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A TYPE II HALF LOGISTIC EXPONENTIATED-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS TO SURVIVAL ANALYSIS

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ABSTRACT
Statisticians have created and proposed new families of distribution by extending or generalizing existing distributions. These families of distributions are made more flexible in fitting different types of data by adding one or more parameters to the baseline distributions. In this article, we present a new family of distributions called Type II half-logistic exponentiated-G family of distributions. We discuss some of the statistical properties of the proposed family such as explicit expressions for the quantile function, probability weighted moments, moments, generating function, survival and order statistics. The new family's sub-models were discussed. We discuss the estimation of the model parameters by maximum likelihood. Two real data sets were employed to show the usefulness and flexibility of the new family.

Keywords: Hazard rate, Survival, Exponentiated–G, Type II Half Logistic-G, Maximum likelihood, Moments, Order Statistics.

INTRODUCTION

Real-life events are frequently described using statistical distributions. Because of its efficiency, the theory of statistical distributions is extensively studied, and new distributions are created. There is a strong desire among statisticians to create statistical distributions that are more flexible. There have been many different types of generalized distributions developed and applied to various phenomena. In many fields, such as economics, engineering, biological studies, and environmental sciences, several continuous univariate distributions have been extensively used for data modeling. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. So, various classes of distributions have been developed by extending common families of continuous distributions. A new improvement for creating and extending traditional distributions is the generated family of continuous distributions. The newly generated families have been extensively researched in a variety of fields, and they provide greater application flexibility. Gupta *et.al.,* (1998) pioneered the exponentiated-G (E-G) class, Eugene *et.al*., (2002) pioneered beta-G, Marshall and Olkin (1997) pioneered Marshall-Olkin-G, Zografos and Balakrishnan (2009) pioneered gamma G distributions, Kumaraswamy Weibull G distribution by Cordeiro *et.al*, (2010), Ristic and Balakrishnan (2011) an Alternative Gamma G distribution, Kumaraswamy-G family by Cordeiro and Castro(2011), Kummer beta generalized family by Cordeiro *et.al*., (2012), a new methods for generating families of continuous distributions by Alzaatreh *et.al*., (2013).

Exponentiated T-X family by Alzaghal *et.al*., (2013), Weibull-G family by Silva *et.al*., (2014), Exponentiated half-logistic family by Cordeiro *et.al*., (2014a), Lomax Generator by Cordeiro *et.al*., (2014b), Logistic-generated distribution by Torabi and Montazari (2014), beta Marshall-Olkin family of distributions by Alizadeh *et.al.*, (2015a), Kumaraswamy OddLog-Logistic-G by Alizadeh *et.al*., (2015b), the logistic-X family by Tahir *et.al*., (2016), the Topp-Leone family of distributions by Al-Shomrani *et.al*., (2016), Exponentiated Marshal-Olkin family of distributions by Dias *et.al*., (2016), Generalized Burr-G family of distributions by Nasir *et.al*.,(2017), Odd Exponentiated Half-Logistic-G family by Afify *et.al.*, (2017), Type II Half-Logistic Family of Distributions by Hassan *et.al.,* (2017), Weibull-X family of distributions by Ahmad *et.al.*, (2018), Marshall-Olkin generalized-G (MOG-G) family of distribution by Yousof*et.al.*, (2018), Exponentiated Kumaraswamy-G by Silva *et.al*., (2019) More recently, Topp-Leone exponentiated-G(TLEx-G) family of distributions by Ibrahim *et.al*., (2020a), Topp Leone Kumaraswamy-G family of distribution by Ibrahim *et.al*., (2020b), Burr X Exponential-G Family by Sanusi *et.al*., (2020a), Topp-Leone Exponential-G family of distributions by Sanusi *et.al*., (2020b), Kumaraswamy-Odd Rayleigh-G family by Falgore and Doguwa (2020a), Inverse Lomax-G (IL-G) family of distributions by Falgore and Doguwa (2020b), Inverse Lomax Exponentiated G family of distributions by Falgore and Doguwa (2020c), Weibull-Odd Frechet family of distributions by Usman *et.al*., (2020), Type II Exponentiated Half-logistic family of distributions by Al-Mofleh *et.al.*, (2020), Odd Weibull-Topp-Leone-G power series family of distributions by Broderick *et.al*., (2021), Exponentiated Half Logistic Odd Lindley-G (EHLOL-G) distribution by Whatmore *et.al*.,(2021). The ability to model both monotonically and non-monotonically increasing, decreasing and constant or more importantly with bathtub-shaped failure rates, even if the baseline failure rate is monotonic, is the motivation for extending distributions for modeling lifetime data. The following are the basic vindications for creating a new family of distributions in practice: to generate skewness for symmetrical models; to generate distributions with negatively skewed, positively skewed, and symmetric; to define special models with all types of hazard rate functions; to make the kurtosis more flexible than that of the baseline distribution, to construct tail weight distributions for modeling various real data sets; to provide consistently better fits than other generated distributions with the same underlying model.

The aim of this paper is to develop and investigate the Type II Half-Logistic Exponentiated-G family of distributions (TIIHLEt-G).

MATERIALS AND METHOD

Type II Half-Logistic Exponentiated-G Family

The Exponentiated G (Et-G) family of distributions as defined in Ibrahim *et.al.*, (2020a) has cumulative distribution function (cdf) given as

$$F_{EG}(x;\alpha,\boldsymbol{\beta}) = \left[H(x;\boldsymbol{\beta})\right]^{\alpha} \tag{1}$$

and its corresponding probability density function (pdf) is defined as

$$f_{EG}(x;\alpha,\boldsymbol{\beta}) = \alpha h(x,\boldsymbol{\beta})\left[H(x;\boldsymbol{\beta})\right]^{\alpha-1} \quad ,x>0 \tag{2}$$

where α > 0 is the shape parameter and h(x; **β**) and H(x; **β**) are the pdf and cdf of the baseline distribution with parameter vector **β**.

Hassan *et.al.*, (2017) proposed the Type II half logistic family (TIIHL) of distributions with cdf defined as:

$$F_{TIIHL}(x;\lambda,\boldsymbol{\xi}) = \frac{2\left[1-G(x,\boldsymbol{\xi})\right]^{\lambda}}{1+\left[G(x,\boldsymbol{\xi})\right]^{\lambda}} \tag{3}$$

and its corresponding pdf is

$$f_{TIIHL}(x,\lambda,\boldsymbol{\xi}) = \frac{2\lambda g(x;\xi)\left[G(x;\boldsymbol{\xi})\right]^{\lambda-1}}{\left[1+\left[G(x;\boldsymbol{\xi})\right]^{\lambda}\right]^{2}} \tag{4}$$

where λ > 0, x > 0 and G(x; ξ) and g(x; ξ) are the cdf and pdf of the baseline distribution with parameter vector ξ.

Proposition

The cdf of a new family of distribution that extends the TIIHL family called Type II Half-Logistic Exponentiated-G Family of distributions is given as

$$F_{TIIHLEt\_G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2H^{\alpha\lambda}(x;\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]} \tag{5}$$

and the pdf is derived as

$$f_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{\partial F_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta})}{\partial x}$$

$$= \frac{2\lambda\alpha h(x;\boldsymbol{\beta})H^{\alpha-1}(x;\boldsymbol{\beta})\left[H^{\alpha(\lambda-1)}(x;\boldsymbol{\beta})\right]}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^{2}}, x>0, \lambda,\alpha>0 \;\; \text{and } \boldsymbol{\beta} \text{ is parameter vector} \tag{6}$$

Proof:

Let the Exponentiated G family be the baseline family with cdf and pdf given in equations (1) and (2) respectively, then the proposed Type II Half Logistic Exponentiated-G family of probability distributions has cdf given as:

$$F_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) \int_{0}^{\left[H(x;\boldsymbol{\beta})\right]^{\alpha}} \frac{2\lambda\alpha h(t;\boldsymbol{\beta})H^{\alpha-1}(t;\boldsymbol{\beta})\left[H^{\alpha}(t;\boldsymbol{\beta})\right]^{\lambda-1}}{\left[1+\left[H^{\alpha}(t;\boldsymbol{\beta})\right]^{\lambda}\right]^{2}}dt$$

Let y = $\left[H(t;\boldsymbol{\beta})\right]^{\alpha}$; when t = 0, y = 0; when t = x, y = $\left[H(x;\boldsymbol{\beta})\right]^{\alpha}$; ∂y=αh(t;**β**)$\left[H(t;\boldsymbol{\beta})\right]^{\alpha-1}$

Then

$$F_{TIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \int_0^{[H(x;\boldsymbol{\beta})]^\alpha} \frac{2\lambda y^{\lambda-1}}{\left[1+y^\lambda\right]^2}\, dy \tag{7}$$

Let

$$M = \left[1+y^\lambda\right]^2 \text{ when } y=0,\ m=1; \text{ when } H^\alpha(x;\boldsymbol{\beta}), m = \left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^2;\ \partial m = 2\left[1+y^\lambda\right]\lambda y^{\lambda-1}\partial y$$

$$\therefore \partial y = \frac{\partial m}{2\left[1+y^\lambda\right]\lambda y^{\lambda-1}}$$

$$F_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \int_0^{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^2} m^{-\frac{3}{2}}\ \partial m\ = \left[-2m^{-\frac{1}{2}}\right]_1^{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^2} = -2\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]^{-1}+2$$

$$= 2 - \frac{2}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]} = 2\left[\frac{1+H^{\alpha\lambda}(x;\boldsymbol{\beta})}{1+H^{\alpha\lambda}(x;\boldsymbol{\beta})}\right] - \frac{2}{1+H^{\alpha\lambda}(x;\boldsymbol{\beta})} = \frac{2+2H^{\alpha\lambda}(x;\boldsymbol{\beta})-2}{1+H^{\alpha\lambda}(x;\boldsymbol{\beta})}$$

Thus

$$F_{TIIHLEt-G}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2H^{\alpha\lambda}(x;\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\boldsymbol{\beta})\right]} \tag{8}$$

Important Representation

In this section, we derived a useful representation for the TIHLEt−G pdf and cdf. Due to the fact that the generalized binomial series is

$$\left(1+Z\right)^{-\beta} = \sum_{i=0}^{\infty}(-1)^i\binom{\beta+i-1}{i}z^i, \tag{9}$$

For $|z|<1,$ and β is a positive real non integer. The density function of the TIIHLEt-G family is then obtained by using the binomial theorem (9) to (6).

$$f(x) = \sum_{i=0}^{\infty} 2\lambda\alpha h(x)(-1)^i(i+1)H_{(x)}^{\alpha\lambda(i+1)-1} \tag{10}$$

Then, the pdf (10) can be written as:

$$f(x) = \sum_{i=0}^{\infty}\vartheta_i h(x)H_{(x)}^{\alpha\lambda(i+1)-1} \tag{11}$$

Where

$$\vartheta_i = 2\lambda\alpha(-1)^i(i+1)$$

In addition, an expansion for the $\left[F(x)\right]^h$ is produced, with h being an integer, and the binomial expansion is worked out once more.

$$\left[F(x)\right]^h = \sum_{j=0}^{h}2^h(-1)^j\binom{h+j-1}{j}H_{(x)}^{\alpha\lambda(j+h)} \tag{12}$$

Moreso, the binomial expansion is applied to $H^{\alpha\lambda(j+h)}(x)$ by adding and subtracting 1, so $\left[F(x)\right]^h$ can now be expressed as follows

$$\left[F(x)\right]^h = \sum_{j=0}^{h}\sum_{m=0}^{\infty}2^h(-1)^{j+m}\binom{h+j-1}{j}\binom{\alpha\lambda(j+h)}{m}\left[1-H(x)\right]^m \tag{13}$$

For m is a real, then $\left[F(x)\right]^h$ is as follows

$$\left[F(x)\right]^{h} = \sum_{p=0}^{\infty} t_p H(x)^p \qquad (14)$$

Where

$$t_p = \sum_{j=0}^{h}\sum_{m=0}^{\infty} 2^h (-1)^{j+m+p} \binom{h+j-1}{j}\binom{\alpha\lambda(j+h)}{m}\binom{m}{p}$$

Statistical Properties

In this section, we derived statistical properties of the new family of distribution.

Probability weighted moments

Greenwood *et.al.*, (1979) introduced a class of moments known as probability weighted moments (PWMs). This class is used to derive inverse form estimators for the parameters and quantiles of a distribution. The PWMs, represented by $\tau_{r,s}$, can be derived for a random variable X using the following relationship.

$$\tau_{r,s} = E\left[X^r F(X)^s\right] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s dx \qquad (15)$$

The PWMs of TIIHLEt-G is derive by substituting (11) and (14) into (15), and replacing h with s, as follows

$$\tau_{r,s} = \int_0^{\infty}\sum_{i=0}^{\infty}\sum_{p=0}^{\infty} t_p \vartheta_i x^r h(x) H^{p+\alpha\lambda(i+1)-1} dx \qquad (16)$$

Now ,

$$\tau_{r,s} = \sum_{i=0}^{\infty}\sum_{p=0}^{\infty} t_p \vartheta_i \tau_{r,\,p+\alpha\lambda(i+1)-1} \qquad (17)$$

Where,

$$\tau_{r,\,p+\alpha\lambda(i+1)-1} = \int_0^{\infty} x^r h(x) H^{p+\alpha\lambda(i+1)-1}(x)\, dx$$

Another formula will be obtained by applying the quantile function, as follows:

$$\tau_{r,s} = \int_0^{1}\sum_{i,p=0}^{\infty} t_p \vartheta_i (Q_H(u))^r u^{p+\alpha\lambda(i+1)-1} \partial u \qquad (18)$$

Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the r^{th} moment for the new family.

$$\mu_r^{'} = E(x^r) = \int_0^{\infty} x^r f(x) dx \qquad (19)$$

By using the important representation of the pdf in equation (11), we have

$$\mu_r^{'} = \int_0^{\infty}\sum_{i=0}^{\infty} \vartheta_i x^r h(x) H^{\alpha\lambda(i+1)-1}(x)\, \partial x \qquad (20)$$

Then, $\mu_r^{'} = \sum_{i=0}^{\infty} \vartheta_i \tau_{r,\alpha\lambda(i+1)-1}$

Where $\tau_{r,\alpha\lambda(i+1)-1}$ is the PWMs.

Also, moment can be obtained, based on the parent quantile function, as follow;

$$\mu_r^{'} = \sum_{i=0}^{\infty} \vartheta_i \int_0^1 (Q_H(u))^r u^{\alpha\lambda(i+1)-1} \partial u \tag{21}$$

Moment generating function (mgf)

The Moment Generating Function of x is given as:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'} = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^r}{r!} \vartheta_i \tau_{r,\alpha\lambda(i+1)-1} \tag{22}$$

Another formula will be obtained by applying the quantile function, as follows:

$$M_u(t) = \sum_{i=0}^{\infty} \vartheta_i \int_0^1 e^{(tQ_H(u))} u^{\alpha\lambda(i+1)-1} \partial u \tag{24}$$

Mean deviation

The mean deviation about the mean $\left[\sigma_2 = E(|x - \mu_1^{'}|)\right]$ and about the median $\left[\sigma_1 = E(|x - M|)\right]$ of x given as

$$\sigma_1 = 2\mu_1^{'} F(\mu^{'}) - 2J(\mu_1^{'}) \tag{25}$$

$$\sigma_2 = \mu_1^{'} - 2J(M) \tag{26}$$

Where $\mu_1^{'} = E(x)$, is the mean, M = Median(x) = Q(0.5), is the median and $j(c) = \int_0^x xf(x)dx$ which is the first incomplete moment.

Additional form is derived as follows, depending on the parent quantile function.

$$j(c) = \sum_{i=0}^{\infty} \vartheta_i \int_0^x Q_H(u) u^{\alpha\lambda(i+1)-1} \partial u \tag{27}$$

Reliability function

The reliability function which is also known as survivor function that gives the probability that a patient will survive longer than specified period of time. It is defined as

$$R(x;\lambda,\alpha,\boldsymbol{\beta}) = 1 - F(x;\lambda,\alpha,\boldsymbol{\beta}) \tag{28}$$

$$R(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{1 - H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}{1 + H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})} \tag{29}$$

Obtaining survival probabilities for different values of time x provides crucial summary information from survival data.

Hazard function

The hazard function is the probability of an event of interest occurring within a relatively short time frame and is defined as:

$$T(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta}) H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta}) H^{\alpha(\lambda-1)}(x;\lambda,\alpha,\boldsymbol{\beta})}{1 - H^{2\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})} \tag{30}$$

The hazard function also known as conditional failure rate, gives the instantaneous potential per unit time for the event of interest to occur, given that the individual has survived up to time x.

Quantile Function

The quantile function is a vital tool to create random variables from any continuous probability distribution. As a result, it has a significant position in probability theory. For x, thequantile function is F(x) = u, where u is distributed as U(0,1).The TIIHLEt-G family is easily simulated by inverting equation (5) which yields the Quantile function Q(u) defined as:

$$Q(u) = H^{-1}\left\{\frac{u}{2-u}\right\}^{\frac{1}{\alpha\lambda}} \quad (31)$$

Where H^{-1} is the quantile function of the baseline cdf H(x; β). The first quartile, the median and the third quartile are obtained by putting u = 0.25, 0.5 and 0.75, respectively in equation (4.6).

Order Statistics

Many areas of statistics including reliability and life testing, have made substantial use of order statistics. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with their corresponding continuous distribution function F (x). Let $X_1, X_2,.., X_n$ be n independently distributed and continuous random variables from the TIHLEt-G family of distribution. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, r = 1, 2, 3, ..., n denote the cdf and pdf of the r^{th} order statistics $X_{r:n}$ respectively. David (1970) gave the probability density function of $X_{r:n}$ as:

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)}F^{r-1}(x)\left[1-F(x)\right]^{n-r} f(x) \quad (32)$$

By substituting equation (5) and equation (6) into equation (32), we have,

$$f_{r:n}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{1}{B(r,n-r+1)}\left[\frac{2H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}{1+H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}\right]^{r-1}\left[\frac{1-H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}{1+H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}\right]^{n-r} \times \frac{2\lambda\alpha h(x)H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha(\lambda-1)}(x;\lambda,\alpha,\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^2} \quad (33)$$

The equation above is called the r^{th} order statistics for the TIIHLEt-G family of distributions.
Let r = n, then the probability density function of the maximum order statistics is

$$f_{n:n}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2n\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha(\lambda-1)}(x;\lambda,\alpha,\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^2}\left[\frac{2H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}{1+H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}\right]^{n-1} \quad (34)$$

Also, let r = 1, then the probability density function of the minimum order statistics is

$$f_{1:n}(x;\lambda,\alpha,\boldsymbol{\beta}) = \frac{2n\lambda\alpha h(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha-1}(x;\lambda,\alpha,\boldsymbol{\beta})H^{\alpha(\lambda-1)}(x;\lambda,\alpha,\boldsymbol{\beta})}{\left[1+H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})\right]^2}\left[\frac{1-H^{\alpha\lambda}(x;\lambda,\alpha,\boldsymbol{\beta})}{1+H^{\alpha\lambda}}\right]^{n-1} \quad (35)$$

Sub-Models

In this section, we describe two sub-models of the TIIHLEt-G family namely, TIIHLEt- Exponential and TIIHLEt-Log-logistic respectively.

Type II Half-Logistic Exponentiated Exponential (TIIHLEtE) Distribution

The cdf and pdf of Exponential distribution which is our baseline distribution with parameter θ are:

$$H(x;\theta) = 1-e^{-\theta x} \quad (36)$$

And

$$h(x;\theta) = \theta e^{-\theta x}, \quad x>0, \theta>0 \quad (37)$$

The TIIHLEtE has cdf and pdf as follows:

$$F_{TIIHLEtE}(x;\lambda,\alpha,\theta) = \frac{2\left[1-e^{-\theta x}\right]^{\alpha\lambda}}{1+\left[1-e^{-\theta x}\right]^{\alpha\lambda}}, x>0, \lambda,\alpha,\theta>0 \quad (38)$$

And

$$f(x;\lambda,\alpha,\theta) = \frac{2\lambda\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{\alpha-1}\left[1-e^{-\theta x}\right]^{\alpha(\lambda-1)}}{\left[1+\left[1-e^{-\theta x}\right]^{\alpha\lambda}\right]^2} \quad (39)$$

Furthermore, the following are the reliability function, hazard rate function and the quantile function respectively:

$$R(x;\lambda,\alpha,\theta)=\frac{1-\left[1-e^{-\theta x}\right]^{\alpha\lambda}}{1+\left[1-e^{-\theta x}\right]^{\alpha\lambda}} \tag{40}$$

$$T(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{\alpha-1}\left[1-e^{-\theta x}\right]^{\alpha(\lambda-1)}}{1-\left[1-e^{-\theta x}\right]^{2\alpha\lambda}} \tag{41}$$

$$x=\frac{-1}{\theta}\ln\left[1-\left[\frac{u}{2-u}\right]^{\frac{1}{\alpha\lambda}}\right] \tag{42}$$

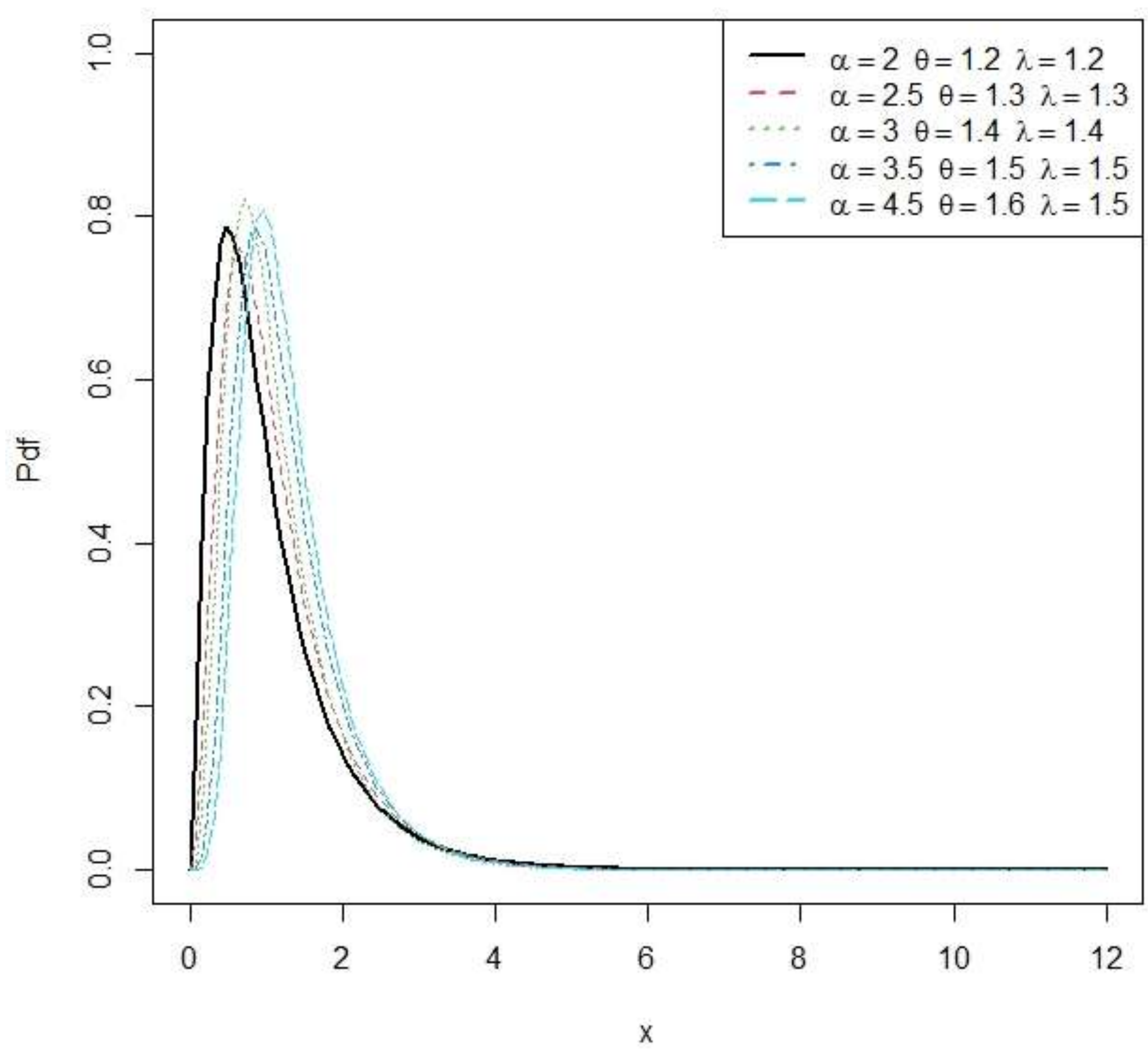

Figure1: Plots of PDF of the TIIHLEtE distribution for different parameter values.

It can be seen from figure 1 that the TIIHLEtE distribution has a positively skewed shape.

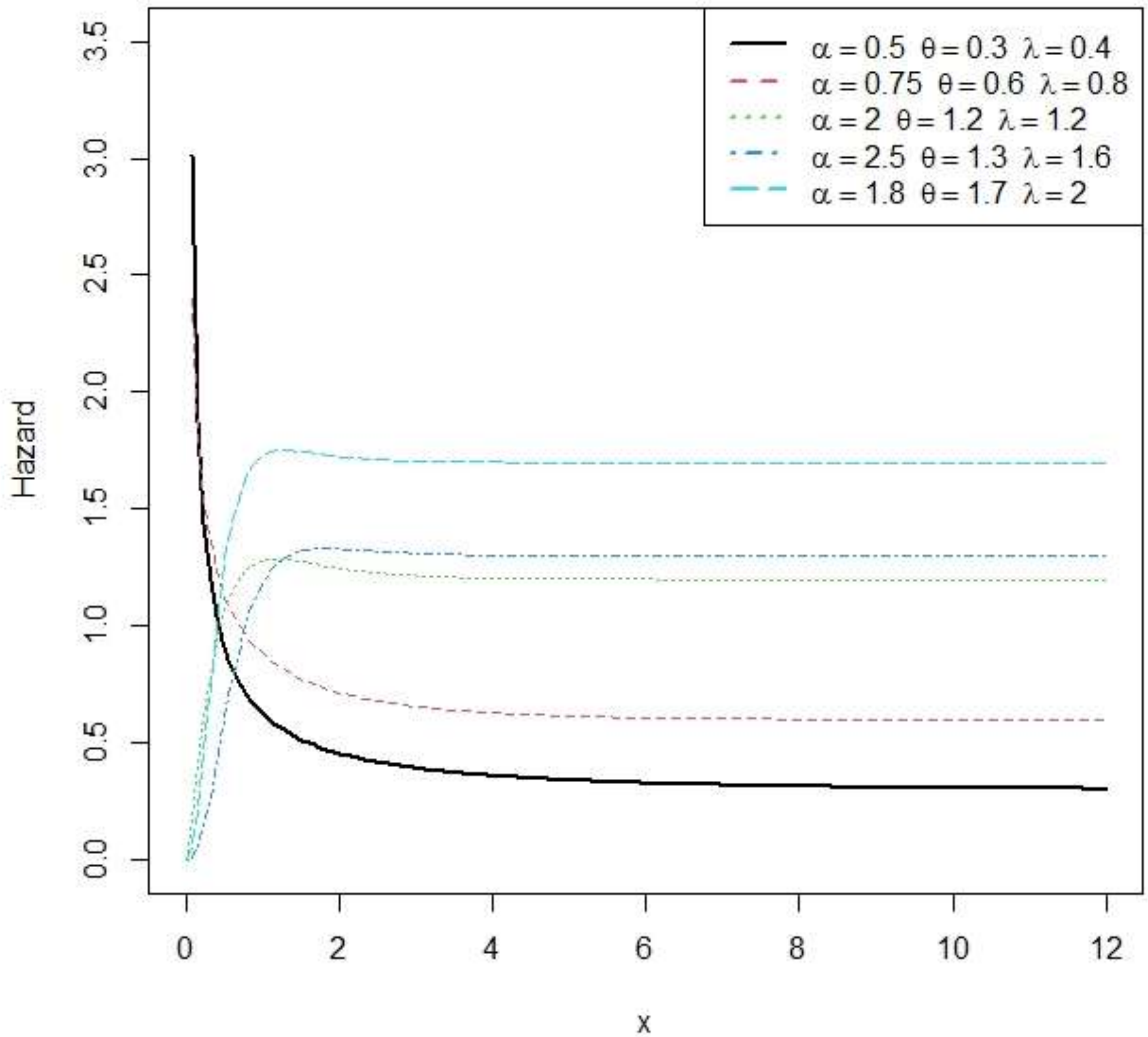

Figure 2: Plots of hazard of the TIIHLEtE distribution for different parameter values

The hazard function of the TIIHLEtE distribution has a monotonically increasing, decreasing, and constant failure rate, as shown in the figures 2.

Type II Half-Logistic Exponentiated Log-Logistic (TIIHLEtLL) Distribution

cdf and pdf of Log-Logistic distribution with parameter θ as our baseline distribution are:

$$H(x;\theta)=\frac{x^{\theta}}{1+x^{\theta}} \tag{43}$$

$$h(x;\theta)=\frac{\theta x^{\theta-1}}{(1+x^{\theta})^{2}},\quad x>0,\theta>0 \tag{44}$$

Now

$$H^{\alpha}(x;\theta,\alpha)=\left[\frac{x^{\theta}}{1+x^{\theta}}\right]^{\alpha\lambda}=\left[x^{-\theta}+1\right]^{-\alpha\lambda}$$

The TIIHLEtLL distribution has CDF and PDF given as;

$$F_{TIHLEtLL}(x;\lambda,\alpha,\theta)=\frac{2\left[x^{-\theta x}+1\right]^{-\alpha\lambda}}{1+\left[\left[x^{-\theta x}+1\right]^{-\alpha\lambda}\right]}, x>0,\lambda,\alpha,\theta>0 \quad (45)$$

$$f_{TIHLEtLL}(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x^{\theta-1}\left[x^{-\theta}+1\right]^{1-\alpha}\left[1+x^{\theta}\right]^{-2}\left[x^{-\theta}+1\right]^{-\alpha(\lambda-1)}}{\left[1+\left[x^{-\theta}+1\right]^{-\alpha\lambda}\right]^{2}} \quad (46)$$

Moreso, the following are the reliability function, hazard rate function and the quantile function respectively:

$$R(x;\lambda,\alpha,\theta)=\frac{1-\left[x^{-\theta}+1\right]^{-\alpha\lambda}}{1+\left[x^{-\theta}+1\right]^{-\alpha\lambda}} \quad (47)$$

$$T(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta x^{\theta-1}\left[1+x^{\theta}\right]^{-2}\left[x^{-\theta}+1\right]^{1-\alpha}\left[x^{-\theta}+1\right]^{-\alpha(\lambda-1)}}{1-\left[x^{-\theta}+1\right]^{2\alpha\lambda}} \quad (48)$$

$$x=\left[\left[\frac{u}{2-u}\right]^{\frac{1}{-\alpha\lambda}}-1\right]^{\frac{-1}{\theta}} \quad (49)$$

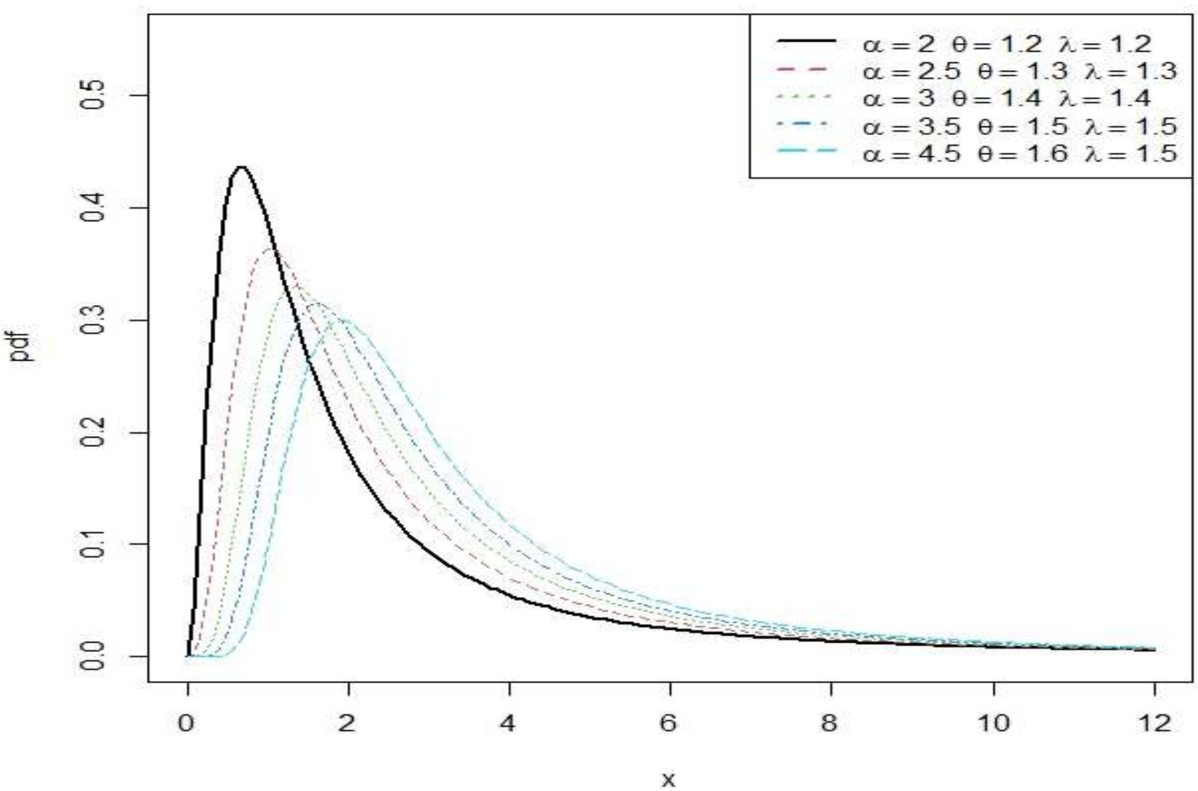

Figure 3: Plots of Pdf of the TIIHLEtLL distribution for different parameter values.

It can be seen from figure 3 that the TIIHLEtLL distribution has a positively skewed shape.

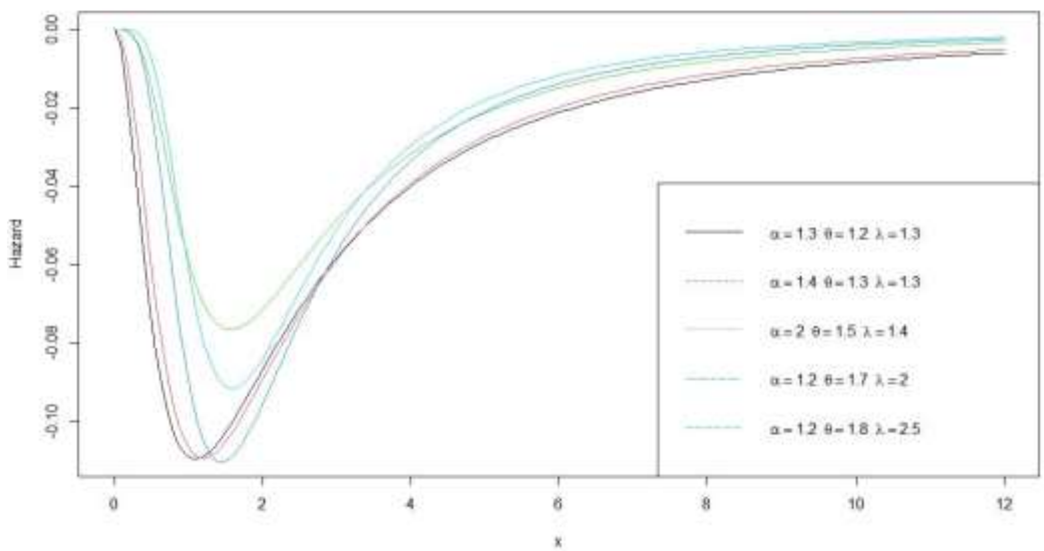

Figure 4: Plots of hazard of the TIHlEtLL distribution for different parameter values.

The TIIHLEtLL distribution hazard plots in figure 4 illustrate that the hazard function has a bathtub-shaped failure rate.

Parameter Estimation

In this paper, we explore the maximum likelihood technique to estimate the unknown parameters of the TIIHLEt-G family for complete data. Maximum likelihood estimates (MLEs) have appealing qualities that may be used to generate confidence ranges and provide simple approximations that function well in finite samples. In distribution theory, the resulting approximation for MLEs is easily handled, either analytically or numerically. Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from the TIIHLEt-G family. Then, the likelihood function based on observed sample for the vector of parameter $(\lambda, \alpha, \beta)^T$ is given by

$$L(\phi) = n\log(2) + n\log(\lambda) + n\log(\alpha) + \sum_{i=0}^{n}\log h(x,\beta) + (\alpha-1)\sum_{i=0}^{n}\log H(x,\beta)$$

$$+\alpha(\lambda-1)\sum_{i=0}^{n}\log H(x,\beta) - 2\sum_{i=0}^{n}\log\left[1+H^{\alpha\lambda}(x,\beta)\right] \quad (50)$$

The components of score vector $U = (U_\lambda, U_\alpha, U_\beta)^T$ are given as

$$U_\lambda = \frac{n}{\lambda} + \sum_{i=0}^{n}\log\left[H(x,\beta)\right] - 2\sum_{i=0}^{n}\frac{H^{\alpha\lambda}(x,\beta)\log\left[H^{\alpha}(x,\beta)\right]}{\left[1+H^{\alpha\lambda}(x,\beta)\right]} \quad (51)$$

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=0}^{n}\log H(x,\beta) + (\lambda-1)\sum_{i=0}^{n}\log H(x,\beta) - 2\lambda\sum_{i=0}^{n}\frac{H^{\alpha(\lambda-1)}(x,\beta)H^{\alpha}(x,\beta)\log\left[H(x,\beta)\right]}{\left[1+H^{\alpha\lambda}(x,\beta)\right]} \quad (52)$$

$$U_\beta = \sum_{i=0}^{n}\frac{h(x,\beta)^{\beta}}{h(x,\beta)} + (\alpha-1)\sum_{i=0}^{n}\frac{H(x,\beta)^{\beta}}{H(x,\beta)} + \alpha(\lambda-1)\sum_{i=0}^{n}\frac{H(x,\beta)^{\beta}}{H(x,\beta)}$$

$$-2\alpha\lambda\sum_{i=0}^{n}\frac{H^{\alpha(\lambda-1)}(x,\beta)H^{\alpha-1}(x,\beta)H(x,\beta)^{\beta}}{\left[1+H^{\alpha\lambda}(x,\beta)\right]} \quad (53)$$

The MLEs are obtained by setting U_λ, U_α and U_β to zero and solving these equations simultaneously. These Equations cannot be solved analytically, necessitating the use of analytical tools to solve them numerically.

Applications to Real Data

In this section, we fit the TIIHLEtE distribution to two real data sets and give a comparative study with the fits of the Type II half logistic exponential (TIHLE) by Elgarhy*et.al.*,(2018), Topp-Leone exponential distribution(TLEx) by Al-Shomrani *et.al*.,(2016), Kumaraswamy exponential distribution(KEx) by Adepoju and Chukwu (2015), Exponentiated exponential Distribution(ExEx) by Gupta and Kundu (1999), and Logistic-X exponential distribution (LoEx) by Oguntunde *et.al.*, (2018), as comparator distributions for illustrative purposes.

The TIHLE distribution proposed by Elgarhy *et.al*.,(2018) has probability density function given as:

$$f(x;\lambda,\alpha)=\frac{2\lambda\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{\lambda-1}}{\left[1+\left[1-e^{-\theta x}\right]^{\lambda}\right]^{2}} \tag{54}$$

The TLEx distribution proposed by Al-Shomrani *et.al*., (2016) has pdf defined as:

$$f(x;\alpha,\theta)=2\alpha\theta e^{-2\alpha x}\left[1-e^{-2\alpha x}\right]^{\theta-1} \tag{55}$$

The KE distribution developed by Adepoju and Chukwu (2015) has pdf defined as:

$$f(x;\alpha,\lambda,\theta)=\alpha\lambda\theta e^{-\alpha x}(1-e^{-\alpha x})\left[1-\left[1-e^{-\alpha x}\right]^{\alpha}\right]^{\lambda-1} \tag{56}$$

The ExEx distribution pioneered Gupta and Kundu (1999) has pdf given as:

$$f(x;\alpha,\theta)=\alpha\theta e^{-\alpha x}\left[1-e^{-\alpha x}\right]^{\theta-1} \tag{57}$$

And the LoEx distribution developed by Oguntunde *et.al.*, (2018) has pdf given as:

$$f(x;\alpha,\lambda)=\frac{\lambda}{\alpha^{\lambda}}x^{-(\lambda+1)}\left[1+(\alpha x)^{-\lambda}\right]^{-2} \tag{58}$$

The two datasets that were used as examples in the application demonstrate the new family of distributions' flexibility, applicability and 'best fit' compared to the above comparator distributions in modeling the data sets experimentally. The R programming language is used to carry out all of the computations.

Data set 1

The first data set as listed below represents the daily confirmed cases of COVID-19 positive cases record in Pakistan from March 24 to April 28, 2020, previously used by Al-Marzouki, *et.al*., (2020):
108, 102, 133, 170, 121, 99, 236, 178, 250, 161, 258, 172, 407, 577, 210, 243, 281, 186, 254,336,342,269,520,414,463,514,427,796,555,742,642,785,783,605,751,806.

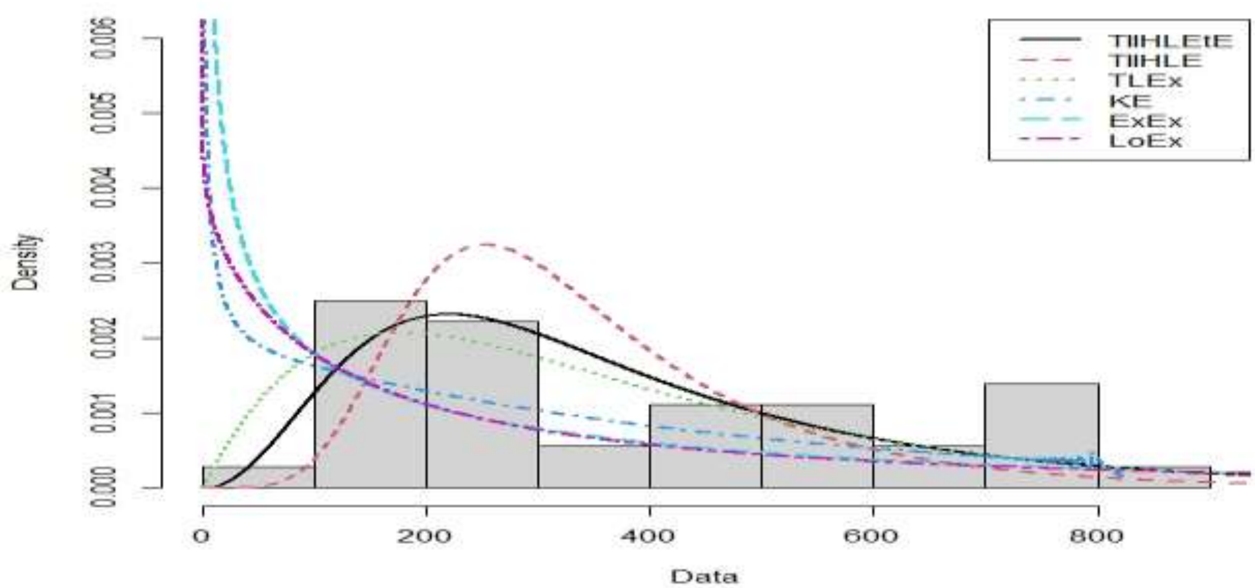

Figure 5: Fitted pdfs for the TIIHLEtE, TIIHLE, TLEx, KE, ExEx, and LoEx distributions to the data set 1

Table 1: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 1

Distributions	α	λ	θ	LL	AIC
TIIHLEtE	1.4789	2.2610	0.003	-242.8838	491.7676
TIIHLE		5.3268	0.0049	-244.8945	493.7891
ToLE	1.3304		0.0016	-247.2426	498.4852
KE	0.1284	0.0521	0.0431	-254.0471	514.0942
ExEx	0.0016		0.4800	-262.3582	528.7164
LoEx	0.0032	0.9270		-263.1668	530.3336

Table 1 presents the results of the Maximum Likelihood Estimation of the parameters of the proposed distribution and the five comparator distributions. Based on the goodness of fit measure, the proposed distribution reported the minimum AIC value, though followed closely by the TIIHLE. The visual inspection of the fit presented in Figure 5, also confirms the superiority of the proposed distribution amongst its comparators. Thus the proposed distribution 'best fit' COVID 19 data set amongst the range of distributions considered.

Data set 2

The second data set shown below represents the mortality rate of the COVID-19 patients in Canada, previously used by Liu *et.al*., (2021):

3.1091, 3.3825, 3.1444, 3.2135,2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685,3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592,2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

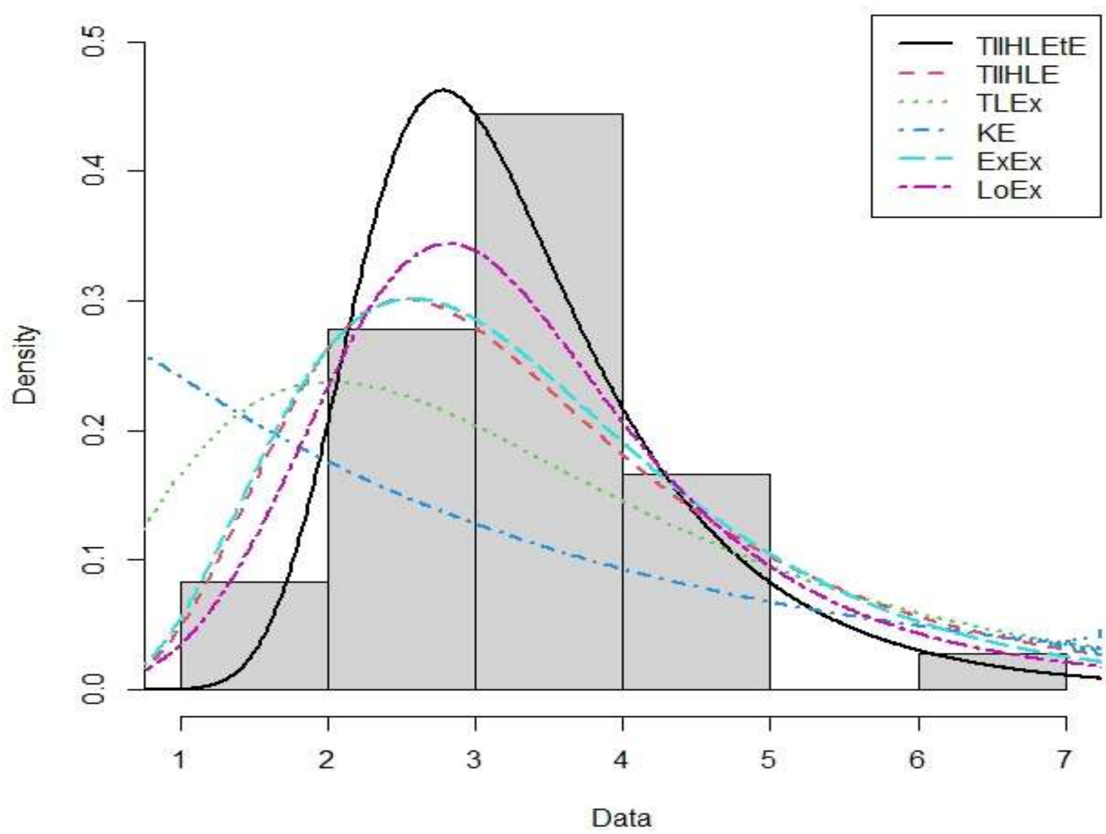

Figure 6: Fitted pdfs for the TIIHLEtE, TIIHLE, TLEx, KE, ExEx, and LoEx distributions to the data set 2

Table 2: MLEs, Log-likelihoods and Goodness of Fits Statistics for the Data Set 2

Distributions	α	λ	θ	LL	AIC
TIIHLEtE	4.9661	4.9266	0.9995	-48.8348	103.6695
TIIHLE		6.5782	0.5936	-54.5742	113.1484
ToLE	2.9205		0.2653	-62.4905	128.981

KE	1.9840	0.0635	5.0215	-77.2137	160.4274
ExEx	0.7605		7.0710	-53.8014	111.6028
LoEx	0.3151	4.1122		-51.1233	106.2466

Table 2 shows the results of the Maximum Likelihood Estimation of the parameters of the TIIHLEtE distribution and the five comparator distributions. Based on the goodness of fit statistic AIC, the new distribution reported the minimum AIC value suggesting that the distribution is the 'best fit' to the mortality rate of the COVID-19 patients. The visual inspection of the fit presented in Figure 6, also reaffirms the superiority of the new distribution amongst its comparators.

Conclusion

A new family of continuous distributions called the Type II Half Logistic Exponentiated-G (TIIHLEt-G) family was proposed and studied. Some of the statistical properties of the proposed family, such as explicit quantile function expressions, probability weighted moments, moments, generating function, survivor functions, and order statistics were investigated. Some of the new family's sub-models were discussed. The method of maximum likelihood was used to estimate the model parameters. In comparison to well-known models, two real data sets were used to highlight the importance and flexibility of one of the sub model. The findings reveal that the new model appears to be superior to the existing models considered and, therefore, provides new distribution to model data in many applications.

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