefkovitch, 1965), accounting game (Vysotskaya, 2018), robotic and automation (Ivanov, Ivanova, and Meleshkova, 2020; Stocco, Salcudean, and Sassani, 1999) and gene expression analysis (Shiflet and Shiflet, 2011).

NOTE ON HISTORY OF DETERMINANT THEORY

Determinant (resultant) was discovered over two centuries before the term "matrix" was coined, which is the backbone of Linear algebra (Bernstein, 2009). A determinant is a scalar value that represents certain aspects of a square matrix's linear transformation and is derived by computing its members, which can be denoted as $\det(A)$ or $|A|$. Determinant provides information about a matrix (its eigenvalues and eigenvectors): Geometrically, it provides the absolute value of area and volume in n-dimensional space, preserving transformation and can be used to create equations for curves, planes, and other geometric figures; and algebraically, it determines whether the system of n-linear equations in n-unknowns has a unique solution and a good indicator whether a square matrix has an inverse, see (Karim, 2013; Muir, 1911a; Rice and Torrence, 2006). The properties of determinants come from the characteristics of the matrices (Browne, 2018). Thus, setting prerequisites for linear equations' nontrivial solutions is the leading application of determinants (Weber and Arfken, 2003).

The determinant is famously known for square matrices. Some methods of computing determinants are fast and simple for lesser dimensions, especially for 2×2 and 3×3 matrices. However, for larger dimensions, Chio's condensations, Dodgson's condensation method, Laplace expansion method, triangle's rule, Gaussian elimination procedure, LU decomposition, QR decomposition, Bareiss algorithm, and Cholesky decomposition are considered. Nowadays, there is an extension of determinant to rectangular matrices, using Laplace expansion, called determinoids. Other less known types or forms of a determinant are the Dieudonné determinant, Fredholm determinant, Slater determinant, immanant, and functional determinant (Sobamowo, 2016). Based on history, the theory of determinant started in 16th century, but we give the chronological order of the contributions till the 21st century. General methods/formulas for evaluating determinants are not new, as they can be attributed to the 18th century. They are, in fact, the modification of the old methods, perhaps except for a few special matrices. Contributors to determinants (and its theory) are many but not all contribute immersive to the subject matter. It is either they reiterated what others have done without a new (or less) contribution or their contributions have been debunked due to a lack of mathematical evidence.

16th century: Gerolamo Cardano (1501-1576), an Italian mathematician provided a rule called *regula de modo* - mother of rules - for resolving a system of two linear equations, in his *"ars magna"* (Cardano, 1993). The rule later gave what we are essentially known as Cramer's rule (Cardano and Spon, 1968). His determinants were practically for 2 × 2 matrices and larger ones were discussed by Leibniz (Babarinsa, 2020; Eves, 1969).

17th century: Determinants emerged from two simultaneous quadratic equations in the theory of equations, matrix algebra, geometry, and differential equations, among other fields of mathematics (Kline, 1990). Let

$$\begin{aligned} a_{11}x^2 + a_{12}x + c_{12} &= 0 \\ a_{21}x^2 + a_{22}x + c_{22} &= 0 \end{aligned} \tag{1}$$

Seki Takakazu who is popularly known as Seki Kōwa (1642-1708) initially introduced the concept of determinant to Japan. In 1683, he published his findings in a book titled "*Method of Solving the Dissimulated Problems*" in the absence of a word that describes the determinant (Martzloff, 2008). He gave the solution to Equation (1) by eliminating x^2 as well as a constant term c_{12} and c_{22}. Thus, he arrived at the determinant as

$$(a_{11}a_{22} - a_{21}a_{12})$$

Seki still introduced the concept of determinants and provided broad guidelines for calculating them in accordance with his 2×2 determinant through a process he called *tatamn (folding)*. Instead of using them to solve systems of linear equations, he applied them to equations (Rothman and Fukagawa, 1998). In the same year of 1683, the European counterpart to work on determinants independently was a German mathematician and logician, Leibniz Gottfried Wilhelm (1646-1716). Leibniz referred to specific combinatorial sums of words of a determinant as "*resultants*" (Muir, 1906). He presented a few results on the outcome and used number pairs as coefficients to serve the same purpose of double subscript for rows and columns in the square matrix of a determinant (Miller, 1930). Leibniz and Seki knew the properties of determinants and that determinants can be expanded using any column which we now called Laplace expansion – though both did not publish the findings (Debnath, 2013a).

18th century: The development of resultant (determinant) was out of sight to mathematicians for over a century until a Scottish mathematician, Colin Maclaurin (1698-1746) in 1748, offered the first results on two, three, and four simultaneous equations that had been published in a book titled "*Treatise of Algebra*" (MacLaurin, 1748). Although the publication of his findings was made two years after his death which gave Cramer the edge to introduce the method (Tweedie, 1915). Nevertheless, Boyer (1966) showed that Cramer's rule was published two years earlier in Colin Maclaurin's posthumous. Hedman (1999) analyzed a document that offers convincing proof that Maclaurin was imparting "Cramer's rule" to his pupils more than 20 years before Cramer published it. While asserting the "opposite" coefficient, Kosinski (2001) contended that the rule he chose to assign the appropriate sign to each summand was incorrect. Cramer remedied this by counting the number of transpositions, or dérangements, in the permutation. Maclaurin missed the general rule for solving linear equations, according to Günther (1908), because of poor notation.

In 1750, Swiss mathematician Gabriel Cramer (1704-1752) hinted that resultants are useful in analytical geometry (Habgood and Arel, 2010). Cramer gave the general rule for solving n linear simultaneous equations in n unknowns $x_1, x_2, x_3 \ldots x_n$ defined by

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= c_3 \\ \vdots + \vdots + \vdots + \cdots + \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= c_n \end{aligned}\right\}$$

$$Ax = c \qquad (2)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \text{ and } c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

the $n \times n$ matrix A (coefficient matrix) has a nonzero determinant, c the constant term (nonhomogeneous term), and the vector $x = (x_1, x_2, \ldots x_n)^T$ is the column vector of the variables; $\forall\, A, c \in \mathbb{F}$

Theorem 1 (Cramer's rule) *Let $Ax = c$ be an $n \times n$ linear system with A an $n \times n$ matrix, if $|A| \neq 0$ and the column (constant) vector c replaces the ith column vector a_i of A, then the i th entry x_i of the unique solution $x = (x_1, x_2, \ldots, x_n)$ is given by*

$$x_i = \frac{|A_i|}{|A|} \qquad (3)$$

where $i = 1,2,\ldots,n$, $|A| = |a_{ij}|$ is the nth - order determinant with a_{ij} as its elements and $|A_i|$ is the nth - order determinant obtained from $|A|$ by replacing its ith column with the column containing the non-homogeneous terms $c_1, c_2, c_3, \ldots, c_n$. In his work "*Introduction to the Analysis of Algebraic Curves*," Cramer went on to further describe how to calculate the terms using his formula for figuring out the sign and getting the numerator. (Cramer, 1750; Robinson, 1970).

Although Swiss scientist Leonhard Euler (1707–1783) demonstrated that a system of linear equations need not have a solution, researchers have linked the solution of a system of linear equations to Cardano (Tucker, 1993). Cardano's methods were practically based on 2×2 resultants, the rule later gave what we knew as Cramer's rule (Cardano, Witmer, and Ore, 2007). At least three drawbacks of Cramer's rule include its failure when the coefficient matrix's determinant is 0, the number of determinant calculations it necessitates (if determinant values are calculated through minors), and is also numerically unstable (Chapra and Canale, 1998; Debnath, 2013a; Higham, 2002; Vein and Dale, 1999). Therefore, Cramer's rule has asymptotic complexity of $O(n.n!)$ via minors, but it has been shown that it is possible to apply Cramer's rule in $O(n^3)$ time (Habgood and Arel, 2012; Shores, 2007). Cramer's rule for solving systems of linear equations has historical and theoretical significance despite its high processing cost (Brunetti and Renato, 2014). Due to roundoff error, Moler (1974) claimed that Cramer's rule is insufficient even for 2×2 linear systems; however, Dunham (1980) provided an example to refute this claim. Other efficient iterative and numerical techniques which include the Gauss–Jordan elimination have replaced Cramer's rule for solving linear systems of equations (Hoffman and Frankel, 2001; Watkins, 2004). Nowadays, much advancement has been made on Cramer's rule to solve simple and large-scale linear systems, Quaternionic systems, minimum-norm least-squares solution of linear equations, matrix iteration, condensed Cramer's rule for the solution of restricted matrix equation where inverse was not employed as well as integrating Dodgson condensation and Sylvester's determinant identity with Cramer's rule, see (Benzi, 2009b; Gu and Xu, 2008; Ji, 2012; Kyrchei, 2008; Ufuoma, 2013).

A French mathematician Étienne Bézout (1730-1783) in 1764 gave methods of calculating resultants (determinants) by combining his rule of term formation and his rule of signs into one. He requires the permutations, unlike Cramer and Leibniz finding the permutations in any way, to be found by a process, and contributed to the recurrent law of formation of the new functions. Then he proved that the nontrivial solutions of a system exist provided the determinant of the coefficient matrix is zero, in his *Théorie des équations algébriques* (Bézout, 1779; Godin, Demours, and Cotte, 1774). He stated a reframed theorem of Vandermonde that determinant of a matrix is zero if two rows are identical. According to Bézout, solving simultaneous equations by elimination is similar to solving nth degree equations in one unknown since "*it is known that a determinate equation may always be interpreted as the outcome of two equations in two unknowns when one of the unknowns is eliminated*". Bézout saw that he could determine the form of its solution. Conversely, if the coefficients of a given nth degree equation in one unknown had the form built up from such a special solution, that nth degree equation could be solved. In his treatise "*Sur plusieurs classes d'équations de tous les degrés qui admettentune solution algébrique"*. Bézout stated that the degree of the final equation resulting from any number of complete equations in the same number of unknowns, and of any degrees, is equal to the product of the degrees of the equations. Then he discussed another method of finding the resultant equation by finding polynomials, which we may write $Q_1, \dots, Q_n$ such that $P_1Q_1 + P_2Q_2 + \cdots + P_nQ_n = 0$ is the resultant equation. Each $Q_k (k = 1, 2, \dots, n)$ has indeterminate coefficients, which Bézout explicitly determined for many systems of equations by comparing powers of the unknowns $x, y, z, \cdots$ (Bézout, 1762). This theorem brought about the development of the Bézout matrix, the theory of determinants, and resultants (Muir, 1911b).

The widely extended concept known as the theorem for expressing a determinant as an aggregate of products of complementary minors was first published in 1771 by French mathematician and chemist Alexandre-Théophile Vandermonde (1735–1796) in his "*Mémoiresurl' élimination*" (Vandermonde, 1772). His method can evaluate the determinant of order n (Hadamard, 1897). Thus, the only one fit to be viewed as the founder of the theory of determinants is Vandermonde since he was the first to recognize determinants as independent functions (Campbell, 1980). Vandermonde's matrix is a matrix where each row's terms correspond to a geometric progression. The nth-order Vandermonde determinant is

$$|V_n| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$
$$= (x_2 - x_1)(x_3 - x_2)(x_4 - x_3) \dots (x_j - x_i$$

Thus,

$$|V_n| = \prod_{1 \le i < j \le n} (x_j - x_i) \qquad (4)$$

Where $|V_n|$ is Vandermonde determinant or alternant, the right-hand side is the continued product of all the differences that can be formed from the $\frac{n(n-1)}{2}$ pairs of numbers taken from $x_1, x_2, \dots, x_n$ with the order of the differences taken in the reversed order of the suffixes that are involved. The matrix can be transposed and has applications in Cryptography, polynomial interpolation, and signal processing (Klinger, 1967; Sobczyk, 2002).

Pierre-Simon marquis de Laplace (1749-1827), a French polymath in 1772, gave a notation for a resultant or determinant (Sobczyk, 2002). He created a method to determine the number of terms in this aggregate and provided a rule for how to describe a resultant as an aggregate of terms made up of components (minors) that are also resultants. In addition, he named the new functions and provided proof of the theorem on the impact of transposing two adjacent letters in any of the new functions. His theorem may be described as giving an expression of a resultant in the form of an aggregate of terms each of which is a product of a lower degree (Brualdi and Schneider, 1983). Laplace later claimed that the methods employed by Cramer and Bézout were impractical. Laplace expansion is the best for computing determinants as it works for all forms of square matrices except it has a high time of complexity (Bronson, 1988; Cormen, 2009; Franklin, 1968). However, the disadvantage of Laplace expansion is that nowhere does a determinant of order greater than two have to be computed except by expressing it in numerous minors and thus leading to time wastage (Wexler, 1969). Let A be $n \times n$ matrix. A minor is any $(n-m) \times (n-m)$ matrix formed by deleting m rows and m column from A. A complementary minor is the $m \times m$ matrix diagonally adjacent to the minor matrix A. A consecutive minor is a matrix in which the remaining rows and columns in the minor were adjacent to the original matrix (Rice and Torrence, 2007).

Theorem 2 (Laplace expansion) *Suppose* $A = [a_{ij}]$ *is an n × n matrix such that any* $i, j \in (1, 2, \dots, n)$. *Then its determinant of A is given by*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \qquad (5)$$

where

$$(-1)^{i+j} = \begin{cases} + & when\ i = j\ or\ i + j\ is\ even \\ - & when\ i \neq j\ or\ i + j\ is\ odd \end{cases}$$

$for\ i, j \in \mathbb{Z}^+$.

The minor M_{ij} is defined to be the determinant of the $(n-1) \times (n-1)$matrix that results from the matrix by removing the ith row and the jth column and $(-1)^{i+j}$ the checkerboard sign, for $i, j = 1, 2, \dots, n$. The expression $(-1)^{i+j} M_{ij}$ is known as a cofactor, see (Afriat, 2000; Horn and Johnson, 2012; Lancaster and Tismenetsky, 1985). Thus, $(-1)^{i+j}$ can be represented in the checkerboard sign below

$$\begin{bmatrix} (-1)^{i+i} & (-1)^{i+j} & (-1)^{i+j} & (-1)^{i+j} & \cdots & (-1)^{i+j} \\ (-1)^{i+j} & (-1)^{i+i} & (-1)^{i+j} & (-1)^{i+j} & \cdots & (-1)^{i+j} \\ (-1)^{i+j} & (-1)^{i+j} & (-1)^{i+i} & (-1)^{i+j} & \cdots & (-1)^{i+j} \\ (-1)^{i+j} & (-1)^{i+j} & (-1)^{i+j} & (-1)^{i+i} & \cdots & (-1)^{i+j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{i+j} & (-1)^{i+j} & (-1)^{i+j} & (-1)^{i+j} & \cdots & (-1)^{i+i} \end{bmatrix}$$

According to Jeffrey (2010) when evaluating a determinant, the amount of calculation required can be estimated by giving the determinant's order, while not indicating the value of the determinant. In the quest to make determinants more computable, Almalki, Alzahrani, and Alabdullatif (2013) designed a Laplace expansion-based sequential and parallel technique for finding determinants. Janjia (2005) stated that one of the most significant characteristics of determinants is Laplace expansion theorem. This theorem can be obtained by a proper rearrangement of summands when determinants are stated in terms of permutations.

In 1773, an Italian mathematician and astronomer Joseph Louis Lagrange also known as Giuseppe Luigi Lagrange (1736-1813) for the first time gave a volume interpretation of a determinant. He treated determinants and applied them to elimination theory – bilinear forms. While other contributors focus on the problem of elimination, langrage work, on the other hand, consists of several incidentally obtained algebraic identities. Lagrange's identity and the modern-looking identities are essentially the same and he proved many special cases of general identities. He further gave a theorem in his "*Recherches d'arithmetique*" that a minor determinant adjugates to another determinant (Lagrange, 1775; Weld, 1893).

A German mathematician Carl Friedrich Hindenburg (1741-1808) worked on Cramer and Bezout's point of view in 1784. He wrote his permutation, calculating determinants, in a definite order regarding the sequence of signs by successfully combining the rule of term formation and the rule of signs (Muir, 1911a).

19th century: In 1800, a German mathematician Heinrich August Rothe (1773-1842), made an ill-advised and pointless modification of Cramer's idea of the rule of signs. Though, he made remarkable contributions from the theorems he gave**.** He claimed that by counting the interchanges required to convert one permutation into the other, it is possible to discover the sign of any single permutation when the sign of any other is known – conjugate permutations has the same sign. Rothe further went and stated a theorem that depending on whether m is even or odd, the sign of the new permutation is the same as, or different from, that of the original if one element of a permutation is forced to take up a new position by being passed over m additional elements. (Studnička, 1876)**.**

A year after, the term "determinant " was first introduced in 1801 by German mathematician Johann Carl Friedrich Gauss (1777-1855) in his book titled "*DisquisitionesArithmeticae*" while discussing quadratic forms (Gauss, 1966; Knobloch, 2013). Gauss used the term because the determinant determines the properties of quadratic forms. The new term introduced by Gauss was not 'determinant" but "determinant of a form". Nowadays, the determinant of a form is referring to the discriminant of a quantic. In the theory of numbers, he frequently used determinants. The idea of reciprocal (inverse) determinants was also developed by Gauss (Kani, 2011). In the modern sense, the association of a square matrix and the corresponding polynomial in connection with linear transformation is due to Gauss.

In 1809, a French mathematician Gaspard Monge (1746-1818) used a process of elimination to compute the determinant. His method was quite general because the method possesses numerous other identities of the same kind.

In 1811, a French mathematician and physicist Jacques Philippe Marie Binet (1786-1856) gave an extension of a theorem of Lagrange on determinant which expressed that a sum of products of resultants as a single resultant (Knobloch, 1994; Shallit, 1994). He gave a modern notation for the formula

$$\sum_{k=1}^{k=s}\sum_{h=1}^{h=s}\begin{vmatrix} y_h^1 & y_h^2 & \dots & y_h^n \\ z_k^1 & z_k^2 & \dots & z_k^n \end{vmatrix}\begin{vmatrix} \upsilon_h^1 & \upsilon_h^2 & \dots & \upsilon_h^n \\ \zeta_k^1 & \zeta_k^2 & \dots & \zeta_k^n \end{vmatrix}$$

A French mathematician and engineer Augustin-Louis Cauchy (1789-1857) in 1812 used "determinant" in its modern sense as was the most complete work on determinant. He published a paper in which he used determinants to compute the volume of several solid polyhedral (Vein and Dale, 1999). He gave a multiplication theorem for determinants and new results on minors and adjoints. He introduced the idea of similar matrices (but not the term) and pointed out that the eigenvalues of symmetric matrices are real. He introduced certain matrix terminologies such as terms, characteristics, principal terms, symmetric products, principal product, conjugate, conjugate system, and complementary derived systems. In 1826, Cauchy referred to the coefficients matrix as a "*tableau*" while discussing quadratic forms in n variables. His method produced eigenvalues and eigenvectors, which offered a fresh way to handle quadratic expressions with n variables (Knobloch, 1994). He viewed determinant as a special class of alternating symmetric functions and gave the method as

$$D_n = S(\pm a_{1.1} a_{2.2} \dots a_{n.n}) \qquad (6)$$

His method produced eigenvalues and eigenvectors, which offered a fresh way to handle quadratic expressions with n variables (Knobloch, 1994). However, the eigenvalue problem to solve systems of ordinary differential equations was generalized by French mathematician Jacques Sturm. Later, Cauchy introduced the 2×2 determinant involving partial derivatives – known today as the Jacobian determinant**.** The Jacobian matrix is an $n \times n$ matrix, usually defined and arranged as follows

$$J = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_\_} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \qquad (7)$$

Jacobian determinant has application in polar coordinates, cylindrical polar coordinates, and spherical polar coordinates. In this, he used the word "determinant" in its present sense after considering Laplace, Vandermonde, Gauss, and Bezout's work. He modernized the notation, streamlined what was then known about the subject, and provided a more convincing demonstration for the multiplication theorem than Binet did. With him begins the theory in its generality (Cauchy, 1812).

In 1841, a German mathematician Carl Gustav Jacob Jacobi (1804-1851) gave the definition of determinant which was made algorithmically. Jacobi gave the adjugate determinant of matrix A given as

$|adjA| = |A|^{n-1}$ and $adjA = (A_{ij})^T = (A_{ji})$

where adjugate matrix of A is $adjA$ and A_{ij} are the cofactors of elements a_{ij} (Jacobi, 1896). In the same year (1841), Cayley introduced hyperdeterminant. He published for the first time on the inverse of a matrix. He proposed a theorem which is now known as the Cayley-Hamilton theorem that a matrix must satisfy its characteristic equation (Cayley, 1858). In his memoir, he successfully discovered that there is a close relationship between matrices and linear transformations (Cayley, 1845). Peter Guthrie Tait once said, "*Cayley is forging the weapons for future generations of physicists*" (Bonolis and de Laplace, 2004).

Pierre Frédéric Sarrus (1798–1861), a French mathematician in 1842, gave a memorization scheme to compute only the determinant of a 3×3 matrix, $A = a_{i,j}, \ \forall \ i,j = 1,2,3$. Sarrus rule or basketweave method can be derived from the case of the Leibniz formula, and Laplace expansion. The method considers one to write out the first two columns of the matrix to the right of the third column to yield five columns in a row. Then, add the top-to-bottom diagonal's products and deduct the bottom-to-top diagonal's products (Ahmed and Bondar, 2014; Karim, Ibrahim, and Omar, 2016) to yield

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \qquad (8)$$

In 1866, an English writer Charles Lutwidge Dodgson popularly known as Lewis Carroll (1832-1898) gave a method of computing the determinant of a square matrix by condensation. The method proves to be effective as well as minimizes errors before arriving at the solution (Leggett, Perry, and Torrence, 2009). Dodgson condensation reduces the matrix into 2×2 submatrices for easy computation of determinants. The method reduces the risk of miscalculation as it is bound to divide the determinant of the submatrices by interior elements (Abeles, 1986). The fatal of Dodgson's condensation defect is that the determinant of an interior matrix must not be zero because dividing the determinant of the minors by zero makes the solution indeterminate (Abeles, 1994). The advantage of Dodgson condensation is that the determinant of a square matrix is a rational function of all its connected minors of any two consecutive sizes (Schmidt and Greene, 2011). The fatal defect of Dodgson condensation has a remedy like row (column) permutations, though it may not always work if there are many zero entries in the matrix or the determinant of interior matrix zero - this can happen even if no zeroes appear in the interior of the matrix (Abeles, 2008; Robbins, 2005). In Dodgson's condensation, each smaller matrix contains the 2×2 connected minors of the previous iteration's matrix. The 2×2 connected minors are the determinants of each 2×2 submatrices consisting of adjacent elements of the larger matrix. Beginning with the second stage of iteration, each of these minors is divided by their central element from two stages previous. In this case, Dodgson suggested replacing the zero element with a nonzero element of the matrix by rotating columns or rows and then proceeding with condensation. If all elements of the matrix are zero, then the matrix is trivial, and its determinant is zero. For a given $n \times n$ matrix, a *minor* is any $(n-m) \times (n-m)$ matrix formed by deleting m rows and m columns from A. A *complementary minor* is the resulting $m \times m$ matrix diagonally adjacent to the minor matrix while a *consecutive minor* is one in which the remaining rows and columns in the minor were adjacent in the original matrix. *interior* of A is the $(n-2) \times (n-2)$ consecutive minor that results when the first and last rows and columns of matrix A are deleted, see (Abeles, 1986; Rice and Torrence, 2006, 2007).

Theorem 3 (Dodgson's condensation theorem) *Let A be an $n \times n$ matrix. After k successful condensation, Dodgson produces the matrix*

$$A^{(n-k)} = \begin{pmatrix} |A_{1\ldots k+1,1\ldots k+1}| & |A_{1\ldots k+1,2\ldots k+2}| & \cdots & |A_{1\ldots k+1,n-k\ldots n}| \\ |A_{2\ldots k+2,1\ldots k+1}| & |A_{2\ldots k+2,2\ldots k+2}| & \cdots & |A_{2\ldots k+2,n-k\ldots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-k\ldots n,1\ldots k+1}| & |A_{n-k\ldots n,2\ldots k+2}| & \cdots & |A_{n-k\ldots n,n-k\ldots n}| \end{pmatrix}$$

Whose entries are the determinants of all $(k+1) \times (k+1)$ contiguous submatrices of A.

Then

$$A_n(1,1)A_{n-2}(2,2) = A_{n-1}(1,1)A_{n-1}(2,2) - A_{n-1}(2,1)A_{n-1}(1,2)$$

more precisely,

$$\det(A) = detA_n(1,1) = \frac{detA_{n-1}(1,1)\, detA_{n-1}(2,2) - detA_{n-1}(1,2)\, detA_{n-1}(2,1)}{detA_{n-2}(2,2)} \quad (9)$$

For an $n \times n$ matrix A, let $A_r(i,j)$ denote the r by r minor consisting of r contiguous rows and columns of A, beginning with row i, column j (Amdeberhan and Ekhad, 1997). Note that $A_{n-2}(2,2)$ is the central minor or interior elements; $A_{n-1}(1,1)$, $A_{n-1}(2,2)$, $A_{n-1}(1,2)$ and $A_{n-1}(2,1)$ are the respective northwest, southwest, southeast, northeast, and southwest minors, see (Abeles, 2014; Amdeberhan, 2001; Muir, 1881) and the references therein. According to Bressoud and Propp (1999), *"Although the use of division in Dodgson condensation may appear to be a drawback, it serves as a useful form of error checking for calculations done by hand using integer matrices. When the algorithm is carried out correctly, all the entries of all the intervening matrices are integers, making it impossible to know that a mistake has been made when a division does not come out evenly. The approach is helpful for computer calculations as well, particularly".*

Theorem 4 (Chio's method) *For an $n \times n$ matrix $A = (a_{ij})$ with $a_{nn} \neq 0$, let $F = (f_{ij})$ be the $(n-1) \times (n-1)$ matrix defined by*

$$f_{ij} = \begin{vmatrix} a_{ij} & a_{in} \\ a_{nj} & a_{nn} \end{vmatrix} = a_{ij}a_{nn} - a_{in}a_{nj} \quad (10)$$

Then, $\det(A) = \frac{1}{a_{nn}^{n-2}} det\ F$

For $I, j = 1, \ldots, n-1$.

The process in Equation (10) substitutes every element in the matrix with a 2×2 determinant comprises the a_{ii} element, the highest value in the element's column, the first value in the element's row, and the element being replaced. The computed values of 2×2 determinant replace the $a_{i,j}$ with $a_{i,j}'$. The ith row and the ith column are deleted, therefore decreasing the initial $n \times n$ matrix to an $(n-1) \times (n-1)$ matrix with the equivalent determinant, see (Brualdi and Schneider, 1983; Eves, 1980; Habgood and Arel, 2012). Chio's method will not work if the pivotal element is zero because dividing the determinant of the minors by zero makes the solution indeterminate and the method fail to compute over a small finite field (Robbins, 2005).

A German mathematician, Karl Theodor Wilhelm Weierstrass (1815-1897), gave the axiomatic definition of a determinant.

20th century: During the 20th century, the matrix begins to have tentacles due to its applications in different fields which emerged a new field in mathematics called matrix theory. Since Laplace expansion is a building block for other methods of determinant, only a few of the contributors to the determinant of a matrix in mathematics will be discussed.

Bareiss (1968) worked on improving the computation of determinants by minimizing the complexity time of the condensation. Although Bareiss algorithm or Montante's method is based on row reduction, it can also be proven using Sylvester's identity(Yap, 2000). The Chinese remainder theorem has been used to compute some cases of determinants (Pan, Yu, and Stewart, 1997).

Robbins and Rumsey (1986) made important studies on the iteration of the Dodgson's Determinantal Identity (DDI) to the discovery of Alternating Sign Matrix Conjecture (ASM). The iteration was from the recurrence of the Laurent polynomials (when $\lambda = -1$) to form lambda determinant of matrix (Mills, Robbins, and Rumsey, 1986). An Alternating Sign Matrix has $+1, -1, 0$ as an element in every row and column and thus, the ASM conjecture is given as

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \quad (11)$$

Within a decade Zeilberger (1997) published a combinatorial proof of DDI. A better algorithm than simple Dodgson's condensation is the recurrence of DDI. Though DDI requires more calculation yet the computational complexity of DDI and Dodgson condensation remain the same (Francisco Neto, 2015). Grcar (2012) asserted that several authors including Charles Dodgson reinvented Chio's method of evaluating the determinant. However, Abeles (2014) stated that Dodgson's identity was a result of a theorem of Jacobi while Chio's identity was from a theorem of Sylvester.

Theorem 5 (Jacobi's theorem on adjoint determinant) *Let A be an $n \times n$ matrix, let $[A_{ij}]$ be an $m \times m$ matrix of A, where $m < n$, let $[A'_{ij}]$ be the corresponding $m \times m$ minor of A' and let $[A^*_{ij}]$ be the complementary $(n-m) \times (n-m)$ minor A. Then.*

$$det[A'_{ij}] = (detA)^{m-1}.det[A^*_{ij}] \qquad (12)$$

By Laplace expansion $A.A' = \det(A).I$

Thus,

$$\det(A.A') = \det(A)\det(A') = (detA)^n$$

Likewise, Dodgson's method is a unique case for both Desanot and Muir's law of extensible minors and Jacobi adjoint matrix theorem. More precisely for Dodgson/Muir determinantal identity is

$$det\ A = \frac{\sum_{\sigma\in s_k}(-1)^{l(\sigma)}\prod_{j=1}^{k} detA[\{j,k+1,\dots,n\},\{\sigma(j),k+1,\dots,n\}]}{det\ A[\{k+1,\dots,n\},\{k+1,\dots,n\}]^{k-1}} \qquad (13)$$

From the above equation, if $k = 2$ then it turns out to be DDI. Other special cases where Dodgson's identity was derived are Lagrange, Cauchy and Minding, and Sylvester's identity (Amdeberhan and Ekhad, 1997). It was Brualdi and Schneider (1983) that successfully linked Chio and Sylvester's identity by considering Schur's identity as follows:

Let $A = (a_{ij})$ be a square matrix. If A is partition using block triangularization, then we can factor A into

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & o \\ A_{21} & I \end{bmatrix}.\begin{bmatrix} I & {A_{11}}^{-1}A_{12} \\ 0 & A_{22} - A_{21}{A_{11}}^{-1}A_{12} \end{bmatrix} \qquad (14)$$

Where $A_{11} \neq 0$ is of order K. Multiplying both sides by their respective determinants $|A_{11}|^{n-k-1}$, we therefore have

$$|A_{11}|^{n-k-1}|A| = \left||A_{11}|(A_{22} - A_{21}{A_{11}}^{-1}A_{12})\right| \qquad (15)$$

When $k = 1$ the expression becomes Chio's identity, see (Akritas, Akritas, and Malaschonok, 1996; Eves, 1980).

Chang and Su (1998) devised a method, to reduce the cumbersome method of evaluating determinant, known as the order-reduction formula through condensation which is

$$det\begin{bmatrix} w_{11} & v_1 & w_{12} \\ u_1 & r & u_2 \\ w_{21} & v_2 & w_{22} \end{bmatrix} = r\det\left[\begin{bmatrix} w_{11} & -w_{12} \\ -w_{21} & w_{22} \end{bmatrix} - \frac{1}{r}\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}[u_1 \quad -u_2]\right] \qquad (16)$$

Provided that $r \neq 0$. Where W $(w_{11} \dots w_{22})$, r v and u are a square matrix, a scalar (pivot element), a column matrix and a row matrix respectively.

21st century: In the early 21st century, Rezaifar and Rezaee (2007) discussed a new method of computing determinants. They compute the determinant as the result of submatrices derived by discarding row and column in a specific direction or way and resulting in the formula given as

$$|M| = \frac{1}{|M_{11,nn}|}\begin{vmatrix} |M_{11}| & |M_{1n}| \\ |M_{n1}| & |M_{nn}| \end{vmatrix} \qquad (17)$$

where

$$M = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} M_{nn} = \begin{bmatrix} x & x & x & \\ x & x & x & \\ x & x & x & \\ & & & \end{bmatrix} M_{11,nn} = \begin{bmatrix} & & & \\ & x & x & \\ & x & x & \\ & & & \end{bmatrix}$$

$$M_{n1} = \begin{bmatrix} & x & x & x \\ & x & x & x \\ & x & x & x \\ & & & \end{bmatrix}, M_{11} = \begin{bmatrix} & & & \\ & x & x & x \\ & x & x & x \\ & x & x & x \end{bmatrix} \text{ and } M_{1n} = \begin{bmatrix} & & & \\ x & x & x & \\ x & x & x & \\ x & x & x & \end{bmatrix}$$

Salihu (2012) gave a method that is based on Dodgson-Chio's condensation. By calculating four unique determinants of $(n-1) \times (n-1)$ order, which can be derived from determinants of nn order, we can resolve Salihu's method. If we remove the first row and first column, first row and last column, last row and first column, or last row and last column, we should refer to these elements as unique elements, and one determinant of $(n-2) \times (n-2)$ the order, which is formed from $n \times n$ order determinant with elements $a_{i,j}$ with $i, j \neq 1$, on the condition that the determinant of $(n-2) \times (n-2) \neq 0$. However, Salihu's method is not different from Rezaifar and Rezaee's method. Though Rezaifar and Rezaee first published the new method and gave comprehensive proof as well as the algorithm using MATLAB and FORTRAN, they failed to formulate a theorem. Salihu went further to coin a new term called "*unique elements*" for $M_{11,nn}$. He got his idea from Chio's condensation and Dodgson condensation method, while Rezaifar and Rezaee got theirs from inversing matrices in a linear equation. It may be noted that Salihu was unaware of Rezaifar and Rezaee's article as he did not cite it in his paper. The common thing among the findings of Salihu and, Rezaifar and Rezaee is that their method reduces $n \times$ n matrix into four $(n-1) \times (n-1)$ matrices and one $(n-2) \times (n-2)$ matrix.

Furthermore, Taheri, Boostanpour, and Mohammadi (2013) claimed to have gotten a novel algorithm for the determinant calculation of $n \times n$ matrix, called *TaBe*. They were, in fact, unaware of Salihu nor Rezaifar and Rezaee's work, because their work is termed to be a reinvention of Rezaifar and Rezaee's method.

Urbańska (2008) devised faster combinatorial algorithms for determinants and Pfaffian. Improvements are made on a fast algorithm to compute determinants of special matrices such as circulant matrix, Pentadiagonal matrix, Divisor matrix, Bezout matrix, and Toeplitz matrix, see (Chen, 2014; Cinkir, 2014; El-Mikkawy, 2008).

Over a century of discovering Cramer's rule and Dodgson's condensation, no one has successfully linked the two methods together until recently when Ufuoma (2013) described the relationship between Cramer's rule and Dodgson's condensation with lucid understanding. She, as well, gave proof of her new method is the same as classical Cramer's rule. Without loss of generality, this method is used for the system of the linear equation of the form $Dx = b$.

Let $S_1 = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 & a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n & a_{n1} & \cdots & a_{n(n-1)} \end{bmatrix}$. That is, $S_1 = [D \quad b \quad D']$

Where D' is the array of numbers that remains after removing the last column of D. Thus,

$$|D_n| = \begin{vmatrix} b_1 & a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} a_{11} & \cdots & a_{1(n-1)} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & b_n \end{vmatrix}$$

Therefore,

$$x_n = (-1)^{n-1} \frac{|D_n|}{|D|} \quad (18)$$

Chang (2014) discussed an integrated method of condensation in his *"determinant of a matrix by order condensation"*. The method was easy for hand calculation by reducing the number of steps in the calculation. He provided the MATLAB code for the method; however, he did not prove his acclaimed method but rather gave examples.

In 2016, Sobamowo (2016)gave an extension of the Sarrus rule to 4×4 matrices. His method is the most successful of the Sarrus rule to 4×4 matrices.

CONCLUSION

The theory of determinant came into existence from the contributions of different authors, most of which have roles in the establishment of matrix theory since determinant provides information about the matrix. Nowadays, it is almost impossible to discuss determinant without considering its matrix. Evidently, determinant and matrix have played an important role beyond the field of mathematics.

CONFLICTS OF INTEREST

We, authors, have no potential conflicts of interest to disclose.

REFERENCES

Abeles, F. (1986). Determinants and linear systems: Charles L. Dodgson's view. *The British Journal for the history of science, 19*(03), 331-335.

Abeles, F. (1994). The mathematical pamphlets of Charles Lutwidge Dodgson and related pieces. Lewis Carroll Society of North America.

Abeles, F. (2008). Dodgson condensation: The historical and mathematical development of an experimental method. *Linear Algebra and its Applications, 429*(2-3), 429-438. doi:10.1016/j.laa.2007.11.022

Abeles, F. (2014). Chiò's and Dodgson's determinantal identities. *Linear Algebra and its Applications, 454*, 130-137. doi:http://dx.doi.org/10.1016/j.laa.2014.04.010

Afriat, S. (2000). Determinants and Matrices.Linear Dependence, Springer, Boston.

Ahmed, A. A. and Bondar, K. (2014). Modern method to compute the determinants of matrices of order 3. *Journal of Informatics and Mathematical Sciences, 6*(2), 55-60.

Aitken, A. C. (1956). Determinants and matrices. Edinburgh; New York: Oliver and Boyd; Interscience Publishers.

Akritas, A. G., Akritas, E. K. and Malaschonok, G. I. (1996). Various proofs of Sylvester's (determinant) identity. *Mathematics and Computers in Simulation, 42*(4–6), 585-593. doi:http://dx.doi.org/10.1016/S0378-4754(96)00035-3

Almalki, S., Alzahrani, S. and Alabdullatif, A. (2013). New parallel algorithms for finding determinants of N× N matrices. Paper presented at the Computer and Information Technology (WCCIT), 2013 World Congress on.

Amdeberhan, T. (2001). Determinants through the Looking Glass. *Advances in Applied Mathematics, 230*, 225-230. doi:10.1006/aama.2001.0732

Amdeberhan, T. and Ekhad, S. B. (1997). A Condensed Condensation Proof of a Determinant Evaluation Conjectured by Greg Kuperberg and Jim Propp. *Journal of Combinatorial Theory. Series A 78*, 169-170.

Athloen, S. and McLaughlin, R. (1987). Gauss-Jordan reduction: A brief history. *American Mathematical Monthly 94*, 130-142.

Babarinsa, O. (2020). Algebra in African Indigenous History. In A. Nhemachena, N. Hlabangane, and J. Matowanyika (Eds.), *Decolonising Science, Technology, Engineering and Mathematics (STEM) in an Age of Technocolonialism: Recentring African Indigenous Knowledge and Belief Systems* (pp. 199-212). Cameroon: LangaaRPCIG.

Babarinsa, O., Arif, M. and Kamarulhaili, H. (2019). Potential applications of hourglass matrix and its quadrant interlocking factorization. *ASM Science Journal, 12*(5), 72 - 79.

Babarinsa, O. and Kamarulhaili, H. (2018). Quadrant interlocking factorization of hourglass matrix.In the AIP Conference Proceedings of the 25th National Symposium on Mathematical Sciences (SKSM25).

Bareiss, E. H. (1968). Sylvester's identity and multistep integer-preserving Gaussian elimination. *Mathematics of computation, 22*(103), 565-578.

Bell, E. T. (2014). Men of mathematics: Simon and Schuster.

Benzi, M. (2009a). The Early History of Matrix Iterations: With a Focus on the Italian Contribution. Paper presented at the SIAM Conference on Applied Linear Algebra, Monterey Bay, Seaside, California.

Benzi, M. (2009b). Key moments in the history of numerical analysis. Paper presented at the SIAM Applied Linear Algebra Conference.

Berkhout, A. (2008). Signal Models in Seismic Processing. Handbook of Signal Processing in Acoustics (pp. 1559-1570): Springer.

Bernardes, A. and Roque, T. (2018). History of matrices Mathematics, Education and History: Springer. pp. 209-227

Bernstein, D. S. (2009). Matrix mathematics: theory, facts, and formulas. Princeton University Press.

Bézout, E. (1779). Théoriegénérale des équationsalgébriques; par m. Bézout: de l'imprimerie de Ph.-D. Pierres, rue S. Jacques.

Bézout, É. (1762). Sur plusieurs classes d'équations de tous les degrés qui admettentune solution algébrique. Histoire de l'Académieroyale des sciences, partieMémoires, 17-52.

Bôcher, M. and Duval, E. P. R. (1922). Introduction to higher algebra. Macmillan.

Bonolis, L. and de Laplace, P. S. (2004). From the Rise of the Group Concept to the Stormy Onset of Group Theory in the New Quantum Mechanics. A saga of the invariant characterization of physical objects, events, and theories. *Rivista del NuovoCimento, 27*(4-5), 39.

Boyer, C. B. (1966). Colin Maclaurin and Cramer's rule. *Scripta Mathematica, 27*(4), 377-379.

Bressoud, D. and Propp, J. (1999). How the alternating sign matrix conjecture was solved. *Notices-American Mathematical Society, 46*(i), 637-646.

Bronson, R. (1988). Schaum's outline of theory and problems of matrix operations. New York: McGraw-Hill.

Browne, E. T. (2018). Introduction to the Theory of Determinants and Matrices. UNC Press Books.

Brualdi, R. A. and Schneider, H. (1983). Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley. *Linear Algebra and its Applications, 52-53*(C), 769-791.

Brunetti, M. and Renato, A. (2014). Old and New Proofs of Cramer' s Rule.*Applied Mathematical Sciences, 8*(133), 6689-6697.

Burton, D. M. (2003). The history of mathematics. Mc Graw Hill.

Bylina, B. and Bylina, J. (2009). Influence of preconditioning and blocking on accuracy in solving Markovian models. *International Journal of Applied Mathematics and Computer Science, 19*(2), 207-217.

Campbell, H. G. (1980). Linear algebra with applications: Prentice Hall.

Cardano, G. (1993). Ars Magna or The Rules of Algebra: Transl. and ed. Retrieved from Dover: New York.

Cardano, G. and Spon, C. (1968). Ars magna (1545). *Opera Omnia, 4*, 221-302.

Cardano, G., Witmer, T. R. and Ore, O. (2007). The Rules of Algebra: Ars Magna. Courier Corporation., New York.

Cauchy, A. L. (1812). Mémoiresur les fonctions qui ne peuventobtenirquedeuxvaleurségales et de signescontraires par suite des transpositions opérées entre les variables qu'ellesrenferment. *Journal de l'Ecolepolytechnique,* 17, 29-112.

Cayley, A. (1845). On the theory of linear transformations. *Cambridge Math. J, 4*(1845), 1-16.

Cayley, A. (1858). A memoir on the theory of matrices. *Philosophical transactions of the Royal society of London, 148*, 17-37.

Chang, F. C. (2014). Determinant of matrix by order condensation. *Theory and Applications of Mathematical Science,1*, 73-78.
Chang, F. C. and Su, C.T. (1998). More on quick evaluation of determinants. *Applied Mathematics and Computation, 93*(1), 97-99. doi:http://dx.doi.org/10.1016/S0096-3003(97)10044-3

Channell, R. E. (1977). A compendium, the women of mathematics. (Master of Arts), Emporia State University, Kansas.
Chapra, S. C. and Canale, R. P. (1998). Numerical methods for engineers (Vol. 2). McGraw-Hill.

Chen, X.B. (2014). A fast algorithm for computing the determinants of banded circulant matrices. *Applied Mathematics and Computation, 229*, 201-207. doi:10.1016/j.amc.2013.12.048

Churchhouse, R. F. (2002). Codes and ciphers: Julius Caesar, the Enigma, and the Internet. Cambridge University Press.

Cinkir, Z. (2014). A fast elementary algorithm for computing the determinant of Toeplitz matrices. *Journal of Computational and Applied Mathematics, 255*, 353-361. doi:10.1016/j.cam.2013.05.014

Cormen, T. H. (2009). Introduction to algorithms: Cambridge, Mass. MIT Press.

Cramer, G. (1750). Introduction à l'analyse des lignescourbesalgébriques. *Europeana*, 656-659.

D'Andrea, A., Ferri, F. and Grifoni, P. (2010). An overview of methods for virtual social networks analysis.*Computational social network analysis*, 3-25.

Debnath, L. (2013a). A brief historical introduction to determinant with applications. *International Journal of Mathematical Education in Science and Technology, 44*(3), 388-407.

Debnath, L. (2013b). The modern origin of matrices and their applications. *International Journal of Mathematical Education in Science and Technology, 45*(4), 528-551. doi:10.1080/0020739x.2013.851808

Degos, J.G. (2015). Brief History of Matrices, As a Tool of Consolidated Financial Statements. *MuhasebeveFinansTarihiAraştırmalarıDergisi,* 8: 51-78.

Descartes, R. (1886). La géométrie de René Descartes, vol. 1. Hermann.

Djungu, S.J. A. and Manneback, P. (2020). SpeedSiteRank: PageRank algorithm distributed in websites. *International Journal of Computer Science Issues, 17*(2): 13-18.

Dossey, J. A., Otto, A. D., Spence, L. E. and Eynden, C. V. (2001). Discrete mathematics. Harlow: Addison Wesley.

Dunham, C. B. (1980). Cramer's rule reconsidered or equilibration desirable. *ACM SIGNUM Newsletter, 15*(4), 9-9.

Eberly, D. H. (2001). 3D game engine design : a practical approach to real-time computer graphics. San Francisco: Morgan Kaufmann.

El-Mikkawy, M. E. (2008). A fast and reliable algorithm for evaluating nth order pentadiagonal determinants. *Applied Mathematics and Computation, 202*(1), 210-215.

Eves, H. (1969). An introduction to the history of mathematics. New York: Holt, Rinehart and Winston.

Eves, H. W. (1980). Elementary matrix theory: Courier Corporation.

Feng, Q. and Zhou, Y. (2014). Soft discernibility matrix and its applications in decision making. *Applied Soft Computing, 24*, 749-756.

Francisco Neto, A. (2015). A note on a determinant identity. *Applied Mathematics and Computation, 264*, 246-248. doi:http://dx.doi.org/10.1016/j.amc.2015.04.079

Franklin, J. N. (1968). Matrix theory. Englewood Cliffs, N.J.: Prentice-Hall.

Gauss, C. F. (1966). Disquisitionesarithmeticae (Vol. 157). Yale University Press.

Godin, L., Demours, P. and Cotte, L. (1774). Table alphabétique des matièrescontenuesdansl'Histoire et les Mémoires de l'Académie Royale des Sciences (Vol. 8): Comp. des libraires.

Grcar, J. F. (2011). How ordinary elimination became Gaussian elimination. *Historia Mathematica, 38*(2), 163-218.

Grcar, J. F. (2012). Review of The Chinese Roots of Linear Algebra by Roger Hart. *Bull. Amer. Math. Soc, 49*(4), 589.

Gu, C. and Xu, Z. (2008). Condensed cramer rule for computing a kind of restricted matrix equation. *Journal of Applied Mathematics and Informatics, 26*(5), 1011-1020.

Günther, S. (1908). Geschichte der Mathemat. G.J. Göschen. Berlin.

Gutman, I. (1977). Acyclic systems with extremalHückel π-electron energy. *Theoreticachimicaacta, 45*(2), 79-87.

Habgood, K. and Arel, I. (2010). Revisiting Cramer's rule for solving dense linear systems.In the Proceedings of the 2010 Spring Simulation Multiconference.

Habgood, K. and Arel, I. (2012). A condensation-based application of Cramer’s rule for solving large-scale linear systems. *Journal of Discrete Algorithms, 10*, 98-109. doi:10.1016/j.jda.2011.06.007

Hadamard, J. (1897). Mémoiresurl'élimination. *Actamathematica, 20*(1), 201-238.

Hart, R. (2011). The Chinese roots of linear algebra. Baltimore: The John Hopkins University Press.

Hawe, P., Webster, C. and Shiell, A. (2004). A glossary of terms for navigating the field of social network analysis. *Journal of epidemiology and community health, 58*(12), 971-975.

Hawkins, T. (1974). The theory of matrices in the 19th century.In the Proceedings of the international congress of mathematicians, Vancouver.

Hedman, B. A. (1999). An Earlier Date for “Cramer's Rule”. *Historia Mathematica, 26*(4), 365-368.

Henry, N. and Fekete, J.D. (2007). Henry, Nathalie, and Jean-Daniel Fekete. Matlink: Enhanced matrix visualization for analyzing social networks. In *IFIP* Conference on Human-Computer Interaction, pp. 288-302.

Higham, N. J. (2002). Accuracy and stability of numerical algorithms. Siam.

Hoffman, J. D. and Frankel, S. (2001). Numerical methods for engineers and scientists. CRC press.

Horn, R. A. and Johnson, C. R. (2012). Matrix analysis. Cambridge university press.

Ivanov, S., Ivanova, L. and Meleshkova, Z. (2020). Calculation and Optimization of Industrial Robots Motion.In the Proceedings of the 2020 26th Conference of Open Innovations Association (FRUCT).

Jacobi, C. G. J. (1896). Ueber die bildung und die eigenschaften der determinanten:(De formatione et proprietatibusdeterminantium.): W. Engelmann.

Jaffe, A. (1984). Ordering the universe: the role of mathematics. *SIAM Review, 26*(4), 473-500.

Janjia, M. (2005). A note on Laplace's expansion theorem. *International Journal of Mathematical Education in Science and Technology, 36*(6), 696-698.

Jeffrey, A. (2010). Matrix Operations for Engineers and Scientists: An Essential Guide in Linear Algebra. Springer Science and Business Media.

Ji, J. (2012). A condensed Cramer’s rule for the minimum-norm least-squares solution of linear equations. *Linear Algebra and its Applications, 437*(9), 2173-2178.

Kani, E. (2011). Idoneal numbers and some generalizations. *Mathematics Annalesmathématiques du Québec, 35*(2), 197-227.

Karim, S. (2013). New Sequential and Parallel Division Free Methods for Determinant of matrices,Ph.D thesis, Universiti Utara Malaysia, Malaysia.

Karim, S., Ibrahim, H. and Omar, Z. (2016). Some modifications of Sarrus’s rule method via permutation for finding determinant of 4 by 4 square matrix.In the Proceedings of the AIP Conference 2016.

Kariuki, S. and Löwe, K. (2007). Integrating human factors into process hazard analysis. *Reliability Engineering and System Safety, 92*(12), 1764-1773.

Kendall, D. G. (1949). Stochastic processes and population growth. *Journal of the Royal Statistical Society. Series B (Methodological), 11*(2), 230-282.

Kippenhahn, R. (1999). Code breaking: History and exploration. Universities Press.

Kline, M. (1990). Mathematical thought from ancient to modern times (Vol. 3). Oxford University Press.

Klinger, A. (1967). The Vandermonde Matrix. *The American Mathematical Monthly, 74*(5), 571-574. doi:10.2307/2314898

Knobloch, E. (1994). From Gauss to Weierstrass: determinant theory and its historical evaluations, The intersection of history and mathematics (pp. 51-66): Springer.

Knobloch, E. (2013). Leibniz’s Theory of Elimination and Determinants. In Seki, Founder of Modern Mathematics in Japan. Springer, Tokyo, pp. 229-244

Kosinski, A. (2001). Cramer's rule is due to Cramer. *Mathematics Magazine, 74*(4), 310-312.

Kronecker, L. (1903). Vorlesungen über die Theorie der Determinanten, Erster Band, Bearbeitet und fortgeführt von K. Hensch, BG Teubner, Leipzig.

Kruh, L. and Deavours, C. (2002). The commercial enigma: beginnings of machine cryptography. *Cryptologia, 26*(1), 1-16.

Kyrchei, I. (2008). Cramer's rule for quaternionic systems of linear equations. *Journal of Mathematical Sciences, 155*(6), 839-858.

Kyrchei, I. I. (2015). Cramer's rule for generalized inverse solutions. *Advances in Linear Algebra Research, Nova Sci. Publ., New York,* 79-132.

Lagrange, J. L. (1775). Recherchesd'arithmetique:*Nouveaux Mémoires de l’Académie de Berlin,*

Lancaster, P. and Tismenetsky, M. (1985). The theory of matrices: With applications. Orlando: Academic Press.

Lefkovitch, L. (1965). The study of population growth in organisms grouped by stages. *Biometrics*, 1-18.

Leggett, D., Perry, J. E. and Torrence, E. (2009). Generalizing Dodgson's method: a" double-crossing" approach to computing determinants. *arXiv preprint arXiv: …*, 1-14. doi:10.4169/college.math.j.42.1.043

Lengyel, E. (2012). Mathematics for 3D game programming and computer graphics. Boston: Course Technology PTR.

Lenhart, S. M. and Travis, C. C. (1986). Global stability of a biological model with time delay. In the Proceedings of the American Mathematical Society, pp: 75-78.

MacBeth, C. and Li, X.Y. (1996). Linear matrix operations for multicomponent seismic processing. *Geophysical Journal International, 124*(1), 189-208.

MacDuffee, C. (1934). The theory of matrices. *Bull. Amer. Math. Soc, 40*, 372-373.

MacLaurin, C. (1748). A treatise of algebra. London: A. Millar, and J. Nourse.

Mangal, A., Malik, H. and Aggarwal, G. (2020). An Efficient Convolutional Neural Network Approach for Facial Recognition.In the proceedings of the 2020 10th International Conference on Cloud Computing, Data Science and Engineering (Confluence).

Martzloff, J. C. (2008). Mathematics in Japan Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures (pp. 1396-1400): Springer.

Mehra, J. and Rechenberg, H. (1982). The historical development of quantum theory. New York: Springer-Verlag.

Miller, G. A. (1930). On the History of Determinants. *The American Mathematical Monthly, 37*(5), 216-219. doi:10.2307/2299112

Mills, W. H., Robbins, D. P. and Rumsey, H. (1986). Self-complementary totally symmetric plane partitions. *Journal of Combinatorial Theory, Series A, 42*(2), 277-292. doi:http://dx.doi.org/10.1016/0097-3165(86)90098-1

Moler, C. (1974). Cramer's rule on 2-by-2 systems. *ACM SIGNUM Newsletter, 9*(4), 13-14.

Muir, T. (1881). The Law of Extensible Minors in Determinants. *Transactions of the Royal Society of Edinburgh, 30*(01), 1-4.

Muir, T. (1906). The theory of determinants in the historical order of development (Vol. 1): Macmillan and Company, limited.

Muir, T. (1911a). History of the Theory of Determinants (Vol. I-III). London: MacMillan.

Muir, T. (1911b). The Theory of Determinants in the Historical Order of Development (Vol. II): Macmillan and Company, limited.

Pan, V. Y., Yu, Y. and Stewart, C. (1997). Algebraic and numerical techniques for the computation of matrix determinants. *Computers and Mathematics with Applications, 34*(1), 43-70.

Pickover, C. A. (2011). A passion for mathematics: numbers, puzzles, madness, religion, and the quest for reality. John Wiley and Sons.

Rao, C. R. and Mitra, S. K. (1972). Generalized inverse of a matrix and its applications.In the Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability.pp. 601-620.

Raptis, M., Kirovski, D. and Hoppe, H. (2011). Real-time classification of dance gestures from skeleton animation.In the Proceedings of the 2011 ACM SIGGRAPH/Eurographics symposium on computer animation.

Rezaifar, O. and Rezaee, H. (2007). A new approach for finding the determinant of matrices. *Applied Mathematics and Computation, 188*(2), 1445-1454. doi:10.1016/j.amc.2006.11.010

Rice, A. and Torrence, E. (2006). Lewis Carroll ' s Condensation Method for Evaluating Determinants. *Mathematics Association of America*(November), 12-15.

Rice, A. and Torrence, E. (2007). Shutting up like a telescope": Lewis Carroll's" Curious Condensation Method for Evaluating Determinants. *The college mathematics journal, 38*(2), 85-95.

Robbins, D. and Rumsey, H. (1986). Determinants and alternating sign matrices. *Advances in Mathematics, 62*(2), 169-184. doi:10.1016/0001-8708(86)90099-X

Robbins, D. P. (2005). A conjecture about Dodgson condensation. *Advances in Applied Mathematics, 34*(4), 654-658.

Robinson, S. M. (1970). A short proof of Cramer's rule. Mathematics Magazine, 94-95.

Rohil, H. and Kaushik, P. (2014). Adjacency Matrix based Face Recognition Approach. *International Journal of Computer Applications, 98*(20).

Rothman, T. and Fukagawa, H. (1998). Japanese temple geometry. *Scientific American, 278*(5), 84-91.

Russell, B. and Whitehead, A. N. (1913). Principia mathematica to* 56. Cambridge UK: Cambridge University Press.

Saaty, T. L. (2003). Decision-making with the AHP: Why is the principal eigenvector necessary. *European journal of operational research, 145*(1), 85-91.

Salihu, A. (2012). New Method to Calculate Determinants of n × n (n ≥ 3) Matrix , by Reducing Determinants to 2nd Order.*International Journal of Algebra 6*(19), 913-917.

Schmidt, A. D. and Greene, J. R. (2011). Dodgson's Determinant: A Qualitative Analysis. *Journal of Linear Algebra, 2*(13), 34-54.

Searle, S. R. (2000). The infusion of matrices into statistics. *IMAGE: Bulletin of international linear algebra society, 24*, 25–32.

Shafarevich, I. and Remizov, A. (2013). Matrices and Determinants. *Linear Algebra and Geometry SE - 2*, 25-77 LA - English.

Shallit, J. (1994). Origins of the analysis of the Euclidean algorithm. *Historia Mathematica, 21*(4), 401-419. doi:https://doi.org/10.1006/hmat.1994.1031

Shen, K., Crossley, J. N., Lun, A. W. C. and Liu, H. (1999). The nine chapters on the mathematical art: Companion and commentary. Beijing: Oxford University Press.

Shiflet, A. B. and Shiflet, G. W. (2011). Introducing Matrix Operations through Biological Applications. *Journal Of Computational Science, 2*(1),15-20.

Shores, T. S. (2007). Applied linear algebra and matrix analysis: Springer Science and Business Media.

Sobamowo, M. (2016). On the Extension of Sarrus' Rule to Matrices: Development of New Method for the Computation of the Determinant of Matrix. *International Journal of Engineering Mathematics, 2016*.

Sobczyk, G. (2002). Generalized Vandermonde determinants and applications. *AportacionesMatematicas, SerieComunicaciones, 30*, 203-213.

Stocco, L. J., Salcudean, S. E. and Sassani, F. (1999). On the use of scaling matrices for task-specific robot design. *IEEE Transactions on Robotics and Automation, 15*(5), 958-965.

Studnička, F. J. (1876). AL Cauchy alsformalerbegründer der determinanten-theorie: Eineliterarische-historischestudie (Vol. 8): Verl der Königl. BöhmischenGesellschaft der Wissenschaften.

Sueur, J., Aubin, T. and Simonis, C. (2008). Equipment review: seewave, a free modular tool for sound analysis and synthesis. *Bioacoustics, 18*(2), 213-226.

Sylvester, J. J. (1867). Thoughts on inverse orthogonal matrices, simultaneous signsuccessions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 34*(232), 461-475.

Sylvester, J. J. and Baker, H. F. (2012). *The collected mathematical papers of James Joseph Sylvester* (Vol. 3): Cambridge University Press.

Taheri, S. M., Boostanpour, J. and Mohammadi, B. (2013). A Novel Algorithm for determinant calculation of N× N matrix. In the Proceedings of the 2013 International Conference on Advances in Computing, Communications and Informatics (ICACCI).

Tarokh, V., Seshadri, N. and Calderbank, A. R. (1998). Space-time codes for high data rate wireless communication: Performance criterion and code construction. *Information Theory, IEEE Transactions on, 44*(2), 744-765.

Tarski, A. (1946). Introduction to Logic and the Methodology of Deductive Sciences. Dover Publication, Inc.

Tirkkonen, O. and Hottinen, A. (2002). Square-matrix embeddable space-time block codes for complex signal constellations. *Information Theory, IEEE Transactions on, 48*(2), 384-395.

Tucker, A. (1993). The growing importance of linear algebra in undergraduate mathematics. *The college mathematics journal, 24*(1), 3-9.

Tweedie, C. (1915). A study of the life and writings of Colin MacLaurin. *The mathematical gazette, 8*(119), 133-151.

Ufuoma, O. (2013). A New and Simple Method of Solving Large Linear Systems : Based on Cramer' s Rule but Employing Dodgson' s Condensation.In the Proceedings of the World Congress on Engineering and Computer Science, San Francisco.

Urbańska, A. (2008). Faster Combinatorial Algorithms for Determinant and Pfaffian. *Algorithmica*, *56*(1), 35-50. doi:10.1007/s00453-008-9240-9

Vaishnav, C., Choucri, N. and Clark, D. (2013). Cyber international relations as an integrated system. *Environment Systems and Decisions*, *33*(4), 561-576.

Vandermonde, A.T. (1772). Mémoiresurl'élimination. *Mémoires de l'Académie Paris, II*, 516-532.

Vein, R. and Dale, P. (1999). Determinants and their applications in mathematical physics.. Springer Science and Business Media.

Vysotskaya, A. (2018). Accounting Games: Using Matrix Algebra in Creating the Accounting Models. *Mathematics*, *6*(9), 152-160.

Watkins, D. S. (2004). Fundamentals of matrix computations (Vol. 64). John Wiley and Sons.

Weber, H. J. and Arfken, G. B. (2003). Essential mathematical methods for physicists.ISE: Elsevier.

Wedderburn, J. H. M. (1934). Lectures on matrices (Vol. 17). American Mathematical Soc.

Weld, L. G. (1893). A short course in the theory of determinants. Macmillan and Company.

Wexler, A. (1969). Computation of electromagnetic fields. *Microwave Theory and Techniques, IEEE Transactions on, 17*(8), 416-439.

Weyl, H. (1922). Space--time—matter. Dutton.

Yap, C.K. (2000). Fundamental problems of algorithmic algebra. New York: Oxford University Press.

Yoke, H. P. (1991). Chinese science: the traditional Chinese view. *Bulletin of the School of Oriental and African Studies*, *54*(3), 506-519.

Zeilberger, D. (1997). Dodgson's determinant-evaluation rule proved by two-timing men and women. *Electron. J. Combin, 4*(2), R22.