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ON THE ELZAKI SUBSTITUTION AND HOMOTOPY PERTUBATION METHODS FOR SOLVING PARTIAL DIFFERENTIAL EQUATION INVOLVING MIXED PARTIAL DERIVATIVES

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ABSTRACT
This paper investigated new methods of solving partial differential equations involving mixed partial derivatives that were initially solved by Sujit and Karande in their usual notation by making use of Laplace substitution method. The methods investigated in this paper are Elzaki Substitution and Homotopy perturbation Methods of solving partial differential equations with mixed partial derivatives. Finally, the results obtained showed that Elzaki Substitution method and Homotopy Perturbation method are accurate and efficient method to solve partial differential equations involving mixed partial derivatives
Keywords: Partial Differential Equation, Elzaki transform, Homotopy Perturbation, Mixed partial derivatives

INTRODUCTION

According to Ali and Al-Saif (2008), partial differential equation (PDE) is any equation involving a function of more than one independent variable and at least one partial derivative of that function. The order of a PDE is the order of the highest order derivative that appears in the PDE. The principal part of a PDE is the collection of terms in the PDE containing derivatives of order equal to the order of the PDE. A partial derivative of second or greater order with respect to two or more differential variable, for example $f(x,y)=\dfrac{\partial^2 f(x,y)}{\partial x \partial y}$.

Sujit and Karande (2012) defines partial derivative as derivative of a function of multiple variable when all but the variable of interest are held fixed during differentiation, such that partial derivative involving more than one variable are called mixed partial derivatives. The mixed partial derivatives exist and are continuous at a point x_0 then they are equal at x_0 regardless of the order in which they are taken.

Tarig Elzaki introduced a new integral transform known as Elzaki transform which is modified transform of Sumudu and Laplace transforms. Elzaki can be used to solve ordinary differential equations, partial differential equations, partial integro-differential equations, system of partial differential equations and wave equations. The main advantage of Elzaki transform is that it eliminates the need of linearization, perturbation or any other transformation. Elzaki transform are widely used for solving ordinary and partial differential equations. The existing methods for solving partial differential equations involving mixed partial derivatives are time consuming with large computation (Elzaki, 2012).

According to He J. H. (2004), Homotopy Perturbation Method (HPM) was proposed first by Ji-Huan He for solving differential and integral equations. The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which is easily solved. HPM is applied to Voltera's integro differential equation, nonlinear oscillators, bifurcation of nonlinear problems, delay differential equations, nonlinear wave equations, boundary value problems and to other fields.

Although, the method of Laplace substitution method proposed by Sujit H. and Karande B.D. (2012) provides the analytical solution for solving partial differential equations involving mixed partial derivatives, it is however, of high necessity to note the complexity of the method used. In an attempt to provide an easier method of solution with an equivalent level of accuracy and efficiency, we introduce Elzaki substitution and Homotopy Perturbation Method.

METHOD OF SOLUTION

Elzaki Substitution Method

The use of Elzaki substitution method is discussed here. We consider the general form of non-homogeneous mixed partial differential equation with initial conditions is given below

$$Lu(x,y)=Ru(x,y)=H(x,y) \qquad (1)$$

$$u(x,0)=f(x), \quad Uy(0,y)=g(y) \qquad (2)$$

Here $L=\dfrac{\partial}{\partial x \partial y}$, $Ru(x,y)$ is the remaining linear terms in which contains only first order partial derivatives of $u(x,y)$ with respect to either x or y and $h(x,y)$ is the source term. We can write equation (1) in following form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)+Ru(x,y)=h(x,y) \qquad (3)$$

Putting $\frac{\partial u}{\partial y}=U$ in equation (3), we get

$$\frac{\partial U}{\partial x}+Ru(x,y)=h(x,y) \tag{4}$$

Taking Elzaki transform of equation (4) with respect to x, we get

$$\frac{1}{v}E_x[U(x,y)]-vu(0,y)+E_x[Ru(x,y)]=E_x[h(x,y)]$$

$$Ex[U(x,y)]=v^2u_y(0,y)+vE_x[h(x,y)-Ru(x,y)]$$

$$E_x[U(x,y)]=v^2g(y)+vE_x[h(x,y)-Ru(x,y)] \tag{5}$$

Taking inverse Laplace transform of equation (5) with respect to x, we get

$$U(x,y)=g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]] \tag{6}$$

Re substitute the value of $U(x,y)$ in equation (6), we get

$$\frac{\partial u(x,y)}{\partial y}=g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]] \tag{7}$$

This is the first order partial differential equation in the variables x and y. Taking the Elzaki transform of equation (7) with respect to y, we get

$$\frac{1}{v}E_y[u(x,y)]-vu(x,0)=E_y\left[g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]]\right]$$

$$E_y[u(x,y)]=v^2f(x)+vE_y\left[g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]]\right] \tag{8}$$

Taking the inverse Laplace transform of equation (8) with respect to y, we get

$$u(x,y)=f(x)+E_y^{-1}\left[vE_y\left[g(y)+E_x^{-1}[vE_x[h(x,y)-Ru(x,y)]]\right]\right] \tag{9}$$

The last equation (9) gives the exact solution of initial value problem (1).

Homotopy Perturbation Method

The derivation of special solution of follows from Erinle-Ibrahim et al. (2021)

$$\partial_{x^n}{}^n\partial_{y^m}{}^m....,\partial_i{}^i[u(x,y,....,t)]+L[u(x,y,....,t)]+N[u(x,y,....,t)]=f(x,y,....,t) \tag{10}$$

We will assume that $H(x,....,t)$ is the solution of the linear part of ; we can record an illustration to appropriate the value of the selected singular points for example at $X(x,y,....,t)$ and then the corrected solution can be written as follows

$$U(\alpha,\beta,....,i)=H(\alpha,\beta,....,i)+\int_0^{\alpha},...,\int_0^{t}\lambda(x,y,....,t)$$

$$X\left(\partial_{x^n}{}^n\partial_{y^m}{}^m,....,\partial_{t^i}{}^i[u(x,y,....,t)]+L[u(x,t,....,t)]+N[u(x,y,....t)]-f(x,y,....t)\right)dx,....,dt$$

We will point out that $u(x,y,....,t)$ is the Lagrange multiplier and the second expression on the right is called the modification. The method has been modified into an iteration method in the subsequent approximants.

$$U_{n+1}(\alpha,\beta,....,i)=H(\alpha,\beta,....,i)+\int_0^{\alpha},....,\int_0^{\alpha}\lambda(x,y,....t)$$

$$X\left(\partial_{x^n}{}^n\partial_{y^m}{}^m,....,\partial_{t^i}{}^i[u(x,y,....,t)]+L[u(x,t,....t)]+N[u(x,y,....,t)]-f(x,y,....,t)\right)dx,....,dt$$

Beside $H(\alpha,\beta,....,i)$ as preliminary guestimate with likely nonentities and $u(x,y,....,t)$ is pondered as a circumscribed adaptation meaning $\delta u(x,y,....,t)=0$.

Indeed for random $(\alpha, \beta, ...)$, the above equation can be reformulated as follows,

$$U_{n+1}(x, y, ..., t) = H(x, y, ..., t) + \int_0^x, ..., \int_0^i \lambda(x, y, ..., t)$$

$$X\left(\partial_{x^n}{}^n \partial_{y^m}{}^m, ..., \partial_{t^i}{}^i [u_n(x, y, ..., t)] + L[u(x, y, ..., t)] + N[u(x, y, ..., t)] - f(x, y, ..., t)\right) dx, ..., dt.$$

Case 1:
Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-y} \cos x \tag{11}$$

With initial conditions

$$u(x, 0) = 0. \quad u_y(0, y) = 0 \tag{12}$$

Now, Using Elzaki Substitution Method

In the above initial value problem $Lu(x, y) = \frac{\partial^2 u}{\partial x \partial y}$, $h(x, y) = e^{-y} \cos x$ and general linear term $Ru(x, y)$ is zero Equation (10) we can write in the following form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = e^{-y} \cos x \tag{13}$$

Putting $\frac{\partial u}{\partial y} = N$ in equation (12), we get

$$\frac{\partial N}{\partial x} = e^{-y} \cos x \tag{14}$$

This is the non-homogeneous partial differential equation of first order. Taking Elzaki transform on both sides of equation (13) with respect to x, we get

$$\frac{1}{v} E_x[N(x, y)] - vN(0, y) = E_x\left[e^{-y} \cos x\right]$$

$$\frac{1}{v} E_x[N(x, y)] - vu_y(0, y) = e^{-y}\left(\frac{v^2}{1+v^2}\right)$$

Since $u_y(0, y) = 0$, then we have

$$\frac{1}{v} E_x[N(x, y)] = e^{-y}\left(\frac{v^2}{1+v^2}\right)$$

Multiply both sides by v, and then we have

$$E_x[N(x, y)] = e^{-y}\left(\frac{v^3}{1+v^2}\right)$$

Taking inverse Elzaki transform of equation with respect to x, we get

$$E_x^{-1}[N(x, y)] = E_x^{-y}\left[e^{-y}\left(\frac{v^3}{1+v^2}\right)\right]$$

$$N(x, y) = e^{-y} \sin x \tag{15}$$

This is the partial differential equation of first order in the variables x and y. Taking Laplace transform of equation (15) with respect to y, we get

$$\frac{\partial u(x,y)}{dy} = e^{-y}\sin x \tag{16}$$

$$\frac{1}{v}E_y[u(x,y) - vu(x,0)] = E_y\left[e^{-y}\sin x\right]$$

Since $u(x,0)=0,$ then we have (17)

$$\frac{1}{v}E_y[u(x,y)] = E_y\left[e^{-y}\sin x\right]$$

$$\frac{1}{v}E_y[u(x,y)] = \sin x\left(\frac{v^2}{1+v}\right)$$

Multiply both sides by v, and then we have

$$E_y[u(x,y)] = \sin x\left(\frac{v^3}{1+v}\right)$$

$$E_y[U(x,y)] = \sin x\left[v^2 - \frac{v^2}{1+v}\right]$$

Taking inverse Elzaki transform of equation (17) with respect to y, we get

$$E_y^{-1}\left[E_y[u(x,y)]\right] = \sin x E_y^{-1}\left[v^2 - \frac{v^2}{1-v}\right]$$

$$u(x,y) = \sin x\left(1-e^{-y}\right) \tag{18}$$

This is the required exact solution of equation (10).

Using Homotopy Perturbation Method

$$\frac{\partial^2 u}{\partial x\partial y} = e^{-y}\cos x$$

With initial condition $u(x,0)=0, u_y(0,y)=0$

$$U = h(x) + \int_0^y\int_0^x e^{-y}\cos x\,dxdy$$

Let $U = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$, now we have

$$u_0 + \varepsilon u_1 + \varepsilon^2 u_2 = h(x) + \int_0^y\int_0^x e^{-y}\cos x\,dxdy$$

$$\varepsilon^0 : u_0 = h(x) + \int_0^y\int_0^x e^{-y}\cos x\,dxdy$$

$$\varepsilon^1 : u_1 = 0$$

$$\varepsilon^2 : u_2 = 0$$

$$u(x,y)=u_0+u_1+u_2$$
$$=h(x)+\int_0^y\int_0^x e^{-y}\cos x dxdy$$
Since $h(x)=u(x,0)=0$
$$u(x,y)=\int_0^y e^{-y}\sin x dy$$
$$u(x,y)=-e^{-y}\sin x+e^{-(0)}\sin x$$
$$u(x,y)=\left(1-e^{-y}\right)\sin x. \tag{19}$$

Case 2:
Consider the partial differential equation
$$\frac{\partial^2 u}{\partial y\partial x}=\sin x\sin y \tag{20}$$
With initial conditions
$$u(x,0)=1+\cos x, \qquad u_y(0,y)=-2\sin y. \tag{21}$$
Using Elzaki Substitution Method
In the above example assume that $u_x(x,y)$ and $u_y(x,y)$ both are differentiable in the domain of definition of function $u(x,y)$ this implies that $\frac{\partial u}{\partial x\partial y}=\frac{\partial u}{\partial y\partial x}$ given initial conditions (20) force to write the equation (18) in following form and use the substitution $\frac{\partial u}{\partial y}=N$
$$\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\sin x\sin y \tag{22}$$
$$\frac{\partial N}{\partial x}=\sin x\sin y \tag{23}$$
Taking the Elzaki transform of equation (22) with respect to x, we get
$$\frac{1}{v}E_x[N(x,y)]-vN(0,y)=E_x[\sin x\sin y]$$
$$\frac{1}{v}E_x[N(x,y)]-vu_y(0,y)=\sin y\frac{v^3}{1-v^2}$$
Recall that $u_y(0,y)=-2\sin y$ and multiplying through by v in equation (23)
$$E_x[N(x,y)]=-2v^2\sin y+\left(\frac{v^4}{1+v^2}\right)\sin y$$
$$E_x[N(x,y)]=-2v^2\sin y+\left(\frac{-v^2}{1+v^2}+v^2\right)\sin y \tag{24}$$
Taking inverse Elzaki transform of equation (24) with respect to x, we get
$$E_x^{-1}[E_x[N(x,y)]]=-2\sin y+(-\cos x+1)\sin y$$
$$[N(x,y)]=-2\sin y+(-\cos x+1)\sin y$$

$$[N(x,y)] = (-2 - \cos y + 1)\sin y$$
$$[N(x,y)] = (-\cos y - 1)\sin y$$
$$[N(x,y)] = -\sin y(\cos x + 1)$$
$$\frac{\partial u(x,y)}{\partial y} = -\sin y(\cos x + 1) \qquad (25)$$

This give a partial differential equation of first order in the variable x and y .

Now we take the Elzaki transform of equation (25) with respect to y

$$\frac{1}{v}E_y[u(x,y)] - vu(x,0) = E_y[-\sin y(\cos x + 1)]$$
$$u(x,0) = 1 + \cos x$$

then we have

$$\frac{1}{v}E_y[u(x,y)] - v(1+\cos x) = -(\cos x + 1)E_y[\sin y]$$
$$\frac{1}{v}E_y[u(x,y)] - v(1+\cos x) = -(\cos x + 1)\left(\frac{v^3}{1+v^2}\right)$$

Multiply through by v we get

$$E_y[u(x,y)] - v^2(1+\cos x) = -(\cos x + 1)\left(\frac{v^4}{1+v^2}\right)$$
$$E_y[u(x,y)] = -(\cos x + 1)\left(\frac{v^4}{1+v^2}\right) + v^2(1+\cos x)$$
$$E_y[u(x,y)] = -(\cos x + 1)\left(\frac{-v^2}{1+v^2} + v^2\right) + v^2(1+\cos x)$$

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$$E_y[u(x,y)] = (\cos x + 1)\left(\frac{v^2}{1+v^2} - v^2 + v^2\right)$$
$$E_y[u(x,y)] = (\cos x + 1)\left(\frac{v^2}{1+v^2}\right)$$

Taking inverse Elzaki transform of equation (24) with respect to y , we get

$$E^{-1}\left[E_y[u(x,y)]\right] = (\cos x + 1)E^{-y}\left(\frac{v^2}{1+v^2}\right)$$

$$u(x,y) = (1+\cos x)\cos y \qquad (26)$$

This is the required exact solution of equation .

Applying Homotopy Perturbation Method

$$\frac{\partial^2 u}{\partial y \partial x} = \sin x \sin y$$

Initial condition $u(x,0)=1+\cos x, u_y(0,y)=2\sin y$

$$u(x,y)=h(x)+\int_0^y\int_0^x \sin x\sin y \tag{27}$$

Where $h(x)=(x,0)+u(0,y)-u(0,0)$

$$h(x)=1+\cos x+(-2\cos y)-2$$

$$h(x)=1+\cos x-2\cos y-2$$

Now $u(x,y)=h(x)+\int_0^y\int_0^x \sin x\sin y dxdy$

By comparing coefficient

$$\varepsilon^0 : u_0 = h(x)+\int_0^y\int_0^x \sin x\sin y dxdy$$

$$\varepsilon^1 : u_1 = 0$$

$$\varepsilon^3 : u_2 = 0$$

Now $u(x,y)=u_0+u_1+u_2$

$$u(x,y)=h(x)+\int_0^y\int_0^x \sin x\sin y dxdy$$

$$u(x,y)=h(x)+\int_0^y [-\cos x\sin y]_0^x dy$$

$$u(x,y)=h(x)+\int_0^y [-\cos x\sin y-(-\sin y)]dy$$

$$u(x,y)=h(x)+\int_0^y \sin y-\cos x\sin y dy$$

$$u(x,y)=h(x)+[-\cos y+\cos x\cos y]_0^y$$

$$u(x,y)=h(x)+[(-\cos y+\cos x\cos y)-(-\cos 0+\cos x\cos 0)]$$

$$u(x,y)=h(x)-\cos y+\cos x\cos y+1-\cos x$$

Since $h(x)=1+\cos x+2\cos y-2$ put equation

$$u(x,y)=1+\cos x+2\cos y-2-\cos y+\cos x\cos y+1-\cos x$$

$$u(x,y)=\cos y+\cos x\cos y$$

$$u(x,y)=\cos y(1+\cos x) \tag{28}$$

Which is the required exact value**.**

Numerical Simulation

With the aim to provide more efficient methods to solve partial differential equation involving mixed partial derivatives, The researchers make use of two different methods other than the existing Laplace Substitution Method. The methods are Elzaki Substitution Method and Homotopy Perturbation Method. The two solutions compared is the same as the exact solution in each cases. The results of the solution are presented in graphs as follows:

X	y	U(x,y)
0.1	0.1	0.00950040570
	0.2	0.01809672825
	0.3	0.02587500256
	0.4	0.03291307621
	0.5	0.03928138859
0.2	0.1	0.01890588648
	0.2	0.03601263998
	0.3	0.05149147065
	0.4	0.06549729584
	0.5	0.07817029053
0.3	0.1	0.02812246590
	0.2	0.05356872531
	0.3	0.07659345299
	0.4	0.09742708815
	0.5	0.1162781408
0.4	0.1	0.03705805493
	0.2	0.07058956964
	0.3	0.1009301388
	0.4	0.1283834212
	0.5	0.1532241782

Table 1: Tabular representation of the solution of Case *1 (x = 0.1…0.4, y = 0.1...0.5)*

X	y	U(x,y)
0.1	0.1	1.985037454
	0.2	1.955236905
	0.3	1.905900275
	0.4	1.837520519
	0.5	1.750780866
0.2	0.1	1.970174493
	0.2	1.940597075
	0.3	1.891629853
	0.4	1.823762091
	0.5	1.737671900
0.3	0.1	1.945567951
	0.2	1.916359941
	0.3	1.868004296
	0.4	1.800984170
	0.5	1.715969205
0.4	0.1	1.911463691
	0.2	1.882767674

	0.3	1.835259665
	0.4	1.769414349
	0.5	1.685889629

Table 2: Tabular representation of the solution of Case *1 (x = 0.1…0.4, y = 0.1...0.5)*

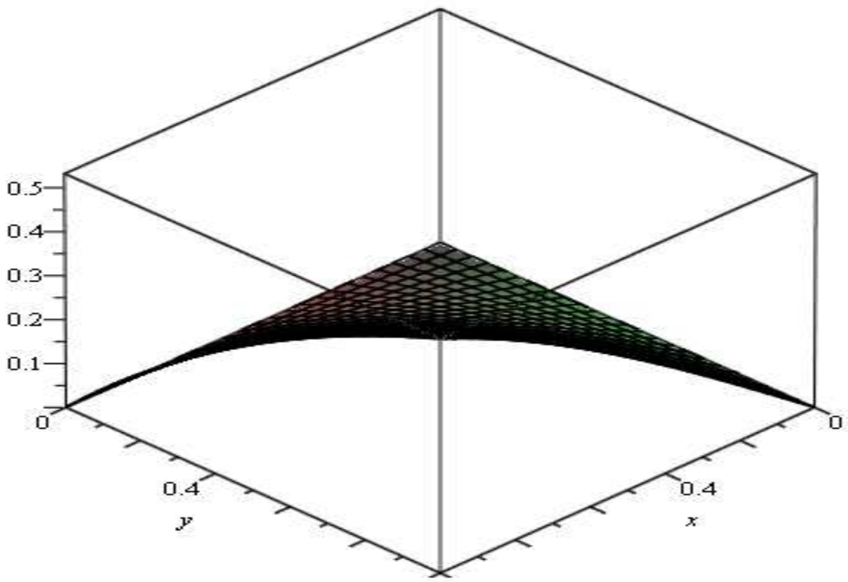

Figure 1: Graphical representation of the solution of Case *1 (x = 0.1…1, y = 0.1...1)*

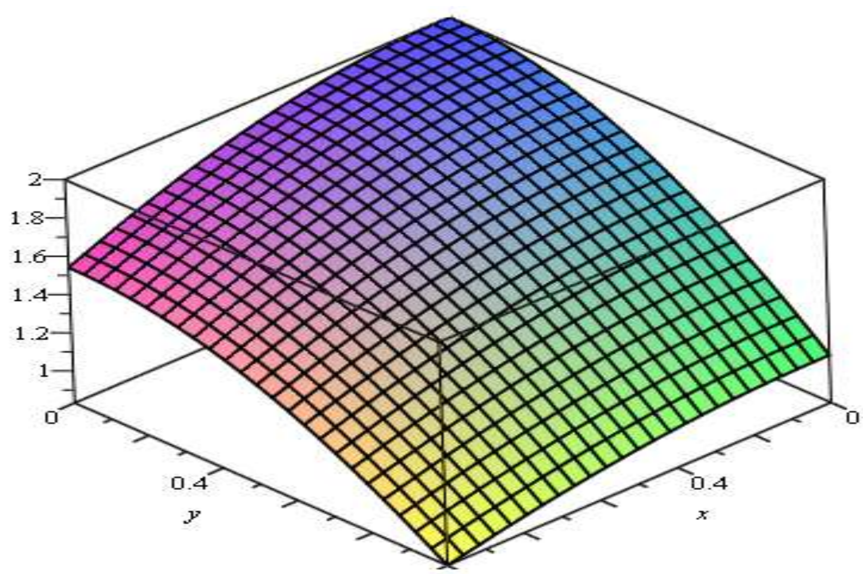

Figure 2: Graphical representation of the solution of Case 2 *(x = 0.1…1, y = 0.1…1)*

Discussion of Findings

The results were presented in both tabular and graphical form. Table 1 showed the tabular representation of the solution of $\frac{\partial^2 u}{\partial x \partial y} = e^{-y}\cos x$ while Table 2 showed the tabular representation of the solution of $\frac{\partial^2 u}{\partial y \partial x} = \sin x \sin y$ both at x= 0,…0.4

and y=0,…0.5. Also Figure 1 shows the graphical representation of the solution of $\frac{\partial^2 u}{\partial x \partial y} = e^{-y}\cos x$ with initial conditions $u(x,0)=0.$ $u_y(0,y)=0$. The results displays shows the values of *u(x,y) at x = y = 0…1.*

Figure 2 shows the graphical representation of the solution of $\frac{\partial^2 u}{\partial y \partial x} = \sin x \sin y$ with initial conditions $u(x,0)=1+\cos x,$ $u_y(0,y)=-2\sin y.$. The results displays shows the values of *u(x,y) at x = y = 0…1.*

CONCLUSION

In this study, we attempted using the Elzaki Substitution Method and Homotopy Perturbation Method to solve the partial differential equations involving mixed partial derivative. Both methods used are semi-analytic and therefore provided the exact solution to the partial differential equation involving mixed partial derivative. The graphical solution was presented using Maple 13 software. The results show that Elzaki Substitution Method and Homotopy Perturbation Method are accurate and efficient method to solve the partial differential equations involving mixed partial derivative. Thus these methods can also be used in place of the Laplace substitution method.

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