FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 5 No. 2, June, 2021, pp 574-578
DOI: https://doi.org/10.33003/fjs-2021-0502-672

ON COEFFICIENT BOUNDS AND FUNCTIONALS OF ANALYTIC FUNCTIONS ON A UNIT DISC

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ABSTRACT
We introduce and investigate a new subclasses of the function class Σ of biunivalent functions defined in the open unit disk, which are associated with linear combinations of some geometric expressions, satisfying subordinate conditions. Coefficients and Fekete-Szegö functional for the class are obtained.
AMS Subject Classification: 30C45

Keywords: Analytic functions, univalent functions, subordination, Hadamard product, linear combination

INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

Where $z\epsilon C$.

We denote by S the subclass of $\mathcal{A}$ consisting of functions which are univalent and analytic (holomorphic) in the unit disk E.

Definition 1

Let $f(z)$ and $g(z)$ be analytic in E, we say that $f(z)$ is subordinate to $g(z)$ writhen as $f(z) \prec g(z)$, if there exist an analytic function $w(z)$ (not necessarily univalent) in E, satisfying $w(0) = 0$ and $|w(z)| < 1$ such that

$$f(z) = g(w(z)), \quad |z| < 1, \tag{2}$$

Definition 2

The class P is the class of all functions of the form

$$p(z) = 1 + z + p_2 z^2 + \cdots \tag{3}$$

which are analytic in E such that for $z \in E$, $Rep(z) > 0, p(0) = 1$. Using subordination principle, $p(z) \in P$ iff

$$p(z) \prec \frac{1+z}{1-z}\,. \tag{4}$$

Its known that , if $f(z)$ is analytic univalent function from a domain D_1 on to a domain D_2, then the inverse function g(z) defined by

$$g(f(z)) = z,\ z \in D_1.$$

is an analytic and univalent mapping from D_2 onto D_1. It is also well known by Koebe one- quarter theorem that the image of E under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus clearly every univalent function in E has an inverse f^{-1} satisfying the following conditions:

$$f^{-1}(f(z)) = z,\ z \in E.$$

and

$$f^{-1}(f(w)) = w,\ \left(|w| < r_\circ(f); r_\circ(f) \geq \frac{1}{4}\right).$$

The inverse of the function $f(z)$ has a series expansion of the form:

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \ldots \tag{5}$$

A function $f(z)$, which is univalent in a neighborhood of the origin , and its inverse $f^{-1}(w)$ satisfy the following condition;

$$f^{-1}(f(w)) = w,$$

or, equivalently

$$w = f^{-1}(w) + a_2[f^{-1}(w)]^2 + a_3[f^{-1}(w)]^3 + \cdots \tag{6}$$

Thus we have

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{7}$$

We denote by Σ the class of all functions $f(z)$ which are biunivalent in E and are given by (1)

Notable example of the class Σ are ; $\frac{z}{1-z}$, $-log(1-z)$, $\frac{1}{2}log\left(\frac{1+z}{1-z}\right)$. For more write up on biunivalent functions , an interested reader can see [6, 7, 8, 9, 10, 11, 12]

2.Preliminary results

Ruscheweyh [5] introduced the operator $\mathcal{D}^n, n \in \mathbb{N} \cup \{0\}$ defined as

$$\mathcal{D}^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z.(z^{n-1}f(z))^n}{n!} \tag{8}$$

and used it to generalize the concept of starlikeness and convexity of functions in the unit disk by defining classes of functions for which the geometric quantities

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)},\ \ n \in \mathbb{N} \cup \{0\} \tag{9}$$

have some positive real part.

Babalola [1] established the univalency of the geometric combination involving $\mathcal{D}^n$ as stated below as follows:

$$\text{Re}\left\{(1-\lambda)\frac{\mathcal{D}^n f\prime(z)}{f(z)} + \lambda\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}\right\} > \beta,\ z \in E \tag{10}$$

He proves that if $f(z)$ satisfy (10) then $\text{Re}\left(\frac{\mathcal{D}^n f(z)}{z}\right) > 0$ in E. For all such $\lambda \geq 0$ such $\lambda \leq \beta \leq 1$, for $n \geq 1$, univalency is implied by the results.

We denote class of functions satisfying (10) by $B_\lambda^n(\beta)$ that is:

$$B_\lambda^n(\beta) = \left\{f \in A : \text{Re}\left[(1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}\right] > \beta, z \in E\right\} \tag{11}$$

where $\mathcal{D}^n$ is the Ruscheweyh differential operator and $0 \leq \lambda \leq \beta < 1$.

Equivalently, we have the above as:

$$B_\lambda^n(\beta) = \left\{f \in A : (1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} \prec \frac{1+(1-2\beta)z}{1-z}\ \ , z \in E\right\} \tag{12}$$

Definition 3

A function $f \in \sum$ given by the function (1) is said to be in the class $B_\lambda^n\Sigma(\beta)$ if the following conditions are satisfied:

$$\text{Re}\left((1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)}\right) > \beta\ ,0 \leq \beta < 1, \lambda \geq 1 \tag{13}$$

and

$$\text{Re}\left((1-\lambda)\frac{\mathcal{D}^n g(w)}{w} + \lambda\frac{\mathcal{D}^{n+1}g(w)}{\mathcal{D}^n g(w)}\right) > \beta\ ,0 \leq \beta < 1, \lambda \geq 1 \tag{14}$$

where the function g is the inverse of f given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{15}$$

Lemma 1[4]

If $p \in P$, then $|c_k| \leq 2$ for each k given , where P is the family of all functions p analytics in E for which $\text{Re}p > 0, p(z) = 1 + c_z + c_2z^2 + \cdots$ for $z \in U$.

Lemma 2 [2]

Let $p \in P$.

Then we have the sharp inequalities

(i) for any real number σ

$$\left|p_2 - \sigma\frac{p_1^2}{2}\right| \leq \begin{cases} 2(1-\sigma)\ ,\ \text{if } \delta \leq 0 \\ 2\ ,\ \text{if } 0 \leq \delta \leq 2 \\ 2(\sigma - 1)\ , \text{if } \delta \geq 2 \end{cases} \tag{16}$$

(ii) for any complex number δ

$$\left|p_2 - \sigma\frac{p_1^2}{2}\right| \leq \begin{cases} 2, \quad \text{if } \ |1-\delta| \leq 1 \\ 2|1-\sigma|,\ \text{if} |1-\delta| \geq 1 \end{cases} \tag{17}$$

Main Results .

Theorem 1

Let $f(z)$ given by (1) be in the class $B_\lambda^n\sum(\beta)$, $0 \leq \beta < 1, \lambda \geq 1$. then

$$|a_2| \leq \frac{2(1-\beta)}{1+(1-\lambda)n} \tag{18}$$

$$|a_3| \leq \begin{cases} \frac{4(1-\beta)}{G} - \frac{4(1-\beta)^2}{(1+n(1-\lambda))^2}, & \text{if} \quad 0 \leq \lambda \leq \frac{n+1}{n-1},\ n \neq 0 \\ \frac{4(1-\beta)}{G} & \text{if} \quad \frac{n+1}{n-1} \leq \lambda \leq \lambda_\circ,\ n \neq 0 \\ \frac{4(1-\beta)^2}{(1+n(1-\lambda))^2} & \text{if} \quad \lambda \leq \lambda_\circ \end{cases} \tag{19}$$

where

$$G = (n+2)[2\lambda + (1-\lambda)(n+1)]$$

and $\lambda_\circ$ is the positive root of

$$\lambda^2 n - \lambda[(1-\beta)(n+2)(n+3) + 2n(n+1)] - (1-\beta)(n+2)(n+1) + 2n + n^2 + 1 = 0\ .$$

Proof:

By (13) and (14) we have

$$(1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} = \beta + (1-\beta)p(z) \tag{20}$$

and

$$((1-\lambda)\frac{\mathcal{D}^n g(w)}{w} + \lambda \frac{\mathcal{D}^{n+1} g(w)}{\mathcal{D}^n g(w)} = \beta + (1-\beta)q(z) \tag{21}$$

expanding (20) and (21) we have

$$1 + [\lambda a_2 + (1-\lambda)(n+1)a_2]z + \left[[(n+2)a_3 - (n+1)a_2^2] + (1-\lambda)\frac{(n+1)(n+2)}{2!}a_3\right]z^2 + \cdots =$$

$$1 + (1-\beta)p_1 z + (1-\beta)p_2 z^2 + \cdots \tag{22}$$

$$1 + [\lambda a_2 + (1-\lambda)(n+1)a_2]w +$$

$$\left[(1-\lambda)\frac{(n+1)(n+2)}{2!}(2a_2^2 - a_3) + \lambda[a_2^2(n+3) - (n+2)a_3]\right]w^2 =$$

$$1 + (1-\beta)q_1 w + (1-\beta)q_2 w^2 \tag{23}$$

Equating coefficients

$$[\lambda + (1-\lambda)(n+1)]a_2 = (1-\beta)p_1 \tag{24}$$

$$\lambda(n+2)a_3 - \lambda(n+1)a_2^2 + (1-\lambda)\frac{(n+1)(n+2)}{2!}a_3 = (1-\beta)p_2 \tag{25}$$

$$-[(1-\lambda)(n+1) + \lambda]a_2 = (1-\beta)q_1 \tag{26}$$

$$(1-\lambda)(2a_2^2 - a_3)\frac{(n+1)(n+2)}{2!} + \lambda[a_2^2(n+3) - (n+2)a_3] = (1-\beta)q_2 \tag{27}$$

From (24) and (26) we have

$$p_1 = -q_1$$

and

$$[1 + n(1-\lambda)]a_2 = (1-\beta)p_1 \tag{28}$$

applying lemma (1) on (p_1) we have the desired result.

$$|a_2| \leq \frac{2(1-\beta)}{1+n(1-\lambda)} \tag{29}$$

Making use of (25),(27)and a_2^2 after a simplification we have

$$a_3 = \frac{(1-\beta)p_2}{G} + \frac{(1-\beta)}{G}\left[\frac{2(1-\beta)G}{(1+n(1-\lambda))^2}\frac{q_1^2}{2} - q_2\right] \tag{30}$$

Applying lemma (1) and (2) on p_1, p_2 and q_2 we have the desired result.

Let S be a class of univalent and analytic functions given by (1) defined in the unit disk. Then the classical Fekete-Szegö inequality [cf [3]], for coefficient of $f \in S$ is given as:

$$|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(\frac{-2\mu}{(1-\mu)}\right). \tag{31}$$

As $\mu \to 1^-$, we have the inequality $|a_3 - \mu a_2^2| \leq 1$. Moreso ,the coefficient functional,

$$\nabla_\mu(f) = a_3 - \mu a_2^2$$

on functions given by (1) in the unit disk E plays an important role in the function theory. For instance , the quantity $a_3 - a_2^2$ represent $s_f(0)$ where s_f denotes the Schwarzian derivative

$$\frac{(\frac{f''}{f'})' - (\frac{f'}{f''})^2}{2}$$

of locally univalent functions f in the unit disk E. Example $\nabla_1(f) = a_3 - a_2^2$. Its observed that the first two non trivial coefficients of the n-th root transform

$$\{f(z^n)\}^{\frac{1}{n}} = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + \cdots \tag{32}$$

of functions represented by (1)are given by

$$c_{n+1} = \frac{a_2}{n} \text{ and } c_{2n+1} = \frac{\left(\nabla_{\frac{(n-1)}{n}}(f)\right)}{n} = \frac{a_3}{n} - \frac{(n-1)a_2^2}{2n^2}\ . \tag{33}$$

Thus it is not out of place to investigate on the inequalities for ∇_μ corresponding to subclasses of normalized univalent functions defined by the functional class Σ in the unit disk. The problem of maximizing the absolute value of the functional $\nabla_\mu(f)$ is called the Fekete-Szegö problem.

Theorem 2

Let $f(z)$ given by (1) be in the class $B_{n,\lambda}\sum(\beta),\ 0 \le \beta < 1, \lambda \ge 1$.Then

(i)For any real number μ

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{4(1-\beta)}{GQ}[Q-(1-\beta)(1-\mu)G]\ , \text{if } \mu \le 1 \\ \frac{4(1-\beta)}{G},\ \text{if } 1 \le \mu \le \frac{2Q+(1-\beta)G}{(1-\beta)G} \\ \frac{4(1-\beta)}{GQ}[(1-\beta)(\mu-1)G - Q],\ \text{if } \mu \ge \frac{2Q+(1-\beta)G}{(1-\beta)G} \end{cases} \tag{34}$$

(ii) For any complex number μ

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{4(1-\beta)}{G}, \quad \text{if } \left|\frac{Q-(\mu-1)(1-\beta)G}{Q}\right| \le 1 \\ \frac{4(1-\beta)}{G}\left|\frac{Q-(\mu-1)(1-\beta)G}{Q}\right|,\ \text{if} \left|\frac{Q-(\mu-1)(1-\beta)G}{Q}\right| \ge 1 \end{cases} \tag{35}$$

where,

$$G = (n+1)[2\lambda + (1-\lambda)(n+1)]$$

,

$$Q = [1 + (1-\lambda)n]^2$$

and

$$\delta = \frac{(\mu-1)(1-\beta)G}{Q}$$

Proof.

From (25),(27) and (28) we have the following;

$$a_2 = \frac{(1-\beta)p_1}{1+n(1-\lambda)} \tag{36}$$

$$a_3 = \frac{(1-\beta)^2 p_1^2}{[1+n(1-\lambda)]^2} + \frac{(1-\beta)(p_2-q_2)}{(n+2)[2\lambda+(1-\lambda)(n+1)]} \tag{37}$$

and

$$a_3 - \mu a_2^2 = \frac{(1-\beta)^2 p_1^2}{[1+n(1-\lambda)]^2} + \frac{2(1-\beta)p_2}{(n+2)[2\lambda+(1-\lambda)(n+1)]} - \frac{\mu(1-\beta)^2 p_1^2}{[1+n(1-\lambda)]^2} \tag{38}$$

$$|a_3 - \mu a_2^2| = \frac{2(1-\beta)}{G}\left|p_2 - \frac{p_1^2}{2}\left[(\mu-1)\frac{G(1-\beta)}{Q}\right]\right| \tag{39}$$

Choose

$$\delta = \frac{(\mu-1)(1-\beta)G}{Q},$$

and applying lemma (2) we have our desired result.

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