



A Structurally Consistent Risk-Adjusted Z-Number AHP-TOPSIS Framework for Multi-Criteria Decision Making

Adamu Zakari Adamu and *Mujahid Abdullahi

Department of Mathematics, Sule Lamido University Kafin hausu, Jigawa State.

*Corresponding author's email: mujahid.abdullahi@slu.edu.ng

ABSTRACT

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Decision-making under Z-number environments faces persistent challenges, including inconsistent uncertainty aggregation and rank instability. This paper proposes a risk-adjusted Z-number multi-criteria decision-making framework for Z-AHP-TOPSIS. The method replaces the traditional linear convex combination with a multiplicative, reliability-qualified magnitude:

$$M(Z) = \{C(A)C(B)\}^\gamma \exp[-\lambda(\sigma_A^2 + \eta\sigma_B^2)],$$

where reliability qualifies rather than compensates restriction, and uncertainty is penalized through an exponential variance term inspired by Pratt's constant absolute risk aversion framework. The formulation is non-additive and Choquet-inspired: it is not claimed to be formally equivalent to a Choquet integral. The magnitude is embedded in Z-AHP for criteria weighting and Z-TOPSIS for alternative ranking. Mathematical properties are analyzed, including monotonicity in central tendency, anti-monotonicity in variance, and non-homogeneity under scaling. Three classical rank-reversal benchmarks are examined: Belton and Gear's alternative-addition problem and Triantaphyllou's distributive- and ideal-mode problems. In these tests, the proposed method yields internally consistent simultaneous and pairwise rankings. However, because it is a new risk-adjusted aggregation, it can change the original distributive or ideal-mode rankings. Thus, the results demonstrate suppression of simultaneous-pairwise reversal on the tested benchmarks, not universal rank-reversal immunity. Sensitivity analysis over λ , γ , and η shows that criteria-weight ordering remains invariant in the tested ranges. The method preserves Z-number restriction and reliability information without fuzzy conversion. Limitations include parameter specification, triangular fuzzy assumptions, a small benchmark set, and the absence of a substantive real-world case. The framework provides a mathematically consistent basis for reliability-aware MCDM, but broader validation is needed.

Keywords: Z-numbers; MCDM; rank reversal; risk aversion; exponential utility; AHP; TOPSIS

INTRODUCTION

The complexity of modern decision-making across diverse fields ranging from industrial engineering to clinical medicine has necessitated the development of robust Multi-Criteria Decision Making (MCDM) frameworks. Traditional MCDM methods such as the Analytic Hierarchy Process (AHP) and the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) were originally designed to handle precise or "crisp" data. However, real-world scenarios are rarely precise; they are often shrouded in ambiguity, human subjectivity, and varying levels of data reliability. Zadeh (2011) introduced Z-numbers as an ordered pair $Z = (A, B)$ where A represents a fuzzy restriction on values of a variable, and B represents the reliability of A . This elegant formulation captures the dual nature of human judgment: what we believe and how confident we are in that belief. Despite the theoretical strength of Z-numbers, their practical application in MCDM frameworks has faced persistent hurdles: inconsistent uncertainty aggregation and rank instability. Current literature reveals that when Z-numbers are transformed into traditional values for calculation, they often suffer from linear distortion. This distortion frequently leads to rank reversal a phenomenon where the addition or removal of an alternative causes the final rankings to change illogically, undermining the entire decision-making process. The rank reversal problem in MCDM was first formally identified by Belton and Gear (1983), who demonstrated that the introduction or removal of alternatives can alter the ranking structure in AHP.

Subsequent research by Triantaphyllou (2000) further illustrated this issue under various normalization modes. These anomalies challenge the validity of MCDM methods in high-stakes decision environments where consistency and stability are paramount.

Despite extensive research on Z-number MCDM methods, several critical gaps remain. The first concerns the linear convex combination used in existing approaches to Z-number magnitude computation, such as the method proposed by Nik Badrul Alam et al. (2023). These approaches treat restriction and reliability as compensatory contributors through convex combinations of the form $Mag(Z) = \lambda Mag(A) + (1 - \lambda) Mag(B)$. This formulation allows higher reliability to compensate for lower restriction values in the aggregation, which may not always align with the hierarchical interpretation of Z-numbers, where reliability qualifies restriction. A further limitation is information loss because current Z-number-based MCDM techniques often require transformation to fuzzy numbers, resulting in a loss of meaning and degradation of decision quality. Ranking irregularities are also encountered in certain scenarios, thereby undermining decision confidence and method validity. Moreover, medical and other high-stakes decision environments require judgments that are both precise and reliable, whereas existing methods fail to adequately integrate uncertainty through risk- or spread-sensitive mechanisms.

The aim of this research is to develop a modified Z-number-based MCDM method that improves decision-making under uncertainty while controlling rank reversal. Specifically, the paper introduces

a unified exponential risk-adjusted magnitude that replaces linear compensation with multiplicative reliability qualification and exponential variance penalization. This magnitude is embedded within Z-AHP for criteria weighting and integrated with Z-TOPSIS for alternative ranking. The contribution is threefold: first, it provides a mathematically consistent magnitude formulation; second, it clarifies the relationship between the proposed formulation, non-additive aggregation, and risk-aversion theory; and third, it evaluates the method on classical rank-reversal benchmarks while distinguishing mathematical properties from numerically demonstrated behavior.

The first contribution is a unified exponential magnitude formulation given by:

$$M(Z) = (C(A)C(B))^{\gamma} \exp[-\lambda(\sigma_A^2 + \eta\sigma_B^2)] \quad (1)$$

which replaces convex combination with multiplicative synergy and introduces exponential variance penalization. The research also establishes the mathematical relationship between the proposed magnitude and concepts from Choquet capacity theory (Choquet, 1953) and Pratt's constant absolute risk aversion framework (Pratt, 1964). The proposed magnitude is Choquet-inspired and CARA-inspired, but formal equivalence is not claimed. Furthermore, the proposed method is evaluated using classical benchmark problems to demonstrate that it suppresses simultaneous pairwise rank reversal on the tested cases. Its robustness is also examined across variations in the model parameters.

PRELIMINARIES AND RELATED WORKS

Zadeh (2011) defined a Z-number as an ordered pair $Z = (A, B)$, where A is a fuzzy number representing a restriction on the values of a variable, and B is a fuzzy number representing the reliability of A . When both components are triangular fuzzy numbers, the Z-number is denoted as:

$$Z = ((a_1, a_2, a_3), (b_1, b_2, b_3)) \quad (2)$$

with $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$.

The magnitude of a Z-number, as defined by Yager (2016), is typically computed as a product of the magnitudes of its components:

$$|Z| = M(A) \cdot M(B) \quad (3)$$

where $M(\cdot)$ is a defuzzification operator.

Zhang and Pedrycz (2019) extended fuzzy AHP using Z-numbers, merging the reliability component with restriction through different multiplication operations. Yager (2016) proposed an AHP model based on the magnitude of Z-numbers for criteria weighting. Edalatpanah (2020) developed a hybrid fuzzy Z-AHP and Z-TOPSIS model for decision making.

Yazdani and Zavadskas (2017) used fuzzy Z-AHP to derive criteria weights in public service digitalization, combined with Z-WASPAS for ranking. Liu and Teng (2019) presented picture fuzzy Z-AHP and Z-TOPSIS methods, adding hesitation and reliability layers. Qin and Liu (2018) developed a Z-number representation of ELECTRE II with hybrid fuzzy linguistic threshold mechanisms. Deng (2020) proposed a hybrid TODIM-ELECTRE model translating linguistic Z-numbers to basic probability assignments. Herrera (2018) introduced PROMETHEE with linguistic Z-numbers, computing pairwise preference indices with TODIM's value/loss transformations.

Oprić (2019) proposed Z-VIKOR for workplace hazard prioritization. Liu (2020) developed decomposed fuzzy Z-TOPSIS for transfer-center location selection. Liao and Xu (2017) presented practical ways to manipulate and compare Z-numbers within TOPSIS and TODIM frameworks. Rani (2019) proposed a novel ranking method for Z-numbers combining vectorial distance and spread measures within a hybrid AHP-WASPAS framework.

Pramanik (2020) developed Z-number fuzzy AROMAN using Z-SAW and Z-WPM for personnel selection.

Li et al. (2018) introduced Z-DEMATEL framework with Z-number influence matrices. Liao and Xu (2019) proposed hybrid Z-DEMATEL-ANP models. Liu and Zhang (2020) developed Z-DEMATEL-TOPSIS frameworks integrating causal analysis with ranking.

The rank reversal problem in AHP was first formally identified by Belton and Gear (1983), who demonstrated that the introduction or removal of alternatives can alter the ranking structure. In their Example 1, three judgment matrices over alternatives A, B, and C were considered with corresponding normalized weight vectors. When an additional alternative D was introduced, the rank of A and B reversed, even though the pairwise weight ratios associated with alternatives A, B, and C remained unchanged and D was, in fact, a copy of B.

Triantaphyllou (2000) further illustrated this issue under various normalization modes. In one example, a decision matrix with three alternatives and three criteria was ranked simultaneously as $A_3 > A_2 > A_1$. When alternatives were compared two at a time, the resulting ranking became $A_1 > A_2 > A_3$ the reverse of the original ranking.

Normalization occurs at two distinct stages in AHP: local weight extraction from comparison matrices and synthesis of global rankings (Dijkstra, 2013).

Arithmetic Mean (Sum) Method (Dijkstra, 2013):

$$w_i = \frac{1}{n} \sum_{j=1}^n \frac{a_{ij}}{\sum_{k=1}^n a_{kj}} \quad (4)$$

Geometric Mean Method (Dijkstra, 2013):

$$w_i = \frac{(\prod_{j=1}^n a_{ij})^{1/n}}{\sum_{k=1}^n (\prod_{j=1}^n a_{kj})^{1/n}} \quad (5)$$

Distributive Mode (Saaty, 1980):

$$x_i^{dist} = \frac{x_i}{\sum_{j=1}^n x_j} \quad (6)$$

This mode is susceptible to rank reversal as the denominator changes when alternatives are added or removed.

Ideal Mode (Saaty, 1994):

$$x_i^{ideal} = \frac{x_i}{\max_{1 \leq j \leq n} x_j} \quad (7)$$

This mode is recommended for independent choice problems as the reference point remains constant.

Despite extensive literature on Z-number MCDM methods, several gaps remain: existing linear convex combination approaches treat restriction and reliability as compensatory contributors instead of taking reliability as a qualifier; ranking irregularities are encountered in certain scenarios; most methods rely on transformation to fuzzy numbers, losing the reliability component; and there is insufficient integration of risk sensitivity and uncertainty spread.

This research aims to address these gaps by developing a modified Z-number framework that uses a synergic Z-number magnitude with reliability as a qualifier and explicitly integrates uncertainty through risk- or spread-sensitive mechanisms.

Pratt (1964) defined the Coefficient of Absolute Risk Aversion (CARA) as:

$$A(x) = -\frac{u''(x)}{u'(x)} \quad (8)$$

For constant absolute risk aversion $A(x) = a > 0$, this yields the differential equation:

$$-\frac{u''(x)}{u'(x)} = a \quad (9)$$

whose canonical solution is:

$$u(x) = -\frac{1}{a}e^{-ax} \quad (10)$$

This exponential utility function penalizes variance of uncertain outcomes. For a risky prospect:

$$E[U(X)] \approx U(E[X]) - \frac{1}{2}A\sigma^2 \quad (11)$$

This relationship establishes that utility decreases exponentially with variance, providing a qualitative basis for incorporating uncertainty penalization into decision models. A full expected-utility foundation for fuzzy Z-number variance would require specifying a utility over fuzzy outcomes and is beyond the scope of this paper. The proposed term should therefore be described as CARA-inspired rather than formally equivalent.

Choquet (1953) introduced capacities and the Choquet integral, allowing non-additive measures and interaction among criteria. A Choquet capacity is a set function satisfying: Boundary conditions: $\mu(\emptyset) = 0$, $\mu(X) = 1$ Monotonicity: If $A \subseteq B$ then $\mu(A) \leq \mu(B)$ Positive synergy (superadditivity) occurs when:

$$\mu(A \cup B) > \mu(A) + \mu(B) \quad (12)$$

This multiplicative interaction concept is fundamental to transforming the base value of the proposed model from compensatory to synergic. The multiplicative product term $\{C(A)C(B)\}^\gamma$ is non-additive and can be interpreted as Choquet-inspired because it does not assume additive compensation between restriction and reliability. However, it is not claimed to be a formal Choquet integral. Establishing such an equivalence would require specifying a capacity μ on all subsets and proving $M(Z) = \int f d\mu$. This is left as a possible theoretical extension.

From Pratt's work, we derive qualitative correspondences: Expected value $E[X]$ corresponds to central tendency $C(A)C(B)^\gamma$ Variance penalty σ^2 corresponds to total fuzziness $\sigma_A^2 + \eta\sigma_B^2$ Risk aversion parameter a corresponds to decision sensitivity λ Exponential utility $e^{-a\sigma^2}$ corresponds to attenuation $\exp[-\lambda(\cdot)]$

From Choquet's work, we derive: Capacity $\mu(S)$ corresponds to reliability weight $C(B)^\eta$ Interaction term corresponds to multiplicative form capturing synergy Non-additivity corresponds to exponentiation for nonlinear influence

PROPOSED RISK-ADJUSTED Z-NUMBER MAGNITUDE

The proposed unified magnitude formula extends the work of Nik Badrul Alam et al. (2023) through two major transformations:

Step 1: Changing the Base Term from Convex Sum to Multiplicative Synergy

The original formula $Mag(Z) = \lambda Mag(A) + (1 - \lambda) Mag(B)$ is replaced with $C(A)C(B)^\gamma$. This represents a nonlinear aggregation where the product captures dependence (synergy) between restriction and reliability, analogous to a Choquet integral in a non-additive capacity.

Step 2: Introducing an Exponential Variance Penalty

Following Pratt's (1964) constant absolute risk aversion framework, we introduce $\exp[-\lambda(\sigma_A^2 + \eta\sigma_B^2)]$. This exponential term attenuates outcomes based on uncertainty, making the method risk-sensitive.

Unified Formula:

$$M(Z) = (C(A)C(B)^\gamma) \exp[-\lambda(\sigma_A^2 + \eta\sigma_B^2)] \quad (13)$$

For a triangular fuzzy number $\tilde{x} = (l, m, u)$

$$\text{Central Value: } C(l, m, u) = \frac{l + m + u}{3} \quad (14)$$

$$\text{Variance: } \sigma^2(l, m, u) = \frac{(u-l)^2 + (m-l)^2 + (u-m)^2}{18} \quad (15)$$

Table 1: Parameter Interpretation

Parameter	Role / Interpretation
γ	Reliability amplification/controls how reliability magnifies restriction
λ	Uncertainty-penalty strength/controls overall risk sensitivity
η	Reliability-spread penalty/controls the weight assigned to reliability uncertainty

MATHEMATICAL PROPERTIES AND THEORETICAL ANALYSIS

Theorem 1. For any Z-number $Z = (A, B)$, a magnitude function that is monotonic with respect to central tendency and anti-monotonic with respect to variance is required for consistent risk-aware ranking.

Proof. Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be two Z-numbers with $C(A_1) > C(A_2)$ but $\sigma_{A_1}^2 + \eta\sigma_{B_1}^2 > \sigma_{A_2}^2 + \eta\sigma_{B_2}^2$. Any ranking function must balance these competing effects. The exponential form naturally achieves this balance while preserving the multiplicative synergy between components. $\square$

Theorem 2. There exist Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ such that the convex linear combination magnitude ranks $Z_1 > Z_2$ in some settings and $Z_2 > Z_1$ in other settings, while the unified risk-adjusted exponential magnitude ranks $Z_2 > Z_1$ in all positive settings.

Proof. Let $Z_1 : A_1 = (8, 9, 10)$, $B_1 = (0.2, 0.4, 0.6)$ and $Z_2 : A_2 = (6, 7, 8)$, $B_2 = (0.8, 0.9, 1.0)$.

Their centroids are $C(A_1) = 9.00$, $C(B_1) = 0.40$, $C(A_2) = 7.00$, $C(B_2) = 0.90$.

Under the convex linear magnitude $M_C(Z) = \lambda C(A) + (1 - \lambda)C(B)$, if $\lambda = 0.85$,

$$M_C(Z_1) = 0.85(9.00) + 0.15(0.40) = 7.71$$

$$M_C(Z_2) = 0.85(7.00) + 0.15(0.90) = 6.09$$

so $Z_1 > Z_2$. If $\lambda = 0.10$

$$M_C(Z_1) = 0.10(9.00) + 0.90(0.40) = 1.26,$$

$$M_C(Z_2) = 0.10(7.00) + 0.90(0.90) = 1.51,$$

So $Z_2 > Z_1$. Thus the convex combination can reverse ranking as λ changes.

For the proposed magnitude, the variances are $\sigma^2(A_1) = \sigma^2(A_2) = \frac{1}{3}$

$$\sigma^2(B_1) = \frac{(0.6-0.2)^2 + (0.4-0.2)^2 + (0.6-0.4)^2}{18} = \frac{0.24}{18} = 0.0133,$$

$$\sigma^2(B_2) = \frac{(1.0-0.8)^2 + (0.9-0.8)^2 + (1.0-0.9)^2}{18} = \frac{0.06}{18} = 0.0033.$$

Since $C(A_2)C(B_2) = 7.00(0.90) = 6.30 > C(A_1)C(B_1) = 9.00(0.40) = 3.60$,

and Z_2 has lower total variance, for any $\lambda > 0$, $\eta > 0$ and $\gamma \geq 1$, $M_U(Z_2) > M_U(Z_1)$. Thus the proposed magnitude is uncertainty-averse and does not reverse under the same change in λ . $\square$

Corollary 1. The convex linear combination magnitude is not risk-consistent with respect to increasing uncertainty, whereas the unified risk-adjusted exponential magnitude is strictly uncertainty-averse.

Starting from the CARA condition, $-\frac{u''(x)}{u'(x)} = a$, rearranging yields $\frac{u''(x)}{u'(x)} = -a$. Integrating gives $\ln u'(x) = -ax + C_1$. Exponentiating yields $u'(x) = C_2 e^{-ax}$. Integrating again gives $u(x) = -\frac{C_2}{a} e^{-ax} + C_3$. Choosing the canonical form with $C_2 = 1$, $C_3 = 0$ yields $u(x) = -\frac{1}{a} e^{-ax}$.

Table 2: Qualitative correspondences

Econometric Utility	Z-Magnitude Equivalent
$E[X]$	$C(A)C(B)^\gamma$
σ^2	$\sigma_A^2 + \eta \sigma_B^2$
$e^{-a\sigma^2}$ Utility Function	$\exp[-\lambda(\cdot)]$ Magnitude Function

Property 1. (Scale sensitivity and risk-adjusted non-homogeneity). For $c > 0$, scaling the restriction component A by c gives $M(cA, B) = c^\gamma \{C(A)C(B)\}^\gamma \exp[-\lambda(c^2 \sigma_A^2 + \eta \sigma_B^2)]$. Thus, $M(cA, B) \neq cM(A, B)$ in general. Linear homogeneity holds only in the degenerate case where variance penalization is removed, e.g. $\lambda = 0$ and $\gamma = 1$. Under the proposed exponential variance penalty, positive scaling is attenuated rather than linearly preserved. This is consistent with risk-averse behavior: larger uncertainty reduces the effective magnitude.

Property 2. (Uncertainty Aversion). For two Z-numbers with equal central values, the one with lower variance receives higher magnitude.

Proof. The exponential term is strictly decreasing in variance, ensuring that higher uncertainty results in lower magnitude. $\square$

Property 3. (Rank Reversal Resistance). The method maintains ranking stability under alternative addition or removal.

Proof. Demonstrated through application to classical rank reversal problems. This is a numerical demonstration, not a general proof. $\square$

Z-AHP-TOPSIS DECISION PROCEDURE

The following steps are proposed for deriving criteria weights using the unified magnitude:

Step 1. Construct pairwise comparison matrices (PCM) for restriction A and reliability B using linguistic scales.

Step 2. Calculate centers and variances for each entry using: $C(l, m, u) = \frac{l+m+u}{3}$ and $\tau^2(l, m, u) = \frac{(u-l)^2 + (m-l)^2 + (u-m)^2}{18}$.

Step 3. Compute: $M_{ij} = C(A_{ij})C(B_{ij})^\gamma \exp[-\lambda(\sigma^2(A_{ij}) + \eta \sigma^2(B_{ij}))]$ to form the magnitude matrix.

Step 4. Aggregate the magnitude matrix using the geometric mean method: $GM_i = (\prod_{j=1}^n M_{ij})^{1/n}$.

Step 5. Normalize to obtain final weights: $w_i = \frac{GM_i}{\sum_{k=1}^n GM_k}$.

NUMERICAL EXPERIMENTS AND COMPARATIVE ANALYSIS

Definition of reference ranking and rank reversal

In this paper, rank reversal is defined as disagreement between the simultaneous ranking and the pairwise ranking produced by the same decision method. The original distributive- or ideal-mode ranking is not treated as a fixed reference when the aggregation formula itself is changed. The proposed risk-adjusted magnitude is a new aggregation rule; therefore, it may legitimately change the original ranking while still avoiding internal simultaneous-pairwise reversal.

Belton and Gear (1983) problem

The proposed formula is applied to the classical rank reversal problem to demonstrate its capacity to maintain ranking stability. The variance for each alternative is computed from the overall weight vectors used as centroids.

Original Problem: Three judgment matrices over alternatives A, B, C. Corresponding normalized weight vectors. The overall weight vector. Variances.

Applying the proposed formula:

$A = (0.451) \exp(-0.04321) = 0.4319$

$B = (0.470) \exp(-0.00437) = 0.4499$

$C = (0.079) \exp(-0.0001) = 0.0790$

Ranking: $B > A > C$

With Alternative D Added:

Overall weight vector. Variances: $\sigma_1^2 = 0.06992$, $\sigma_2^2 = 0.0117$, $\sigma_3^2 = 0.0002$, $\sigma_4^2 = 0.0117$.

Applying the proposed formula:

$A = (0.365) \exp(-0.06992) = 0.1814$

$B = (0.289) \exp(-0.0117) = 0.2856$

$C = (0.057) \exp(-0.0002) = 0.0570$

$D = (0.289) \exp(-0.0117) = 0.2856$

Ranking: $B > A > C$ is preserved.

Problem 2: Triantaphyllou (2000) Distributive Mode

Table 3: Decision matrix for Problem 2

Alternative	C1	C2	C3	Combined Priorities
A1	9/19	2/12	2/7	0.305
A2	5/19	1/12	4/7	0.344
A3	5/19	9/12	1/7	0.351

Original distributive-mode simultaneous ranking: $A_3 > A_2 > A_1$.

Centroids and Variances for original data:

$x_1 = \frac{9+2+2}{3} = 4.3333$, $\sigma_1^2 = 10.8889$

$x_2 = \frac{5+1+4}{3} = 3.3333$, $\sigma_2^2 = 2.8889$

$x_3 = \frac{5+9+1}{3} = 5$, $\sigma_3^2 = 10.6667$

Applying the proposed formula:

$A_1 = (0.305) \exp(-10.8889) = 5.6863 \times 10^{-6}$

$A_2 = (0.344) \exp(-2.8889) = 1.9100 \times 10^{-2}$

$A_3 = (0.351) \exp(-10.6667) = 8.1812 \times 10^{-6}$

Proposed simultaneous ranking: $A_2 > A_3 > A_1$

Pairwise Comparisons:

1. A_1 vs A_2 : $A_1 = 9.6494 \times 10^{-6}$, $A_2 = 1.1258 \times 10^{-5} \Rightarrow A_2 > A_1$

2. A_1 vs A_3 : $A_1 = 9.7297 \times 10^{-6}$, $A_3 = 1.1158 \times 10^{-5} \Rightarrow A_3 > A_1$

3. A_2 vs A_3 : $A_2 = 0.0286$, $A_3 = 1.1321 \times 10^{-5} \Rightarrow A_2 > A_3$

Combined pairwise ranking: $A_2 > A_3 > A_1$.

Thus, under the proposed method, the simultaneous and pairwise rankings agree. The proposed transformation does not reproduce the original distributive-mode ranking. This change is due to the risk-adjusted magnitude, not to alternative addition or removal. Therefore, the claim should be stated as: The proposed method removes the simultaneous-pairwise inconsistency in this benchmark, but it changes the original distributive-mode ordering.

Problem 3: Triantaphyllou (2000) Ideal Mode

Table 4: Decision matrix for Problem 3

Alternative	C1	C2	C3	Combined Priorities
A1	9/9	5/8	2/8	0.5398
A2	1/9	8/8	5/8	0.6850
A3	8/9	2/8	8/8	0.6730

Original ideal-mode simultaneous ranking: $A_2 > A_3 > A_1$
 Variances: $x_1 = 5.3333$, $\sigma_1^2 = 8.2233$; $x_2 = 4.6667$, $\sigma_2^2 = 8.2222$; $x_3 = 6$, $\sigma_3^2 = 8$
 Applying the proposed formula:
 $A_1 = (0.5398) \exp(-8.2233) = 0.00014$
 $A_2 = (0.6850) \exp(-8.2222) = 0.00018$
 $A_3 = (0.6730) \exp(-8) = 0.00023$
 Proposed simultaneous ranking: $A_3 > A_2 > A_1$
 Pairwise Comparisons:

1. A_1 vs A_2 : $A_1 = 0.0001613$, $A_2 = 0.0002252 \Rightarrow A_2 > A_1$
 2. A_1 vs A_3 : $A_1 = 0.000186$, $A_3 = 0.0002463 \Rightarrow A_3 > A_1$
 3. A_2 vs A_3 : $A_2 = 0.000181$, $A_3 = 0.0002325 \Rightarrow A_3 > A_2$
 Combined pairwise ranking: $A_3 > A_2 > A_1$.
 Again, the proposed method is internally consistent between simultaneous and pairwise evaluation, but it does not preserve the original ideal-mode ranking.

Comparison with existing methods

Table 5: Comparison with existing methods

Aspect	Existing Methods	Proposed Method	Status
Aggregation	Linear convex combination	Multiplicative reliability-qualified aggregation	Proposed design feature
Uncertainty treatment	Often ignored or linearly treated	Exponential variance penalty	CARA-inspired risk adjustment
Information preservation	May convert to fuzzy/crisp values	Retains Z-number restriction and reliability	Design feature
Rank-reversal control	Limited	Reduced simultaneous pairwise reversal in tested benchmarks	Demonstrated on three benchmarks, not universal proof
Theoretical foundation	Often heuristic	Non-additive aggregation and risk-aversion analogy	Choquet-inspired and CARA-inspired, not formal equivalence

Given the pairwise comparison entry from Nik Badrul Alam et al. (2023):

Restriction Component: $A_{12} = (1/5, 1/3, 1)$
 Reliability Component: $B_{12} = (0.2, 0.3, 0.4)$
 Centres: $C(A_{12}) = 0.511$, $C(B_{12}) = 0.300$
 Variances: $\sigma^2(A_{12}) = 0.0612$, $\sigma^2(B_{12}) = 0.0033$
 Unified Magnitude for $\lambda = 1.0$, $\gamma = 1.0$, $\eta = 1.0$
 $M(Z_{12}) = (0.511 \times 0.300) \times \exp[-1.0 \times (0.0612 + 0.0033)] = 0.1533 \times 0.9375 = 0.144$

Table 6: Magnitude matrix $\lambda = 1.0$, $\gamma = 1.0$, $\eta = 1.0$

	C1	C2	C3	C4	C5	C6
C1	1.000	0.144	0.112	0.112	1.287	0.144
C2	0.552	1.000	0.144	0.144	1.287	0.112
C3	0.657	0.552	1.000	0.552	4.260	0.112
C4	0.657	0.552	0.144	1.000	1.287	0.112
C5	0.045	0.045	0.024	0.045	1.000	0.552
C6	0.144	0.112	0.112	0.112	0.552	1.000

Check: $\sum w_i = 1.001$ (within rounding tolerance).

SENSITIVITY AND ROBUSTNESS ANALYSIS

We examine how criteria weights change when varying λ , γ , and η . The weights are derived from pairwise comparisons of Z-numbers using the exponential magnitude.

Table 7: Final weights

Criterion	GM	Weight
C1	0.332	0.127
C2	0.464	0.178
C3	0.908	0.350
C4	0.594	0.229
C5	0.083	0.032
C6	0.220	0.085

Table 8: Variation of criteria weights with λ ($\gamma = 1$, $\eta = 1$)

Criterion	$\lambda = 0.2$	$\lambda = 0.5$	$\lambda = 0.8$	$\lambda = 1.0$	$\lambda = 1.5$
C1	0.095	0.109	0.119	0.127	0.138
C2	0.162	0.171	0.175	0.178	0.180
C3	0.414	0.387	0.365	0.350	0.328
C4	0.259	0.245	0.237	0.229	0.215
C5	0.015	0.022	0.028	0.032	0.039
C6	0.055	0.065	0.076	0.085	0.100

Interpretation: As λ increases, criteria with higher average reliability (e.g., C2) gain weight, while less reliable criteria (e.g., C3) lose weight. This matches the intended uncertainty penalization.

Table 9: Variation of criteria weights with γ ($\lambda = 1.0$, $\eta = 1.0$)

Criterion	$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 2.0$	$\gamma = 5.0$
C1	0.121	0.127	0.135	0.142
C2	0.169	0.178	0.190	0.199
C3	0.334	0.350	0.371	0.388
C4	0.219	0.229	0.242	0.252
C5	0.038	0.032	0.025	0.019
C6	0.085	0.085	0.085	0.085

Interpretation: Increasing γ shifts weight from low-core criteria (e.g., C5) toward high-core criteria (e.g., C3), but the relative ranking of criteria by weight stays the same.

Table 10: Variation of criteria weights with η ($\lambda = 1.0, \gamma = 1.0$)

Criterion	$\eta = 0.5$	$\eta = 1.0$	$\eta = 2.0$	$\eta = 4.0$
C1	0.124	0.127	0.130	0.133
C2	0.174	0.178	0.182	0.185
C3	0.344	0.350	0.357	0.363
C4	0.225	0.229	0.234	0.239
C5	0.035	0.032	0.029	0.026
C6	0.085	0.085	0.085	0.085

Interpretation: The effect is a mild non-linear scaling, and no weight ordering changes throughout the variation of parameters. Across the tested ranges $\lambda \in [0.2, 1.5]$, $\gamma \in [0.5, 5.0]$ and $\eta \in [0.5, 4.0]$, the criteria-weight ordering remains $C_3 > C_4 > C_2 > C_1 > C_6 > C_5$. Thus, the rank order itself not merely the numerical weights is stable within the tested parameter ranges.

DISCUSSION

The proposed unified exponential magnitude formulation makes several theoretical contributions. By replacing the convex sum with the multiplicative structure $(C(A)C(B))^\gamma$, the method captures the synergistic interaction between restriction and reliability. This approach aligns with the spirit of Choquet's capacity theory (Choquet, 1953), in which non-additive measures capture interactions among criteria. The exponential variance penalty $\exp[-\lambda(\sigma_A^2 + \eta\sigma_B^2)]$ provides a rigorous mechanism for uncertainty penalization derived from Pratt's constant absolute risk aversion framework (Pratt, 1964). This mechanism ensures that alternatives with higher uncertainty are appropriately penalized. The method also operates directly on Z-numbers without converting them into fuzzy numbers, thereby preserving both restriction and reliability information throughout the decision process.

The mathematical analysis and application to classical rank reversal problems demonstrate that the proposed method suppresses simultaneous pairwise reversal in the tested benchmarks. The exponential penalty stabilizes rankings by ensuring that uncertainty is appropriately incorporated into the magnitude computation.

The proposed method also has several practical implications. In high-stakes decision environments such as medical diagnosis, where both the value and reliability of judgments are critical, the method provides more defensible decisions by explicitly incorporating uncertainty. It accommodates expert judgments expressed as Z-numbers and preserves the confidence levels that experts attach to their assessments. Furthermore, the parameters λ , γ , and η allow decision-makers to calibrate the method according to their specific risk preferences and decision contexts. The sensitivity analysis confirms that the method maintains consistent rankings across parameter variations, thereby enhancing decision confidence.

Despite these contributions, the method has several limitations. The use of exponential risk-adjusted magnitudes and Z-number transformations increases mathematical complexity compared with traditional fuzzy or crisp MCDM methods. The parameters λ , γ , and η also require careful specification. Although the sensitivity analysis demonstrates stability, the selected parameter values may still affect the outcomes. Moreover, because the model relies on expert inputs for both the restriction and reliability components, its final output remains subject to human factors. The method also assumes triangular fuzzy numbers for both restriction and reliability; therefore, extensions to other fuzzy-number types may require additional development. The small number of benchmark examples and the absence of a substantive real-world case are further limitations that should be addressed in future work.

CONCLUSION

This paper has presented a modified Z-number-based MCDM framework that controls rank reversal through a unified exponential risk-adjusted magnitude. The main innovation is the replacement of linear convex combination with the multiplicative structure $\{C(A)C(B)\}^\gamma$, which treats reliability as a qualifier of restriction. A second innovation is the exponential variance penalty $\exp[-\lambda(\sigma_A^2 + \eta\sigma_B^2)]$, which is inspired by Pratt's constant absolute risk aversion framework. The evidence in this paper should be interpreted at three levels. Mathematically established, the proposed magnitude is monotonic in central tendency and anti-monotonic in variance. It is not linearly homogeneous under scaling; instead, scaling is attenuated by the variance penalty. The convex linear magnitude can reverse under changes in λ , whereas the proposed magnitude is uncertainty-averse under the stated conditions. Demonstrated on benchmark examples, in Belton and Gear's alternative-addition problem, the ranking $B > A > C$ is retained after adding alternative D. In Triantaphyllou's distributive-mode problem, the proposed method gives $A2 > A3 > A1$ in both simultaneous and pairwise evaluation, although this differs from the original distributive ranking $A3 > A2 > A1$. In the ideal-mode problem, the proposed method gives $A3 > A2 > A1$ in both simultaneous and pairwise evaluation, although this differs from the original ideal-mode ranking $A2 > A3 > A1$. Thus, the method suppresses simultaneous-pairwise reversal in the tested benchmarks, but it does not preserve every original ranking. Universal rank-reversal immunity is not proved. Future work should establish general conditions under which the proposed magnitude guarantees ranking stability, test additional benchmark problems, and apply the method to a substantive real-world decision problem. Limitations include dependence on the parameters λ , γ , and η ; the small number of benchmark examples; the assumption of triangular fuzzy numbers; reliance on expert inputs; and the absence of a real-world validation case. Extensions to other fuzzy-number types and formal Choquet-capacity representations remain open.

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