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IMPROVING PRECONDITIONED GAUSS-SEIDEL ITERATIVE METHOD FOR L −MATRICES

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ABSTRACT
The Gauss-Seidel is a well-known iterative method for solving the linear system $Ax = b$. Convergence of this method is guaranteed for linear systems whose coefficient matrix A is strictly or irreducibly diagonally dominant, Hermitian positive definite and invertible H −matrix. In this current work, a preconditioned version of the Gauss-Seidel method is used to accelerate the convergence of this iterative method towards the solution of linear system $Ax = b$ under mild conditions imposed on A. Convergence theorems on preconditioned Gauss-Seidel iterative method are advanced and proved. The superiority of Preconditioned Gauss-Seidel method is demonstrated by solving some numerical examples.

Keywords: Gauss-Seidel method, L −-matrix, iteration matrix, convergence, spectral radius

INTRODUCTION

Iterative solution methods for solving the linear system $Ax = b$ take the general form

$$x^{(k+1)} = Tx^{(k)} + c, \qquad k = 0,1,2,\cdots \tag{1}$$

The convergence of these methods is seldom guaranteed for every matrix; however, quite a lot of theory exist for coefficient matrices arising from the finite difference discretization of elliptic partial differential equations. In comparison to direct methods, there is the widely recognized weakness of lack of robustness associated with iterative methods. This setback hinders the acceptance of iterative methods in real world applications in spite of their inherent appeal for sparse large linear systems. Fortunately, both the efficiency and robustness of iterative methods can be enhanced by the application of preconditioning technique. Preconditioning is just a means of applying a transformation, called the preconditioner, to the original linear system in order to transform it into one which has the same solution, but which is more suitable for numerical solution. The following preconditioned linear system, obtained through application of the transformation P to the linear system $Ax = b$, is considered

$$PAx = Pb \tag{2}$$

where $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is a nonsingular L −matrix, $P = I + S$, where $I \in I^{n\times n}$, the set of $n \times n$ identity matrices, and S is a sparse matrix whose nonzero entries are the negatives of the corresponding entries of A, $b \in \Re(A)$, $\Re(A)$ being the range of A is a column vector and $x \in \mathbb{R}^{n\times n}$ is the vector of unknowns.

In order to effectively solve the preconditioned system (2), several preconditioners have been presented. In 1987, the preconditioner P introduced by Milaszewciz (1987) assumes the form $P = I + S$, where

$$S = (s_{ij}) = \begin{cases} -a_{i1}, & \text{for } i = 2,\cdots,n \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

with the condition that the coefficient matrix A is an L −matrix with

$$a_{i,i+1}a_{i+1,i} > 0 \text{ and } 0 < a_{1i}a_{i1} < 1 \text{ for } i = 2,3,\cdots,n \tag{4}$$

Gunawardena et al. (1991) proposed the preconditioned Gauss-Seidel method with $P = I + S$, where

$$S = (s_{ij}) = \begin{cases} -a_{ii+1}, & \text{for } i = 1,2,\cdots,n-1, \quad j = i+1 \\ 0, & \text{otherwise} \end{cases}$$

Similar preconditioners were proposed by Kotakemori *et al.* (1996), Kohno *et al.* (1997), Kotakemori et al. (2002), Morimoto *et al.* (2003) and Morimoto *et al.* (2004). The preconditioned effect of these preconditioners is seldom observed on the last row of A, because they are formed from a part of upper triangular part of A. The preconditioner of Morimoto *et al.* (2003) was an attempt at providing the preconditioned effect on the last row of A. It takes the form $P_{R_1} = I + R$, where R is defined as

$$R = (r_{nj}) = \begin{cases} -a_{nj}, & 1 \le j \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

The preconditioned matrix PA, denoted by A_{R_1}, is defined by

$$A_{R_1} = (I+R)A = \left(a_{ij}^{R_1}\right), \qquad a_{ij}^{R_1} = \begin{cases} a_{ij}, & 1 \le i < n-1, 1 \le j \le n, \\ a_{nj} - \sum_{k=1}^{n-1} a_{nk}a_{kj}, & 1 \le j \le n. \end{cases}$$

Then, a splitting of the preconditioned matrix A_{R_1} is obtained thus

$$A_{R_1} = M_{R_1} - N_{R_1} = (I - L + R - RL - RU) - U = (I - L - D_R + R - RL - E_R) - U,$$

where D_R, E_R are the diagonal and strictly lower triangular parts of RU, respectively. if $\sum_{k=1}^{n-1} a_{nk}a_{ki} \neq 1$, then $M_{R_1}^{-1}$ exists, and the Gauss-Seidel iterative matrix T_{R_1} is defined by

$$T_{R_1} = (I - D_R - L + R - RL - E_R)^{-1}U.$$

Niki *et al.* (2004) built on Morimoto *et al.* (2003) to propose the preconditioner $P_R = I + S + R$, arising from which the preconditioned matrix A_R assumes the structure

$$A_R = (I+S+R)A = \left(a_{ij}^{R}\right), \qquad a_{ij}^{R} = \begin{cases} a_{ij} - a_{ii+1}a_{i+1j}, & 1 \le i < n, \\ a_{nj} - \sum_{k=1}^{n-1} a_{nk}a_{kj}, & 1 \le j \le n. \end{cases}$$

with the corresponding splitting

$$A_R = M_R - N_R = (I - D - D_R) - (L - R + RL + E + E_R) - (U - S + SU).$$

In a quest to address the shortcomings of the preconditioner (3), Dehghan and Hajarian (2009) introduced two new preconditioners $\bar{P} = I + \bar{S}$ and $\tilde{P} = I + \tilde{S}$, with

$$\bar{S} = \begin{cases} -(a_{i1} + \gamma_i), & \text{for } i = 2, \cdots, n \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$\tilde{S} = \begin{cases} -(a_{in} + \delta_i), & \text{for } i = 1, \cdots, n-1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where $\gamma_2, \gamma_3, \cdots, \gamma_n$ and $\delta_1, \delta_2, \cdots, \delta_{n-1}$ are real parameters. These preconditioners were applied to accelerate the convergence of the successive overrelaxation (SOR) iterative method under mild conditions on the coefficient matrix A. In furtherance of the search for fast converging iterative methods, Ndanusa and Adeboye (2012) attempted an improvement on the SOR method by proposing a preconditioner $P = I + S$, with S having the structure

$$S = \begin{cases} -a_{i1}, & i = 2, \cdots, n \\ -a_{i,i+1}, & i = 1, \cdots, n-1 \\ 0, & \text{otherwise} \end{cases}$$

In this current work, we focus on investigating the validity of the results of Dehghan and Hajarian (2009) applied to Gauss-Seidel method. Throughout this paper, we assume that the coefficient matrix A has a splitting of the form $A = I - L - U$, where I denotes the $n \times n$ identity matrix, and $-L$ and $-U$ are the strictly lower and strictly upper triangular parts of A, respectively.

MATERIALS AND METHODS

Preliminaries

From the general iteration formula for linear systems (1), the Gauss-Seidel method is represented by

$$x^{(k+1)} = (I-L)^{-1}Ux^{(k)} + (I-L)^{-1}b \qquad k = 0,1,2,\cdots \tag{7}$$

Where,

$$T_G = (I-L)^{-1}U \tag{8}$$

is the iteration matrix of the Gauss-Seidel method. Convergence of the method is guaranteed if the spectral radius of the iteration matrix is less than 1, and the smaller it is, the faster the method converges. It is known that the Gauss-Seidel converges faster than the Jacobi method while the SOR converges faster than the Gauss-Seidel method.

Following on the preconditioners (5) and (6), two preconditioned linear systems are introduced as follows:

$\bar{A}x = \bar{b}$ where $\bar{A} = \bar{P}A$ and $\bar{b} = \bar{P}b$

$\tilde{A}x = \tilde{b}$ where $\tilde{A} = \tilde{P}A$ and $\tilde{b} = \tilde{P}b$

A usual splitting of $\bar{A}$ and $\tilde{A}$ is obtained as

$$\bar{A} = \bar{D} - \bar{L} - \bar{U} \text{ and } \tilde{A} = \tilde{D} - \tilde{L} - \tilde{U} \tag{9}$$

respectively, where

$$\left.\begin{array}{lll} \bar{D} = I + \bar{D}_1, & \bar{L} = L - \bar{S} + \bar{L}_1, & \bar{U} = U + \bar{U}_1 \\ \tilde{D} = I + \tilde{D}_1, & \tilde{L} = L + \tilde{L}_1, & \tilde{U} = U + \tilde{U}_1 \end{array}\right\} \tag{10}$$

where $\bar{D}_1(\tilde{D}_1)$, $\bar{L}_1(\tilde{L}_1)$ and $\bar{U}_1(\tilde{U}_1)$ represent the diagonal, strictly lower and strictly upper triangular parts of $\bar{S}U(\tilde{S}A)$ respectively; that is, $\bar{S}U = -\bar{D}_1 + \bar{L}_1 + \bar{U}_1$ and $\tilde{S}A = \tilde{S} - \tilde{S}U - \tilde{S}L = \tilde{D}_1 - \tilde{L}_1 - \tilde{U}_1$. Two forms of the Gauss-Seidel iteration matrix related to $\bar{A}$ and $\tilde{A}$ are described by

$$\bar{T}_{G1} = (\bar{D} - \bar{L})^{-1}\bar{U}, \quad \bar{T}_{G2} = [I - (L - \bar{S} + \bar{L}_1)]^{-1}(U + \bar{U}_1 - \bar{D}_1) \tag{11}$$

and

$$\tilde{T}_{G1} = (\tilde{D} - \tilde{L})^{-1}\tilde{U}, \quad \tilde{T}_{G2} = \left[I - \left(L + \tilde{L}_1\right)\right]^{-1}\left(U + \tilde{U}_1 - \tilde{D}_1\right) \quad (12)$$

Definition 1: A matrix A is said to be reducible if there exists some nonempty set $K \subset \mathbb{N}$ that satisfies

$$a_{ij} = 0 \text{ for any } i \in K, \qquad j \in \mathbb{N}/K$$

If A is not a reducible matrix, then A is called an irreducible matrix.

Convergence Analysis

Lemma 1 (Varga (1981)) Let $A \geq 0$ be an irreducible matrix. Then,

i. A has a positive real eigenvalue equal to its spectral radius.
ii. For $\rho(A)$ there corresponds an eigenvector $x > 0$.
iii. $\rho(A)$ increases when any entry of A increases.
iv. $\rho(A)$ is a simple eigenvalue of A.

Lemma 2 (Varga (1981)) Let A be a nonnegative matrix. Then

i. If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
ii. If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3 (Li and Sun (2000)) Let $A = M - N$ be an M −splitting of A. Then the splitting is convergent, i.e., $\rho(M^{-1}N < 1)$, if and only if A is a nonsingular M −matrix.

Theorems

Theorem 1 Let $T_G, \bar{T}_{G1}, \bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ be defined by (8), (11) and (12). If A is an irreducible L −matrix with $a_{1q}a_{q1} > 0$ and $a_{ns}a_{sn} > 0, \gamma_q \in ((1 - a_{1q}a_{q1})/a_{1q}, -a_{q1}) \cap (0, -a_{q1}), \delta_s \in ((1 - a_{ns}a_{sn})/a_{ns}, -a_{sn}) \cap (0, -a_{sn})$ for $q = 2,3,\cdots,n$ and $s = 1,2,\cdots,n-1$, then $T_G, \bar{T}_{G1}, \bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ are nonnegative and irreducible matrices.

Proof: Since A is an irreducible L −matrix, it implies that $L \geq 0$ and $U \geq 0$. Thus,

$$\begin{aligned} T_G &= (I - L)^{-1}U \\ &= (I + L + L^2 + \cdots + L^{n-1})U \\ &= U + LU + L^2U + \text{nonnegative terms} \end{aligned}$$

Therefore, T_G is nonnegative. We can also obtain that $U + LU + L^2U$ is irreducible for irreducible A; hence T_G is irreducible. Now, from (9) and (10) we obtain

$$\bar{D} = \operatorname{diag}(1, 1 - a_{12}(a_{21} + \gamma_2), 1 - a_{13}(a_{31} + \gamma_3), \cdots, 1 - a_{1n}(a_{n1} + \gamma_n))$$

$$\tilde{D} = \operatorname{diag}(1 - a_{n1}(a_{1n} + \delta_1), 1 - a_{n2}(a_{2n} + \delta_2), \cdots, 1)$$

$$\bar{D}_1 = \operatorname{diag}(0, -a_{12}(a_{21} + \gamma_2), -a_{13}(a_{31} + \gamma_3), \cdots, -a_{1n}(a_{n1} + \gamma_n))$$

$$\tilde{D}_1 = \operatorname{diag}(-a_{n1}(a_{1n} + \delta_1), -a_{n2}(a_{2n} + \delta_2), \cdots, 0)$$

$$\bar{L} = \begin{pmatrix} 0 & & & & \\ \gamma_2 & 0 & & & \\ \gamma_3 & -a_{32} + a_{12}(a_{31} + \gamma_3) & 0 & & \\ \vdots & \vdots & \vdots & 0 & \\ \gamma_n & -a_{n2} + a_{12}(a_{n1} + \gamma_n) & \cdots & \cdots & 0 \end{pmatrix}$$

$$\bar{U} = \begin{pmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ & 0 & -a_{23} + a_{13}(a_{21} + \gamma_2) & \cdots & -a_{2n} + (a_{21} + \gamma_2) \\ & & 0 & \vdots & \vdots \\ & & & 0 & -a_{n-1,n} + a_{1n}\left(a_{n-1,1} + \gamma_{n-1}\right) \\ & & & & 0 \end{pmatrix}$$

$$\tilde{L} = \begin{pmatrix} 0 & & & & \\ -a_{21} + a_{n1}(a_{2n} + \delta_2) & 0 & & & \\ -a_{31} + a_{n1}(a_{3n} + \delta_3) & -a_{32} + a_{n2}(a_{3n} + \delta_3) & 0 & & \\ \vdots & \vdots & \vdots & 0 & \\ -a_{n1} & -a_{n2} & \cdots & \cdots & 0 \end{pmatrix}$$

$$\tilde{U} = \begin{pmatrix} 0 & -a_{12} + a_{n2}(a_{1n} + \delta_1) & -a_{13} + a_{n3}(a_{1n} + \delta_1) & \cdots & \delta_1 \\ & 0 & -a_{23} + a_{n3}(a_{2n} + \delta_2) & \cdots & \delta_2 \\ & & 0 & \vdots & \vdots \\ & & & 0 & \delta_{n-1} \\ & & & & 0 \end{pmatrix}$$

Now,

$$1 - a_{1q}\left(a_{q1} + \gamma_q\right) > 1 - a_{1q}\left(a_{q1} + \frac{1 - a_{1q}a_{q1}}{a_{1q}}\right) = 1 - \frac{1}{a_{1q}} > 0 \quad (13)$$

$$1 - a_{ns}(a_{sn} + \delta_s) > 1 - a_{ns}\left(a_{sn} + \frac{1 - a_{ns}a_{sn}}{a_{ns}}\right) = 1 - \frac{1}{a_{ns}} > 0 \quad (14)$$

From (13) and (14), $\bar{D} > 0, \tilde{D} > 0, \bar{D}_1 \leq 0$ and $\tilde{D}_1 \leq 0$. Also,

$$-a_{ij} + a_{1j}(a_{i1} + \gamma_i) \geq -a_{ij} + a_{1j}(a_{i1} - a_{i1}) \geq -a_{ij} \geq 0 \quad (15)$$

$$-a_{ij} + a_{nj}(a_{in} + \delta_i) \geq -a_{ij} + a_{nj}(a_{in} - a_{in}) \geq -a_{ij} \geq 0 \quad (16)$$

It implies that $\bar{L},\ \bar{U} \geq 0$ and $\tilde{L}, \tilde{U} \geq 0$. Thus from (11) we have

$$\begin{aligned}\bar{T}_{G1} &= (\bar{D} - \bar{L})^{-1}\bar{U}\\ &= [\bar{D}(I - \bar{D}^{-1}\bar{L})]^{-1}\bar{U}\\ &= \bar{D}^{-1}(I - \bar{D}^{-1}\bar{L})^{-1}\bar{U}\\ &= (I - \bar{D}^{-1}\bar{L})^{-1}\bar{D}^{-1}\bar{U}\\ &= [I + \bar{D}^{-1}\bar{L} + (\bar{D}^{-1}\bar{L})^2 + \cdots + (\bar{D}^{-1}\bar{L})^{n-1}]\bar{D}^{-1}\bar{U}\\ &= \bar{D}^{-1}\bar{U} + (\bar{D}^{-1})^2\bar{L}\bar{U} + (\bar{D}^{-1})^3\bar{L}^2\bar{U} + \text{nonnegative terms}\end{aligned}$$

Therefore, $\bar{T}_{G1}$ is a nonnegative and irreducible matrix. Using similar argument, we can show that $\bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ are nonnegative and irreducible matrices.

Theorem 2 Let T_G and $\bar{T}_{G1}$ be defined by (8) and (11) respectively. If $\gamma_q \in ((1 - a_{1q}a_{q1})/a_{1q}, -a_{q1}) \cap (0, -a_{q1})$ and A is an irreducible L −matrix with $a_{1q}a_{q1} > 0$ for $q = 2,3, \cdots, n$, then

i. $\rho(\bar{T}_{G1}) < \rho(T_G)$, if $\rho(T_G) < 1$;
ii. $\rho(\bar{T}_{G1}) = \rho(T_G)$, if $\rho(T_G) = 1$;
iii. $\rho(\bar{T}_{G1}) > \rho(T_G)$, if $\rho(T_G) > 1$.

Proof: Theorem 1 has established T_G to be a nonnegative and irreducible matrix. Therefore, following Lemma 1, then for $(T_G) = \lambda$, there corresponds a positive vector x, such that

$$T_G x = \lambda x \tag{17}$$

That is,

$$Ux = (I - L)\lambda x \tag{18}$$

Or, equivalently

$$U = (I - L)\lambda$$

Then ,

$$\begin{aligned}\bar{T}_{G1}x - \lambda x &= (\bar{D} - \bar{L})^{-1}\bar{U}x - \lambda x\\ &= (\bar{D} - \bar{L})^{-1}[\bar{U} - \lambda(\bar{D} - \bar{L})]x\\ &= (\bar{D} - \bar{L})^{-1}[-\lambda\bar{D} + \lambda\bar{L} + \bar{U}]x\\ &= (\bar{D} - \bar{L})^{-1}[-\lambda(I + \bar{D}_1) + \lambda(L - \bar{S} + \bar{L}_1) + (U + \bar{U}_1)]x\\ &= (\bar{D} - \bar{L})^{-1}[-\lambda\bar{D}_1 + \lambda(\bar{L}_1 - \bar{S}) + \bar{U}_1 - \lambda I + \lambda L + U]x\\ &= (\bar{D} - \bar{L})^{-1}[-\lambda\bar{D}_1 + \lambda(\bar{L}_1 - \bar{S}) + \bar{U}_1 - (I - L)\lambda + U]x\\ &= (\bar{D} - \bar{L})^{-1}[-\lambda\bar{D}_1 + \lambda(\bar{L}_1 - \bar{S}) + \bar{U}_1]x\\ &= (\bar{D} - \bar{L})^{-1}[-\lambda\bar{D}_1 + \bar{D}_1 + \lambda(\bar{L}_1 - \bar{S}) - \bar{L}_1 - \bar{D}_1 + \bar{L}_1 + \bar{U}_1]x\\ &= (\bar{D} - \bar{L})^{-1}[(1 - \lambda)\bar{D}_1 + \lambda(\bar{L}_1 - \bar{S}) - \bar{L}_1 + \bar{S}U]x\\ &= (\bar{D} - \bar{L})^{-1}[(1 - \lambda)\bar{D}_1 - (1 - \lambda)\bar{L}_1 + (-\lambda)\bar{S} + \bar{S}U]x\\ &= (\bar{D} - \bar{L})^{-1}[(1 - \lambda)(\bar{D}_1 - \bar{L}_1) + (-\lambda)\bar{S} + \bar{S}U]x\\ &= (\bar{D} - \bar{L})^{-1}[(1 - \lambda)(\bar{D}_1 - \bar{L}_1) - \lambda\bar{S} + \bar{S}(I - L)\lambda]x\\ &= (\bar{D} - \bar{L})^{-1}[(1 - \lambda)(\bar{D}_1 - \bar{L}_1) - \lambda\bar{S} + \lambda\bar{S} - \lambda\bar{S}L]x\end{aligned}$$

But $\bar{S}L = 0$, so

$$= (\lambda - 1)(\bar{D} - \bar{L})^{-1}[-\bar{D}_1 + \bar{L}_1]x$$

Let $B = (\bar{D} - \bar{L})^{-1}[-\bar{D}_1 + \bar{L}_1]x$. Then $[-\bar{D}_1 + \bar{L}_1] \geq 0$, since $-\bar{D}_1 \geq 0$ and $\bar{L}_1 \geq 0$. Suppose $G = \bar{D} - \bar{L}$, it is evident that $\bar{D}$ is a nonsingular M −matrix and $\bar{L} \geq 0$; hence, the splitting $G = \bar{D} - \bar{L}$ is an M −splitting of G. It is observed that $\bar{D}^{-1}\bar{L}$ is a strictly lower triangular matrix so that $\rho(\bar{D}^{-1}\bar{L}) = 0 < 1$, and by implication of Lemma 3, G is a nonsingular M −matrix. Therefore, $(\bar{D} - \bar{L})^{-1} \geq 0$. Consequently, $B \geq 0$.

1) If $\lambda < 1$, then $\bar{T}_{G1}x - \lambda x \leq 0$. Therefore $\bar{T}_{G1}x \leq \lambda x$. And by Lemma 2, we obtain $\rho(\bar{T}_{G1}) < \lambda = \rho(T_G)$;
2) If $\lambda = 1$, then $\bar{T}_{G1}x - \lambda x = 0$. Therefore $\bar{T}_{G1}x = \lambda x$. And by Lemma 2, we obtain $\rho(\bar{T}_{G1}) = \lambda = \rho(T_G)$;
3) If $\lambda > 1$, then $\bar{T}_{G1}x - \lambda x \geq 0$. Therefore $\bar{T}_{G1}x \geq \lambda x$. And by Lemma 2, we obtain $\rho(\bar{T}_{G1}) > \lambda = \rho(T_G)$.

Theorem 3 Let T_G and $\bar{T}_{G2}$ be defined by (8) and (11) respectively. If $\gamma_q \in ((1 - a_{1q}a_{q1})/a_{1q}, -a_{q1}) \cap (0, -a_{q1})$ and A is an irreducible L −matrix with $a_{1q}a_{q1} > 0$ for $q = 2,3, \cdots, n$, then

i. $\rho(\bar{T}_{G2}) < \rho(T_G)$, if $\rho(T_G) < 1$;
ii. $\rho(\bar{T}_{G2}) = \rho(T_G)$, if $\rho(T_G) = 1$;
iii. $\rho(\bar{T}_{G2}) > \rho(T_G)$, if $\rho(T_G) > 1$.

Proof: Following (18), similar to the proof of Theorem 2, we can get

$$\bar{T}_{G2}x - \lambda x = (\lambda - 1)[I - (L - \bar{S} + \bar{L}_1)]^{-1}\bar{L}_1 x \tag{19}$$

and also we can show

$$[I-(L-\bar{S}+\bar{L}_1)]^{-1}\bar{L}_1 x \geq 0 \quad (20)$$

Therefore

1) If $\lambda < 1$, then $\bar{T}_{G2}x - \lambda x \leq 0$. Therefore $\bar{T}_{G2}x \leq \lambda x$. And by Lemma 2, we obtain $\rho(\bar{T}_{G2}) < \lambda = \rho(T_G)$;
2) If $\lambda = 1$, then $\bar{T}_{G2}x - \lambda x = 0$. Therefore $\bar{T}_{G2}x = \lambda x$. And by Lemma 2, we obtain $\rho(\bar{T}_{G2}) = \lambda = \rho(T_G)$;
3) If $\lambda > 1$, then $\bar{T}_{G2}x - \lambda x \geq 0$. Therefore $\bar{T}_{G2}x \geq \lambda x$. And by Lemma 2, we obtain $\rho(\bar{T}_{G2}) > \lambda = \rho(T_G)$.

Theorem 4 Let T_G and $\tilde{T}_{G1}$ be defined by (8) and (12) respectively. If $\delta_s \in ((1-a_{ns}a_{sn})/a_{ns}, -a_{sn}) \cap (0, -a_{sn})$ and A is an irreducible L −matrix with $a_{ns}a_{sn} > 0$ for $s = 1,2,\cdots,n-1$, then

i. $\rho(\tilde{T}_{G1}) < \rho(T_G)$, if $\rho(T_G) < 1$;
ii. $\rho(\tilde{T}_{G1}) = \rho(T_G)$, if $\rho(T_G) = 1$;
iii. $\rho(\tilde{T}_{G1}) > \rho(T_G)$, if $\rho(T_G) > 1$.

Proof: By (18) we can write

$$\begin{aligned}
\tilde{T}_{G1}x - \lambda x &= (\tilde{D}-\tilde{L})^{-1}\tilde{U}x - \lambda x \\
&= (\tilde{D}-\tilde{L})^{-1}[\tilde{U} - \lambda(\tilde{D}-\tilde{L})]x \\
&= (\tilde{D}-\tilde{L})^{-1}[U + \tilde{U}_1 - \lambda\{(I+\tilde{D}_1) - (L+\tilde{L}_1)\}]x \\
&= (\tilde{D}-\tilde{L})^{-1}[U + \tilde{U}_1 - \lambda(I-L) - \lambda\tilde{D}_1 + \lambda\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[U + \tilde{U}_1 - U - \lambda\tilde{D}_1 + \lambda\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[\tilde{U}_1 - \lambda\tilde{D}_1 + \lambda\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[(-\tilde{D}_1 + \tilde{L}_1 + \tilde{U}_1) + (1-\lambda)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[-(\tilde{S}A) - (\lambda-1)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[-\tilde{S} + \tilde{S}U + \tilde{S}L - (\lambda-1)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[\tilde{S}U - \tilde{S}(I-L) - (\lambda-1)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[\lambda\tilde{S}(I-L) - \tilde{S}(I-L) - (\lambda-1)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[(\lambda-1)\tilde{S}(I-L) - (\lambda-1)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}[\frac{(\lambda-1)}{\lambda}\tilde{S}U - (\lambda-1)\tilde{D}_1 + (\lambda-1)\tilde{L}_1]x \\
&= (\tilde{D}-\tilde{L})^{-1}\frac{(\lambda-1)}{\lambda}[\tilde{S}U - \lambda\tilde{D}_1 + \lambda\tilde{L}_1]x
\end{aligned}$$

In analogy to the proof of Theorem 2, we can prove that $(\tilde{D}-\tilde{L})^{-1}[\tilde{S}U - \lambda\tilde{D}_1 + \lambda\tilde{L}_1]x \geq 0$. Hence

1) If $\lambda < 1$, then $\tilde{T}_{G1}x - \lambda x \leq 0$. Therefore $\tilde{T}_{G1}x \leq \lambda x$. And by Lemma 2, we obtain $\rho(\tilde{T}_{G1}) < \lambda = \rho(T_G)$;
2) If $\lambda = 1$, then $\tilde{T}_{G1}x - \lambda x \leq 0$. Therefore $\tilde{T}_{G1}x = \lambda x$. And by Lemma 2, we obtain $\rho(\tilde{T}_{G1}) = \lambda = \rho(T_G)$;
3) If $\lambda > 1$, then $\tilde{T}_{G1}x - \lambda x \geq 0$. Therefore $\tilde{T}_{G1}x \geq \lambda x$. And by Lemma 2, we obtain $\rho(\tilde{T}_{G1}) > \lambda = \rho(T_G)$.

Theorem 5 Let T_G and $\tilde{T}_{G2}$ be defined by (8) and (12) respectively. If $\delta_s \in ((1-a_{ns}a_{sn})/a_{ns}, -a_{sn}) \cap (0, -a_{sn})$ and A is an irreducible L −matrix with $a_{ns}a_{sn} > 0$ for $s = 1,2,\cdots,n-1$, then

i. $\rho(\tilde{T}_{G2}) < \rho(T_G)$, if $\rho(T_G) < 1$;
ii. $\rho(\tilde{T}_{G2}) = \rho(T_G)$, if $\rho(T_G) = 1$;
iii. $\rho(\tilde{T}_{G2}) > \rho(T_G)$, if $\rho(T_G) > 1$.

Proof: By virtue of the preceding results we can get

$$\tilde{T}_{G2}x - \lambda x == (I-\tilde{L})^{-1}\frac{(\lambda-1)}{\lambda}[\tilde{S}U + \lambda\tilde{L}_1]x$$

And since $(I-\tilde{L})^{-1}[\tilde{S}U + \lambda\tilde{L}_1]x$ is nonnegative, we obtain

1) If $\lambda < 1$, then $\tilde{T}_{G2}x - \lambda x \leq 0$. Therefore $\tilde{T}_{G2}x \leq \lambda x$. And by Lemma 2, we obtain $\rho(\tilde{T}_{G2}) < \lambda = \rho(T_G)$;
2) If $\lambda = 1$, then $\tilde{T}_{G2}x - \lambda x \leq 0$. Therefore $\tilde{T}_{G2}x = \lambda x$. And by Lemma 2, we obtain $\rho(\tilde{T}_{G2}) = \lambda = \rho(T_G)$;
3) If $\lambda > 1$, then $\tilde{T}_{G2}x - \lambda x \geq 0$. Therefore $\tilde{T}_{G2}x \geq \lambda x$. And by Lemma 2, we obtain $\rho(\tilde{T}_{G2}) > \lambda = \rho(T_G)$.

RESULTS AND DISCUSSION

Numerical Examples

We consider two examples in order to illustrate the theorems advanced in this research. These examples are adapted from Dehghan and Hajarian (2009).

Example 1 Let

$$A = \begin{pmatrix} 1.0 & -0.2 & -0.1 & -0.1 & -0.1 & -0.1 \\ -0.5 & 1.0 & 0 & 0 & 0 & -0.5 \\ -0.2 & -0.1 & 1.0 & -0.3 & -0.1 & -0.2 \\ -0.1 & 0 & -0.2 & 1.0 & -0.3 & -0.1 \\ -0.3 & -0.2 & -0.1 & -0.1 & 1.0 & -0.1 \\ -0.2 & -0.3 & -0.2 & -0.1 & -0.2 & 1.0 \end{pmatrix}$$

It is obvious that matrix A satisfies the conditions of Theorems 1 -5. The spectral radii of the corresponding iterative matrices are presented in Tables I and II.

Example 2 Let

$$A = \begin{pmatrix} 1.00 & -0.20 & -0.20 & -0.10 & -0.25 & -0.40 \\ -0.50 & 1.0 & 0 & 0 & 0 & -1.00 \\ -0.30 & -0.50 & 1.00 & -0.05 & -0.25 & -0.10 \\ -0.25 & 0.10 & -0.55 & 1.00 & -0.30 & -0.10 \\ -0.20 & -0.15 & -0.30 & -0.05 & 1.00 & -0.50 \\ -0.30 & -0.25 & -0.25 & -0.10 & -0.30 & 1.00 \end{pmatrix}$$

It is easily seen that A is an L −matrix, even though it is not diagonally dominant. The results of Example 2 are illustrated in Tables III and IV.

Results

Table I. The spectral radii of $T_G, \bar{T}_{G1}, \bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ iterative matrices for $\gamma_q = 0.002$, $\delta_s = 0.002$ ($q = 2,3,\cdots,6$ and $s = 1,2,\cdots,5$)

Iterative matrix	Spectral radius
$\boldsymbol{T_G}$	0.6812179257
$\boldsymbol{\bar{T}_{G1}}$	0.6129514333
$\boldsymbol{\bar{T}_{G2}}$	0.6420664384
$\boldsymbol{\tilde{T}_{G1}}$	0.6069503066
$\boldsymbol{\tilde{T}_{G2}}$	0.6449521320

Table II. The spectral radii of $T_G, \bar{T}_{G1}, \bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ iterative matrices for $\gamma_q = 0.0002$, $\delta_s = 0.0002$ ($q = 2,3,\cdots,6$ and $s = 1,2,\cdots,5$)

Iterative matrix	Spectral radius
$\boldsymbol{T_G}$	0.6812179257
$\boldsymbol{\bar{T}_{G1}}$	0.6122894054
$\boldsymbol{\bar{T}_{G2}}$	0.6416648502
$\boldsymbol{\tilde{T}_{G1}}$	0.6060413240
$\boldsymbol{\tilde{T}_{G2}}$	0.6444718235

Table III. The spectral radii of $T_G, \bar{T}_{G1}, \bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ iterative matrices for $\gamma_q = 0.004$, $\delta_s = 0.004$ ($q = 2,3,\cdots,6$ and $s = 1,2,\cdots,5$)

Iterative matrix	Spectral radius
$\boldsymbol{T_G}$	1.188906006
$\boldsymbol{\bar{T}_{G1}}$	1.245696081
$\boldsymbol{\bar{T}_{G2}}$	1.214534814
$\boldsymbol{\tilde{T}_{G1}}$	1.242353178
$\boldsymbol{\tilde{T}_{G2}}$	1.219085080

Table IV. The spectral radii of $T_G, \bar{T}_{G1}, \bar{T}_{G2}, \tilde{T}_{G1}$ and $\tilde{T}_{G2}$ iterative matrices for $\gamma_q = 0.0003$, $\delta_s = 0.0003$ ($q = 2,3,\cdots,6$ and $s = 1,2,\cdots,5$)

Iterative matrix	Spectral radius
$\boldsymbol{T_G}$	1.188906006
$\boldsymbol{\bar{T}_{G1}}$	1.246848156
$\boldsymbol{\bar{T}_{G2}}$	1.215022866
$\boldsymbol{\tilde{T}_{G1}}$	1.243668402
$\boldsymbol{\tilde{T}_{G2}}$	1.219722139

DISCUSSION

Tables I-IV depict the results of Examples 1 and 2, wherein the theorems advanced in this paper are illustrated. In Tables I and II, the spectral radii of the various iterative matrices are shown to be less than one, which shows that all the methods are convergent. Also, the spectral radii of all the preconditioned methods are proven to be smaller than that of the classical Gauss-Seidel method, which is in conformity with the theorems proposed. Tables III-IV shows that all the spectral radii of the various iterative matrices are more than one, indicating non-convergence. This is also understandably in conformity with the foregoing theorems, due to the fact that the Gauss-Seidel method is divergent for this problem. The non-convergence of the Gauss-Seidel method is attributed to the obvious fact that the matrix is not diagonally dominant.

CONCLUSION

Two preconditioners, introduced in an earlier work for SOR iterative methods, have been extended to precondition the Gauss-Seidel method for solving linear systems with $L-$matrices. Convergence theorems and numerical experiments revealed that the convergence rates of the preconditioned Gauss-Seidel methods are superior to that of the classical Gauss-Seidel method.

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