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ORDER AND CONVERGENCE OF THE ENHANCED 3-POINT FULLY IMPLICIT SUPER CLASS OF BLOCK BACKWARD DIFFERENTIATION FORMULA FOR SOLVING INITIAL VALUE PROBLEM

[1]Muhammad Abdullahi,[2] Hamisu Musa

[1] Department of Mathematical Science, Federal University Dutsinma. Katsina State. Nigeria.
[2] Department of Mathematics and Computer Science, Umaru Musa Yar'Adua University, Katsina. Katsina State. Nigeria.
Corresponding author's email; [1]maunwala@gmail.com, [2]hamisu.musa@umyu.edu.ng

ABSTRACT

This paper studied an enhanced 3-point fully implicit super class of block backward differentiation formula for solving stiff initial value problems developed by Abdullahi & Musa and go further to established the necessary and sufficient conditions for the convergence of the method. The method is zero stable, A-stable and it is of order 5. The method is found to be suitable for solving first order stiff initial value problems.

Keyword: A-Stable, Block, Consistency, Convergence, Order and Zero stability

INTRODUCTION

In science, engineering and social science most of the real life problems are converted into models. Such models brought stiff ordinary differential equations. Block backward differentiation formula is one of the reliable block numerical methods for obtaining solutions of stiff initial value problems. Backward differentiation formula was first discovered by (Curtiss & Hirschfelder, 1952), in his method integration of stiff equations, Cash (1980) extended the work of Curtiss, with integration of stiff system of ODEs using extended backward differentiation formula, Milner (1953) discovered block numerical solution of differential equation, Brugano (1998) with solving differential problem by multistep method, Chu and Hamilton (1987) with parallel solution of ODE's by multistep method, Dalquish (1974) with problem related to numerical method, Ibrahim (2007) developed 2 and 3 point implicit methods for solving stiff initial value problem, both methods are zero and A-stable can handle stiff problem with appreciated results. Among the block backward methods developed are that of Musa (2014, 2012A, 2013 & 2012B) with a new fifth order implicit block method, a new super class of block backward differentiation formula, an accurate block solver and a new variable step size block method . All the methods are both zero and A-stable and performed better in terms of accuracy, maximum error and reduced computational time. Babangida (2016) with convergence of 2-point diagonally implicit super class of block backward differentiation formula. Recent works include Musa and Unwala (2019) with extended 3-point super class of block backward differentiation formula for stiff initial Value problem, Musa and Bala (2019) with development of 3-point diagonally implicit super class of block backward differentiation formula for stiff initial Value problem. Both methods are 3-point, zero and A-stable, with consistency and accuracy and display better in the result analysis.

The 3-point ESBBDF was derived by modifying a new fifth order method of the form

$$\sum_{j=0}^{5} \alpha_{j,i}\, y_{n+j-2} = h\beta_{k,i,}(f_{n+k} - \rho f_{n+k-1}), k = 1,2,3 \tag{1}$$

A non-zero coefficient $\beta_{k-2,i} \neq 0$ was introduced in (1) (where$\beta_{k-2,i} = \rho\beta_{k,i,}$) and a free parameter ρ was choosing as $\rho = -\frac{4}{5}$ to obtains the 3ESBBD formula

$$\sum_{j=0}^{5} \alpha_{j,i}\, y_{n+j-2} = h\beta_{k,i,}(f_{n+k} - \rho f_{n+k-2})k = 1,2,3 \tag{2}$$

Using $\rho = -\frac{4}{5}$ the following method was derived

$$\begin{aligned}
y_{n+1} &= -\frac{29}{70}y_{n-2} - \frac{37}{28}y_{n-1} + \frac{9}{7}y_n + \frac{23}{14}y_{n+2} - \frac{27}{140}y_{n+3} - \frac{15}{7}hf_{n+1} + \frac{12}{7}hf_{n-1} \\
y_{n+2} &= -\frac{27}{265}y_{n-2} + \frac{44}{53}y_{n-1} - \frac{44}{53}y_n + \frac{72}{53}y_{n+1} - \frac{68}{265}y_{n+3} - \frac{60}{53}hf_{n+2} + \frac{48}{53}hf_n \\
y_{n+3} &= \frac{68}{673}y_{n-2} - \frac{435}{673}y_{n-1} + \frac{1240}{673}y_n - \frac{1580}{673}y_{n+1} + \frac{1380}{673}y_{n+2} - \frac{300}{673}hf_{n+3} + \frac{240}{673}hf_{n+1}
\end{aligned} \tag{3}$$

(3) is called Enhanced 3-Point fully implicit super class of block backward differentiation formula for solving first initial value problems.

Detailed of derivation and stability analysis of the method can found in (Abdullahi& Musa 2021)

Order of the Method

In this section, we derive the order of the methods (3) for the values of $\rho = -\frac{4}{5}$

The method (3) can be converted to a general matrix form as follows

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h\sum_{j=0}^{1} D_j^* Y_{m+j-1}, \tag{4}$$

Where C_0^*, C_1^*, D_0^* and D_1^* are square matrices defined by

$$C_0^* = \begin{bmatrix} \frac{29}{70} & \frac{37}{28} & -\frac{9}{7} \\ \frac{27}{265} & -\frac{44}{53} & \frac{44}{53} \\ -\frac{68}{673} & \frac{435}{673} & -\frac{1240}{673} \end{bmatrix}, C_1^* = \begin{bmatrix} 1 & -\frac{23}{14} & \frac{27}{140} \\ -\frac{72}{53} & 1 & \frac{68}{265} \\ \frac{1580}{673} & -\frac{1380}{673} & 1 \end{bmatrix} \tag{5}$$

$$D_0^* = \begin{bmatrix} 0 & \frac{12}{7} & 0 \\ 0 & 0 & \frac{48}{53} \\ 0 & 0 & 0 \end{bmatrix}, \qquad D_1^* = \begin{bmatrix} -\frac{15}{19} & 0 & 0 \\ 0 & \frac{60}{53} & 0 \\ \frac{48}{673} & 0 & \frac{300}{673} \end{bmatrix} \tag{6}$$

and $\mathrm{Y_{m-1}}, \mathrm{Y_m}, \mathrm{F_{m-1}}$ and $\mathrm{F_m}$ are column vectors defined by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{7}$$

Thus, equations (3) can be rewritten as

$$\begin{bmatrix} \frac{29}{70} & \frac{37}{28} & -\frac{9}{7} \\ \frac{27}{265} & -\frac{44}{53} & \frac{44}{53} \\ -\frac{68}{673} & \frac{435}{673} & -\frac{1240}{673} \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & -\frac{23}{14} & \frac{27}{140} \\ -\frac{72}{53} & 1 & \frac{68}{265} \\ \frac{1580}{673} & -\frac{1380}{673} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = h \begin{bmatrix} 0 & -\frac{12}{7} & 0 \\ 0 & 0 & \frac{48}{53} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} -\frac{15}{7} & 0 & 0 \\ 0 & \frac{60}{53} & 0 \\ \frac{68}{673} & 0 & \frac{300}{673} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{8}$$

Let C_0^*, C_1^*, D_0^* and D_1^* be block matrices defined by

$$C_0^* = [C_0, C_1, C_2], \quad C_1^* = [C_3, C_4, C_5], \qquad D_0^* = [D_0, D_1, D_2]$$

$\mathrm{D}_1^* = [D_3, D_4, D_5]$.Where

$$C_0 = \begin{bmatrix} \frac{29}{70} \\ \frac{27}{265} \\ -\frac{68}{673} \end{bmatrix}, C_1 = \begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix}, C_2 = \begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix}, C_3 = \begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix}, C_4 = \begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix}, C_5 = \begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix}, D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix}, D_2 =$$

$$\begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix}, D_3 = \begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix}, D_4 = \begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix}, D_5 = \begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}. \tag{9}$$

Definition1 (Order of the Method): The order of the block method (3) and its associated linear operator are given by

$$L[\mathrm{y}(\mathrm{x}); \mathrm{h}] = \sum_{\mathrm{j}=0}^{5} [\mathrm{C_j} \mathrm{y}(\mathrm{x} + \mathrm{jh})] - \mathrm{h} \sum_{\mathrm{j}=0}^{5} [\mathrm{D_j} y'(\mathrm{x} + \mathrm{jh})] \tag{10}$$

Where p is unique integer such that

$\mathrm{E_q} = 0$, $\mathrm{q} = 0,1, \ldots \mathrm{p}$ and $\mathrm{E_{p+1}} \neq 0$, where the $\mathrm{E_q}$ are constant Matrices

Defined by:

$$\mathrm{E_0} = \mathrm{C_0} + \mathrm{C_1} + \cdots + \mathrm{C_k}$$

$$\mathrm{E_1} = \mathrm{C_1} + 2\mathrm{C_2} + \cdots + \mathrm{kC_k} - (\mathrm{D_0} + \mathrm{D_1} + \cdots + \mathrm{D_k})$$

$$\vdots$$

$$\mathrm{E_q} = \frac{1}{\mathrm{q}!}(\mathrm{C_1} + 2^{\mathrm{q}}\mathrm{C_2} + \cdots + \mathrm{k^q C_k}) - \frac{1}{(\mathrm{q}-1)!}(\mathrm{D_1} + 2^{\mathrm{q}-1}\mathrm{D_2} + \cdots + (\mathrm{k})^{\mathrm{q}-1}\mathrm{D_k}).$$

For $\mathrm{q} = 0$, we have

$$\mathrm{E_0} = \mathrm{C_0} + \mathrm{C_1} + \mathrm{C_2} + \mathrm{C_3} + \mathrm{C_4} + \mathrm{C_5}$$

$$= \begin{bmatrix} \frac{29}{70} \\ \frac{27}{265} \\ -\frac{68}{673} \end{bmatrix} + \begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + \begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + \begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + \begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

$$E_1 = C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 - (D_0 + D_1 + D_2 + D_3 + D_4 + D_5)$$

$$= \begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2\begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3\begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4\begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5\begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix} - \left[\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

$$E_2 = \frac{1}{2!}(C_1 + 2^2C_2 + 3^2C_3 + 4^2C_4 + 5^2C_5) - \frac{1}{1!}(D_1 + 2D_2 + 3D_3 + 4D_4 + 5D_5)$$

$$= \frac{1}{2!}\left[\begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2^2\begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3^2\begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4^2\begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5^2\begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix}\right] - \frac{1}{1!}\left[\begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + 3\begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + 4\begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

$$E_3 = \frac{1}{3!}(C_1 + 2^3C_2 + 3^3C_3 + 4^3C_4 + 5^3C_5) - \frac{1}{2!}(D_1 + 2^2D_2 + 3^2D_3 + 4^2D_4 + 5^2D_5)$$

$$= \frac{1}{3!}\left[\begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2^3\begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3^3\begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4^3\begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5^3\begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix}\right] - \frac{1}{2!}\left[\begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + 2^2\begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + 3^2\begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + 4^2\begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + 5^2\begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (14)$$

$$E_4 = \frac{1}{4!}(C_1 + 2^4C_2 + 3^4C_3 + 4^4C_4 + 5^4C_5) - \frac{1}{3!}(D_1 + 2^3D_2 + 3^3D_3 + 4^3D_4 + 5^3D_5)$$

$$= \frac{1}{4!}\left[\begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2^4\begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3^4\begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4^4\begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5^4\begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix}\right] - \frac{1}{3!}\left[\begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + 2^3\begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + 3^3\begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + 4^3\begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + 5^3\begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

$$E_5 = \frac{1}{5!}(C_1 + 2^5C_2 + 3^5C_3 + 4^5C_4 + 5^5C_5) - \frac{1}{4}(D_1 + 2^4D_2 + 3^4D_3 + 4^4D_4 + 5^4D_5)$$

$$= \frac{1}{5!}\left[\begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2^5\begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3^5\begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4^5\begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5^5\begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix}\right] - \frac{1}{4!}\left[\begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + 2^4\begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + 3^4\begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + 4^4\begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + 5^4\begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (16)$$

$$E_6 = \frac{1}{6!}(C_1 + 2^6C_2 + 3^6C_3 + 4^6C_4 + 5^6C_5) - \frac{1}{5}(D_1 + 2^5D_2 + 3^5D_3 + 4^5D_4 + 5^5D_5) = \frac{1}{6!}\left[\begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2^6\begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3^6\begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4^6\begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5^6\begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix}\right] - \frac{1}{5!}\left[\begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + 2^5\begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + 3^5\begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + 4^5\begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + 5^5\begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix}\right] = \begin{bmatrix} -\frac{65}{4} \\ \frac{10}{19} \\ \frac{3}{7} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

Therefore, the method (3) is of order 5 according to definition (1), with error constant as $E_6 = \begin{bmatrix} -\frac{65}{4} \\ \frac{10}{19} \\ \frac{3}{7} \end{bmatrix}$

Convergence of the Method

In this section, we apply the theorem on convergence by Henrici (1962) to show the convergence of the method (3)

Theorem (1): Henrici (1962) stated the following conditions for convergence of Linear Multi-Step Method (LMM):

1. A necessary condition for convergence of the Linear Multi-step Method (3) is that the modulus of none of the root of the associated polynomial $\rho(\xi)$ exceeds one, and that the roots of modulus one is simple. The condition, thus imposed on $\rho(\xi)$ is called the condition of zero stability.
2. A necessary condition for convergence of the Linear Multi-step Method (3) is that the order of the associated difference operator be at least one. The condition that the order$\rho \geq 1$, is called the condition of consistency.

To prove that the method (3) is convergent, we need to show that conditions (1) and (2) stated in Theorem (1) are satisfied.

Stability Analysis of the Method

Consider the 3-point super class of block backward differentiation formula derived in (3)

Definition 2 (Zero Stability)

A linear Multistep method (3) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one and that any root with modulus one is simple.

The characteristic polynomial of the method (3) is given by:

$$R(t,\bar{h}) = Det(\mathrm{A} * \mathrm{t} - \mathrm{B}) = 0 \tag{19}$$

Where Det. stands for the determinant. Thus,

$$R(t,\bar{h}) = -\frac{1074666933}{1257254257}t^3 - \frac{368762757}{26754416}t^3\bar{h} + \frac{3343095}{6688604}t^3\overline{h^2} + \frac{58263839007}{247478348}t^2\overline{h^2} + \frac{58263839007}{43247478348}t^2 - \frac{89250}{128627}t^3\overline{h^3} - \frac{297613352}{61869587}t^2\bar{h} - \frac{25020464}{24759199}t + \frac{45696}{4759199}t\bar{h} + \frac{142767}{249683} = 0 \tag{20}$$

By putting $\bar{h} = h\lambda = 0$ in (20), we obtain the first characteristic polynomial as:

$$R(t,0) = -\frac{1074666933}{1257254257}t^3 + \frac{58263839007}{43247478348}t^2 - \frac{25020464}{24759199}t + \frac{142767}{249683} = 0 \tag{21}$$

Hence $\mathrm{t} = 1,\ \mathrm{t} = 0.455458367$ and $\mathrm{t} = -0.027686857$ (22)

Therefore, the method (3) is Zero Stable according to definition (2).

Consistency Conditions

Definition 3 (Consistency)

A Linear Multi-Step Method is said to be consistent if its order p is greater than or equal to one. It also follows that a LMM is consistent if and only if:

$$\sum_{j=0}^{K} C_j = 0 \tag{23}$$

and

$$\sum_{j=0}^{K} jC_j = \sum_{j=0}^{K} D_j \tag{24}$$

Where C_j and D_j are constant coefficient matrices. Similarly, it follows that LMM is consistent if and only if $\rho(1) = 0$ and $\rho(1) = \sigma(1)$. Where ρ and σ are the first and second characteristic polynomial respectively.

In this previous section, it has been shown that the method (3) is of order 5, which is greater than 1, that is order $p \geq 1$. Thus, the linear multi-step method (3) is consistent if the

Conditions (23) and (24) stated above are satisfied:

$$\sum_{\mathrm{j}=0}^{5} \mathrm{C_j} = \mathrm{C_0} + \mathrm{C_1} + \mathrm{C_2} + \mathrm{C_3} + \mathrm{C_4} + \mathrm{C_5}$$

$$= \begin{bmatrix} \frac{29}{70} \\ \frac{27}{265} \\ -\frac{68}{673} \end{bmatrix} + \begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + \begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + \begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + \begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

Therefore, the condition (23) is satisfied. Also

$$\sum_{j=0}^{5} jC_j = 0 \cdot C_0 + 1 \cdot C_1 + 2 \cdot C_2 + 3 \cdot C_3 + 4 \cdot C_4 + 5 \cdot C_5$$

$$= 0 \cdot \begin{bmatrix} \frac{29}{70} \\ \frac{27}{265} \\ -\frac{68}{673} \end{bmatrix} + 1 \cdot \begin{bmatrix} \frac{37}{28} \\ -\frac{44}{53} \\ \frac{435}{673} \end{bmatrix} + 2 \cdot \begin{bmatrix} -\frac{9}{7} \\ \frac{44}{53} \\ -\frac{1240}{673} \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ -\frac{72}{53} \\ \frac{1580}{673} \end{bmatrix} + 4 \cdot \begin{bmatrix} -\frac{23}{14} \\ 1 \\ -\frac{1380}{673} \end{bmatrix} + 5 \cdot \begin{bmatrix} \frac{27}{140} \\ \frac{68}{265} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{27}{7} \\ \frac{108}{53} \\ \frac{368}{673} \end{bmatrix} \quad (26)$$

And

$$\sum_{j=0}^{5} D_j = D_0 + D_1 + D_2 + D_3 + D_4 + D_5$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{12}{7} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{48}{53} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{15}{7} \\ 0 \\ \frac{68}{673} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{60}{53} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{300}{673} \end{bmatrix} = \begin{bmatrix} -\frac{27}{7} \\ \frac{108}{53} \\ \frac{368}{673} \end{bmatrix} \quad (27)$$

Therefore, $\sum_{j=0}^{5} jC_j = \sum_{j=0}^{5} D_j$ .Thus, condition in (24) is also met; the method (3) is consistent. Hence, the method (3) is Convergent in accordance with the theorem (1)

CONCLUSION

All the necessary and sufficient conditions for the convergence of a linear Multistep Method has been tested in this paper on the developed method, Enhanced 3-Point fully implicit block backward differentiation formula and the method satisfied the conditions. The order of the method has been investigated; the method is of order 5. The is found to be suitable for solving first order stiff initial value problems (IVPs).

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