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ENERGY SPECTRUM AND SOME USEFUL EXPECTATION VALUES OF THE TIETZ-HULTHÉN POTENTIAL

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ABSTRACT

In this paper, concept of supersymmetric quantum mechanics has been employed to derive expression for bound state energy eigenvalues of the Tietz-Hulthén potential, the corresponding equation for normalized radial eigenfunctions were deduced by ansatz solution technique. In dealing with the centrifugal term of the effective potential of the Schrödinger equation, a Pekeris-like approximation recipe is considered. By means of the expression for bound state energy eigenvalues and radial eigenfunctions, equations for expectation values of inverse separation-squared and kinetic energy of the Tietz-Hulthén potential were obtained from the Hellmann-Feynman theorem. Numerical values of bound state energy eigenvalues and expectation values of inverse separation-squared and kinetic energy the Tietz-Hulthén potential were computed at arbitrary principal and angular momentum quantum numbers. Results obtained for computed energy eigenvalues of Tietz-Hulthén potential corresponding to $Z = 0$ and $V_0 = 0$ are in excellent agreement with available literature data for Tietz and Hulthén potentials respectively. Studies have also revealed that increase in parameter Z results in monotonic increase in the mean kinetic energy of the system. The results obtained in this work may find suitable applications in areas of physics such as: atomic physics, chemical physics, nuclear physics and solid state physics

KEYWORDS: Energy spectrum, SUSYQM, Tietz-Hulthén potential, Hellmann-Feynman theorem, expectation values

INTRODUCTION

Potential energy functions (or more simply potential) are relevant in quantum mechanics because they provide a means of representing the interaction between a physical object and its neighborhood (Yanar et al., 2021). On the other hand, all the necessary information regarding a quantum mechanical system can be obtained from wave functions representing the system under investigation (Tsaur and Wang, 2014; Eyube et al., 2019a). Such information that can be obtained from the wave function of a system include: thermodynamic properties (Dong and Cruz-Irisson, 2012; Eyube et al, 2021), magnetic properties (Horchani et al., 2020) and optical properties (Eshhgi et al., 2019) just to mention but a few.

However, it is important to understand that obtaining the wave function for a particular quantum system requires solving a relativistic or nonrelativistic wave equation (Eyube et al., 2019b; Eyube et al., 2019c). The Schrödinger wave equation is fundamental and is particularly useful in the description of spinless particles (Yahya and Oyewumi, 2016; Eyube et al., 2019b). For a given potential energy model, the solution of Schrödinger equation is dependent on the presence of the centrifugal term in the effective potential of the equation (Yanar et al., 2020; Eyube et al., 2020a). The centrifugal term is given by $\ell(\ell + 1)\hbar^2/2\mu r^2$ where $\ell \geq 0$ is the is the angular momentum quantum number, r is the separation and μ is the reduced mass of the system.

For all values ℓ, only the Mie-type potential, the harmonic oscillator and the Coulombic potential are known to give exact solutions with the Schrödinger equation (Ikot et al., 2014). On the other hand, many other potentials yield exact solution with the Schrödinger equation for $\ell = 0$, the solutions are also referred to as s-wave solutions (Hassanabadi et al., 2013a). Some of the potential models with s-wave solution include: Tietz potential (Nikoofard et al., 2013; Eyube et al., 2021a), Morse potential (Eyube et al., 2020b), Pöschl-Teller potential (Yanar et al., 2020), Manning-Rosen potential (Chen and Jia, 2009; Ikhdair, 2011), Rosen-Morse potential (Liu et al., 2014) and Hulthén potential (Ikhdair, 2009).

Usually, most of the known potential energy models have no exact solution with the Schrödinger equation, in this case, approximate numerical solution (Lucha and Schöberl, 1999) or approximate analytical solution (Wei and Dong, 2008) are considered. On the one hand, approximate analytical solution has the advantage that it can be obtained in closed forms which is more suitable for exploring the physical properties of the system it represents. Numerical solutions are not closed form solutions but their convergence to the actual solution of the equation can be observed (Eyube et al., 2021b). Thus it suffices to conclude that the most accurate analytical solution is one which agrees more closely to a corresponding numerical solution.

Different solution methods of the Schrödinger and other wave equations have been advocated in the literature, among which include: supersymmetric quantum mechanics (Hassanabadi et al., 2013b), Nikiforov-Uvarov method and its parametric form (Nikoofard et al., 2013), exact and proper quantization rule (Eyube et al., 2020c; Eyube et al., 2021b), ansatz solution method (Chen and Jia, 2009) and asymptotic iteration method (Sous, 2019)

Quite a number of potential energy functions have been used to solve the Schrödinger equation, one of such potential models is the Hulthén potential. The eigenenergies and oscillator strength of this potential model has been studied by Varshni (1990). Jia and collaborators (2008) obtained bound state energies of the Hulthén potential via a newly proposed approximation scheme. The approximate bound state solutions of Dirac equation with Hulthén potential including Coulomb-like tensor potential has been studied (Ikhdair and Sever, 2010). The Hulthén potential has also been studied as a combined potential. With the help of Nikiforov-Uvarov method, Meyur and Debnath (2009) solved the Schrödinger equation with Hulthén plus Manning-Rosen potential and obtained results in terms of Jacobi polynomials. In another studies, analytical solutions of Schrödinger equation with Manning-Rosen plus Hulthén potential was obtained within the frameworks of supersymmetric quantum mechanics (SUSYQM).

This work aims at obtaining the energy spectrum and some useful expectation values of a proposed Tietz-Hulthén potential, the Tietz-Hulthén potential is simply a combination of the Hulthén potential plus Tietz potential expressed as

$$V(r) = -\frac{Z e^2 \delta \,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + V_0 \left\{ \frac{\sinh \frac{1}{2}\delta (r - r_e)}{\sinh \frac{1}{2}\delta r} \right\}^2, \tag{1}$$

Where the first and second terms in equation (1) are the Hulthén potential (Jia et al., 2008) and Tietz potential (Eyube et al., 2021a) respectively. e is the electronic charge, δ is the screening parameter that determines the range of the potential, V_0 is the potential strength of the Tietz potential, r_e is the equilibrium bond length and Z is the atomic number. This paper is organized as follows: In section two, the method of SUSYQM is used to derive energy spectrum of the Tietz-Hulthén potential. In section three, radial wave functions of the Tietz-Hulthén potential is obtained by ansatz solution technique. Some important expectation values of the Tietz-Hulthén potential are obtained in section four. Numerical results are discussed in section five. In section six, a brief conclusion of the work is given.

SUSYQM APPROACH TO THE ENERGY SPECTRUM OF TIETZ-HULTHÉN POTENTIAL

The radial part of the Schrödinger equation expressed in three-dimensional spherical coordinates is given as (Eyube et al., 2021a)

$$\mathrm{H}(r)\psi_{n\ell}(r) = E_{n\ell}\,\psi_{n\ell}(r), \tag{2}$$

where n is the principal quantum number, $\psi_{n\ell}(r)$ is the radial wave function and the Hamiltonian operator is

$$\mathrm{H}(r) = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}}{\mathrm{d}r^2} + V_\mathrm{eff}(r), \tag{3}$$

with the effective potential given as

$$V_\mathrm{eff}(r) = V(r) + \frac{L\hbar^2}{2\mu r^2}. \tag{4}$$

Replacing equation (1) into (4) gives

$$V_\mathrm{eff}(r) = -\frac{Z e^2 \delta \,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + V_0 \left\{ \frac{\sinh \frac{1}{2}\delta (r - r_e)}{\sinh \frac{1}{2}\delta r} \right\}^2 + \frac{L\hbar^2}{2\mu r^2}. \tag{5}$$

Equation (2) has exact solution with (5) only for the case of s-wave. However, with the choice of a suitable approximation scheme, the centrifugal term in equation (5) can be transformed to terms which can be absorbed into the Tietz-Hulthén potential, in this way, approximate analytical solution can be obtained. The Pekeris-like approximation scheme gives that for small values of the screening parameter δ, the centrifugal term can be approximated by (Eyube et al., 2021a)

$$\frac{L\hbar^2}{2\mu r^2} \approx \frac{L\hbar^2}{2\mu} \left\{ d_0 + \frac{d_1 \,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{d_2 \,\mathrm{e}^{-2\mathrm{e}^{-\delta r}}}{\left(1-\mathrm{e}^{-\delta r}\right)^2} \right\}. \tag{6}$$

where the constant coefficients d_{-1+j} (j = 1, 2, 3) are obtained from Eyube et al. (2021a) as

$$d_0 = \frac{1}{r_\mathrm{e}^2} - \frac{1}{\delta r_\mathrm{e}^3}\left(3 + e^{-\delta r_\mathrm{e}}\right)\left(1 - e^{-\delta r_\mathrm{e}}\right) + \frac{3}{\delta^2 r_\mathrm{e}^4}\left(1 - e^{-\delta r_\mathrm{e}}\right)^2, \tag{7}$$

$$d_1 = \frac{2\mathrm{e}^{\delta r_\mathrm{e}}}{\delta r_\mathrm{e}^3}\left(2 + e^{-\delta r_\mathrm{e}}\right)\left(1 - e^{-\delta r_\mathrm{e}}\right)^2 - \frac{6\mathrm{e}^{\delta r_\mathrm{e}}}{\delta^2 r_\mathrm{e}^4}\left(1 - e^{-\delta r_\mathrm{e}}\right)^3, \tag{8}$$

$$d_2 = -\frac{\mathrm{e}^{2\delta r_\mathrm{e}}}{\delta r_\mathrm{e}^3}\left(1 + e^{-\delta r_\mathrm{e}}\right)\left(1 - e^{-\delta r_\mathrm{e}}\right)^3 + \frac{3\mathrm{e}^{2\delta r_\mathrm{e}}}{\delta^2 r_\mathrm{e}^4}\left(1 - e^{-\delta r_\mathrm{e}}\right)^4. \tag{9}$$

Substituting equation (6) into (5) and simplifying, one obtains

$$V_{\text{eff}}\left(r\right)=V_0\ \mathrm{e}^{-\delta r_{\mathrm{e}}}+\frac{L\hbar^2 d_0}{2\mu}-\left\{Z e^2\delta+2V_0\sinh\left(\delta r_{\mathrm{e}}\right)+\frac{L\hbar^2\left(d_2-d_1\right)}{2\mu}\right\}\left(\mathrm{e}^{\delta r}-1\right)^{-1}$$
$$+\left\{4V_0\sinh^2\left(\tfrac{1}{2}\delta r_{\mathrm{e}}\right)+\frac{L\hbar^2 d_2}{2\mu}\right\}\mathrm{e}^{\delta r}\left(\mathrm{e}^{\delta r}-1\right)^{-2}. \quad (10)$$

In arriving at the equation (10), the following relation has been used

$$\frac{1}{\left(\mathrm{e}^{\delta r}-1\right)^2}\equiv-\frac{1}{\mathrm{e}^{\delta r}-1}+\frac{\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r}-1\right)^2}. \quad (11)$$

By inserting equation (10) in (3) and replacing the resulting expression into equation (2) leads to the following equation in compact form

$$\frac{\psi''_{n\ell}\left(r\right)}{\psi_{n\ell}\left(r\right)}=-\gamma_{n\ell}-\frac{\gamma_0}{\mathrm{e}^{\delta r}-1}+\frac{\gamma_1\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r}-1\right)^2}. \quad (12)$$

where prime denotes derivatives with respect to argument in brackets and the parameters $\gamma_{n\ell}$, γ_0 and γ_1 are defined as

$$\gamma_{n\ell}=\frac{2\mu}{\hbar^2}\left(E_{n\ell}-V_0\ \mathrm{e}^{-\delta r_{\mathrm{e}}}\right)-L d_0, \quad (13)$$

$$\gamma_0=\frac{2\mu Z e^2\delta}{\hbar^2}+\frac{4\mu V_0}{\hbar^2}\sinh\left(\delta r_{\mathrm{e}}\right)+L\left(d_2-d_1\right), \quad (14)$$

$$\gamma_1=\frac{8\mu V_0}{\hbar^2}\sinh^2\left(\tfrac{1}{2}\delta r_{\mathrm{e}}\right)+L d_2. \quad (15)$$

Assume supersymmetry (SUSY) as one in which $E_{0\ell}=0$, equation (12) is satisfied by the ground state ($n=0$) wave function

$$\psi_{0\ell}\left(r\right)=N_{0\ell}\exp\left(-\int\phi\left(r\right)\mathrm{d}r\right), \quad (16)$$

where $N_{0\ell}$ is the normalization constant for the ground state wave function and ϕ is referred to as superpotential in the context of SUSYQM (ref). Substituting equation (16) in (12), one obtains the nonlinear Riccati differential equation given by

$$\phi^2-\phi'=-\gamma_{0\ell}-\frac{\gamma_0}{\mathrm{e}^{\delta r}-1}+\frac{\gamma_1\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r}-1\right)^2}. \quad (17)$$

In order to solve equation (17), a trial wave function of the form

$$\phi\left(r\right)=-c_0\left(\mathrm{e}^{\delta r}-1\right)^{-1}+c_1, \quad (18)$$

is chosen, where c_0 and c_1 are constants. Upon replacing equation (18) into (17) and comparing coefficients, get

$$c_0=\delta\tau, \quad (19)$$

$$c_1=\frac{\gamma_0}{2c_0}-\frac{c_0}{2}, \quad (20)$$

$$\gamma_{0\ell}=-\left(\frac{\gamma_0}{2c_0}-\frac{c_0}{2}\right)^2, \quad (21)$$

where

$$\tau=\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{\gamma_1}{\delta^2}}\equiv\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{8\mu V_0}{\delta^2\hbar^2}\sinh^2\left(\tfrac{1}{2}\delta r_{\mathrm{e}}\right)+\frac{L d_2}{\delta^2}}, \quad (22)$$

Next a pair of partner potentials is constructed for the Hamiltonian, these partner potentials are defined as (Jia et al., 2009; Xu et al., 2010)

$$\mathrm{V}_-\left(r\right)\equiv\phi^2-\phi'=\left(\frac{\gamma_0}{2c_0}-\frac{c_0}{2}\right)^2-\frac{c_0^2+2c_0c_1}{\mathrm{e}^{\delta r}-1}+\frac{\left(c_0^2-\delta c_0\right)\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r}-1\right)^2}, \quad (23)$$

$$\mathrm{V}_+\left(r\right)\equiv\phi^2+\phi'=\left(\frac{\gamma_0}{2c_0}-\frac{c_0}{2}\right)^2-\frac{c_0^2+2c_0c_1}{\mathrm{e}^{\delta r}-1}+\frac{\left(c_0^2+\delta c_0\right)\mathrm{e}^{\delta r}}{\left(\mathrm{e}^{\delta r}-1\right)^2}. \quad (24)$$

If shape invariant condition of SUSYQM applies, the partner potentials in (23) and (24) are related by (Jia et al., 2009; Xu et al., 2010)

$$V_{+}\left(r,\, a_0\right)=V_{-}\left(r,\, a_1\right)+R\left(a_1\right), \tag{25}$$

where a_1 is a new set of parameters uniquely determined from the old set $c_0 \equiv A$ through the relations $c_1 = c_0 + \delta$, $c_2 = c_0 + 2\,\delta$, …, $c_n = c_0 + n\,\delta$. R (c_1) is the remainder term and is independent of r (Jia et al., 2009; Xu et al., 2010). Equation (25) leads to the relation

$$R\left(c_j\right)=\left(\frac{\gamma_0}{2c_{j-1}}-\frac{c_j}{2}\right)^2-\left(\frac{\gamma_0}{2c_j}-\frac{c_j}{2}\right)^2, \tag{26}$$

where $j \in \square^{+}$. To determine the energy eigenvalues, we define

$$\gamma_{n\ell}^{(-)}=\sum_{j=1}^{n}R\left(c_j\right). \tag{27}$$

Substituting equation (26) in (27) and expanding out the summation leads to

$$\gamma_{n\ell}^{(-)}=\left(\frac{\gamma_0}{2c_0}-\frac{c_0}{2}\right)^2-\left(\frac{\gamma_0}{2c_n}-\frac{c_n}{2}\right)^2\equiv\left(\frac{\gamma_0}{2c_0}-\frac{c_0}{2}\right)^2-\left\{\frac{\gamma_0}{2\left(c_0+\delta n\right)}-\frac{c_0+\delta n}{2}\right\}^2. \tag{28}$$

The energy eigenvalue is obtained from the expression

$$\gamma_{n\ell}=\gamma_{0\ell}+\gamma_{n\ell}^{(-)}. \tag{29}$$

Inserting equations (13), (21) and (28) into (29) and simplifying yields

$$E_{n\ell}=V_0\,\mathrm{e}^{-\delta r_{\mathrm{e}}}+\frac{L\hbar^2 d_0}{2\mu}-\frac{\delta^2\hbar^2}{2\mu}\left(\frac{\eta}{\rho}-\frac{\rho}{2}\right)^2, \tag{30}$$

where

$$\eta=\frac{\gamma_0}{2\delta^2}, \tag{31}$$

$$\rho=n+\tau. \tag{32}$$

RADIAL WAVE FUNCTIONS OF THE TIETZ-HULTHÉN POTENTIAL

Wave functions of the rotating Tietz-Hulthén potential can be obtained via ansatz solution method. Letting

$$x=\mathrm{e}^{-\delta r}, \tag{33}$$

where $x\in\left(0,\,1\right)$, equation (12) is transformed to

$$x\left(1-x\right)\psi_{n\ell}''\left(x\right)+\left(1-x\right)\psi_{n\ell}'\left(x\right)+\left(\frac{\gamma_0-\gamma_{n\ell}}{\delta^2}+\frac{\gamma_{n\ell}}{\delta^2}\frac{1}{x}-\frac{\gamma_1}{\delta^2}\frac{1}{1-x}\right)\psi_{n\ell}\left(x\right)=0. \tag{34}$$

Since equation (34) has singularities at $x = 0$ and $x = 1$, we suppose an ansatz solution of the form

$$\psi_{n\ell}\left(x\right)=N_{n\ell}\,x^{\frac{1}{2}\sigma}\left(1-x\right)^{\frac{1}{2}\omega}\Omega_{n\ell}\left(x\right), \tag{35}$$

where σ is a parameter, $N_{n\ell}$ is the normalization constant, $\Omega_{n\ell}$ (x) is an unknown function of x and the parameter ω is given by equation (22). Replacing equation (35) into (34), leads to the Gauss hypergeometric differential equation

$$x\left(1-x\right)\Omega_{n\ell}''\left(x\right)+\left\{\sigma+1-\left(\sigma+\tau+1\right)x\right\}\Omega_{n\ell}'\left(x\right)-\left\{\left(\sigma+\tau\right)^2+\frac{\gamma_0-\gamma_{\nu J}}{\delta^2}\right\}\Omega_{n\ell}\left(x\right)=0, \tag{36}$$

subject to the requirement that

$$\sigma\,\alpha=\left(-\gamma_{n\ell}\right)^{\frac{1}{2}}. \tag{37}$$

Considering bound state solutions, equation (36) has solution as the hypergeometric function given by

$$\Omega_{n\ell}\left(x\right)={}_2\mathrm{F}_1\left(-n,\, n+\sigma+\tau;\, \sigma+1;\, x\right). \tag{38}$$

Normalization condition of wave functions requires that

$$\int_0^{\infty}\mathrm{P}_{n\ell}\left(r\right)\mathrm{d}\,r=1, \tag{39}$$

where

$$\mathrm{P}_{n\ell}\left(r\right)=\left|\psi_{n\ell}\left(r\right)\right|^{2}, \tag{40}$$

is the probability density function. Upon substituting equations (40), (35) and then (38) into (39) gives

$$N_{n\ell}=\left(\frac{\delta}{M_{n\ell}}\right)^{\frac{1}{2}}, \tag{41}$$

where

$$M_{n\ell}=\int_{0}^{x_0} x^{\sigma-1}\left(1-x\right)^{\omega}\left|{}_2\mathrm{F}_1\left(-n,\ n+\sigma+\tau;\ \sigma+1;\ x\right)\right|^{2}\mathrm{d}\,x. \tag{42}$$

where the upper limit of the definite integral is $x_0=\mathrm{e}^{\delta r_\mathrm{e}}$

EXPECTATION OR MEAN VALUES OF THE TIETZ-HULTHÉN POTENTIAL

Here, the Hellmann-Feynman theorem (HFT) (Eyube et al., 2021b) is used to obtain expression for the expectation or mean values of kinetic energy for the system in Tietz-Hulthén potential. The HFT states that if the Hamiltonian H(q) of a quantum mechanical system is a function of a parameter, q, then H (q, r), $E_{n\ell}(q)$ and $\psi_{n\ell}(q)$ are related by

$$\frac{\partial E_{n\ell}\left(q\right)}{\partial q}=\left\langle\psi_{n\ell}\left(r,\ q\right)\left|\frac{\partial\mathrm{H}}{\partial q}\right|\psi_{n\ell}\left(r,\ q\right)\right\rangle, \tag{43}$$

where $\psi_{n\ell}(r, q)$ is the normalized radial wave function. For the Tietz-Hulthén potential, the Hamiltonian is given by substituting equation (5) into (3), this gives

$$\mathrm{H}=-\frac{\hbar^{2}}{2\mu}\frac{\mathrm{d}^{2}}{\mathrm{d}r^{2}}-\frac{Ze^{2}\,\delta\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}+V_0\left\{\frac{\sinh\frac{1}{2}\delta\left(r-r_e\right)}{\sinh\frac{1}{2}\delta r}\right\}^{2}+\frac{L\hbar^{2}}{2\mu r^{2}}. \tag{44}$$

Differentiating equation (30) partially with respect to L, we find

$$\frac{\partial E_{n\ell}}{\partial L}=\frac{\hbar^{2}d_0}{2\mu}-\frac{\hbar^{2}\sqrt{\Lambda}}{\mu}\left\{\frac{d_2-d_1}{2\rho}-\frac{d_2}{\sigma\left(2\tau-1\right)}\right\}. \tag{45}$$

where

$$\Lambda=\left(\frac{\eta}{\rho}-\frac{\rho}{2}\right)^{2}, \tag{46}$$

$$\sigma^{-1}=\frac{\eta}{\rho^{2}}+\frac{1}{2}. \tag{47}$$

Similarly, differentiating equation (43) with respect to L gives

$$\frac{\partial\mathrm{H}}{\partial L}=\frac{\hbar^{2}}{2\mu r^{2}}. \tag{48}$$

Upon substituting equations (48) and (45) into (43), the expectation value of inverse separation squared is found to be

$$\left\langle r^{-2}\right\rangle=d_0-\sqrt{\Lambda}\left\{\frac{d_2-d_1}{\rho}+\frac{2d_2}{\sigma\left(2\tau-1\right)}\right\}. \tag{49}$$

The expectation value of kinetic energy $\left\langle T\right\rangle$ is given as (Eyube et al., 2021b)

$$\left\langle\mathrm{T}\right\rangle\equiv\left\langle\psi_{n\ell}\left(r,\ q\right)\left|-\frac{\hbar^{2}}{2\mu}\frac{\mathrm{d}^{2}}{\mathrm{d}r^{2}}\right|\psi_{n\ell}\left(r,\ q\right)\right\rangle. \tag{50}$$

Explicit expression for $\left\langle\mathrm{T}\right\rangle$ is obtained by Hellmann-Feynman theorem, by differentiating equations (30) and (44) with respect to μ, one obtains

$$\frac{\partial E_{n\ell}}{\partial\mu}=-\frac{L\hbar^{2}d_0}{2\mu^{2}}+\frac{\delta^{2}\hbar^{2}\Lambda}{2\mu^{2}}-\frac{\sqrt{\Lambda}}{\mu}\left\{\frac{Ze^{2}\delta}{\rho}+\frac{2V_0}{\rho}\sinh\left(\delta r_\mathrm{e}\right)-\frac{8V_0}{\sigma}\frac{\sinh^{2}\left(\frac{1}{2}\delta r_\mathrm{e}\right)}{2\tau-1}\right\}, \tag{51}$$

$$\frac{\partial\mathrm{H}}{\partial\mu}=\frac{\hbar^{2}}{2\mu^{2}}\frac{\mathrm{d}^{2}}{\mathrm{d}r^{2}}-\frac{L\hbar^{2}}{2\mu^{2}r^{2}}. \tag{52}$$

Replacing equations (51) and (52) in (43) and with the help of equation (49), the expectation value of the kinetic energy of the system can simply be written as

$$\langle \mathrm{T}\rangle = -\mu\frac{\partial E_{n\ell}}{\partial \mu} - L\frac{\partial E_{n\ell}}{\partial L}\,. \tag{59}$$

RESULTS AND DISCUSSION

In order to confirm the validity of the equation for energy spectrum of the Tietz-Hulthén potential derived in this work, it is observed that by letting $V_0 = 0$, equation (1) is reduced to Hulthén potential energy function (Varshni, 1990). If in addition, d_{-1+j} (j = 1, 2, 3) are chosen as 0, $\omega\delta^2$ and δ^2 respectively as opposed to that given by equations (7) – (9), the equation (30) is reduced to equation (19) of Jia et al. (2008) which is the expression for energy eigenvalues of the Hulthén potential. On the other hand, setting $Z = 0$, equation (1) gives the potential energy function of Tietz potential (Nikoofard et al., 2013; Eyube et al., 2021a), if the mapping $\delta \rightarrow 2\alpha$ is used, equation (30) reproduces equation (41) of Eyube et al. (2021a), the expression for ro-vibrational energy of the Tietz oscillator. Therefore, we conclude that equation (30) is the correct expression for the energy spectrum of the Tietz-Hulthén potential, where the Tietz and Hulthén potentials are two special cases.

Using equation (30), bound state energies are calculated for arbitrary principal and angular momentum quantum numbers as a function of Z, computed results and are represented by $E_{n\ell}$ (SUSY) in Table 1. Also shown in Table 1 are literature data of bound state energy eigenvalues, $E_{n\ell}$ (PQR) of Tietz oscillator obtained by proper quantization rule (Eyube et al., 2021a). Evidently, results obtained for $E_{n\ell}$ (SUSY) and in favorable agreement with those of $E_{n\ell}$ (PQR). Graphical plot of the variation of bound state energy eigenvalues versus Z for different values principal quantum number (n) is shown in Figure 1, as revealed by the plot, the energies increases with increase in n and also decreases monotonically when Z is gradually increased from zero.

Equations (49) and (59) have been used to compute expectation values for inverse separation-squared and kinetic energy respectively, the results are shown in Table 1. Figure 2 is the plot of expectation values of inverse position-squared as a function of Z for different quantum states, n. from the plot, it is clear that $< r^{-2} >$ increases monotonically as Z is increased from zero and it is smaller for larger values of n. The graphical plot of the variation of $< T >$ versus Z for different values of n is shown in Figure 3. Here, it is noted that $< T >$ increases monotonically as Z is gradually increased and is independent of the state of the system.

Table 1: Bound state energy eigenvalues and expectation values as a function of n, ℓ and Z for μ = 5, r_e = 2 and e = 1 (all physical quantities in atomic unit)

state		$Z = 0$					$Z = 1$	
n	ℓ	$E_{n\ell}$ (SUSY)	$E_{n\ell}$ (PQR)	$\langle r^{-2}\rangle$	$\langle T\rangle$	$E_{n\ell}$ (SUSY)	$\langle r^{-2}\rangle$	$\langle T\rangle$
0	0	0.059306	0.059294	0.145249	0.024491	-0.509761	0.735332	0.123987
	1	0.083224	0.083203	0.098535	0.018236	-0.388677	0.498844	0.092325
	2	0.112446	0.112405	0.054250	0.011595	-0.240735	0.274656	0.058706
	3	0.135995	0.135919	0.028440	0.007110	-0.121507	0.143996	0.035999
	5	0.164181	0.163988	0.008657	0.002893	0.021216	0.043842	0.014653
1	0	0.116300	0.116235	0.066639	0.030376	-0.221216	0.337385	0.153794
	1	0.127612	0.127529	0.048081	0.024291	-0.163945	0.243437	0.122992
	2	0.142489	0.142371	0.028872	0.017069	-0.088617	0.146189	0.086431
	3	0.155544	0.155374	0.016455	0.011517	-0.022513	0.083328	0.058328
	5	0.172885	0.172562	0.005695	0.005424	0.065315	0.028855	0.027488
2	0	0.144553	0.144408	0.035918	0.026688	-0.078163	0.181870	0.135141
	1	0.150770	0.150599	0.026955	0.022247	-0.046684	0.136495	0.112660
	2	0.159350	0.159131	0.017147	0.016609	-0.003232	0.086839	0.084122
	3	0.167328	0.167041	0.010359	0.011910	0.037174	0.052471	0.060339
	5	0.178755	0.178281	0.003943	0.006192	0.095074	0.019988	0.031401
3	0	0.160580	0.160332	0.021517	0.022167	0.003004	0.108975	0.112276
	1	0.164355	0.164074	0.016594	0.019007	0.022123	0.084051	0.096282
	2	0.169745	0.169404	0.010999	0.014804	0.049428	0.055721	0.075010
	3	0.174972	0.174547	0.006936	0.011094	0.075908	0.035148	0.056233
	5	0.182898	0.182254	0.002840	0.006215	0.116096	0.014411	0.031550
5	0	0.177139	0.176622	0.009476	0.015204	0.086907	0.048038	0.077086
	1	0.178831	0.178268	0.007570	0.013516	0.095485	0.038385	0.068547
	2	0.181357	0.180711	0.005299	0.011130	0.108297	0.026880	0.056477
	3	0.183949	0.183192	0.003544	0.008854	0.121449	0.017988	0.044966
	5	0.188212	0.187170	0.001612	0.005516	0.143103	0.008200	0.028091

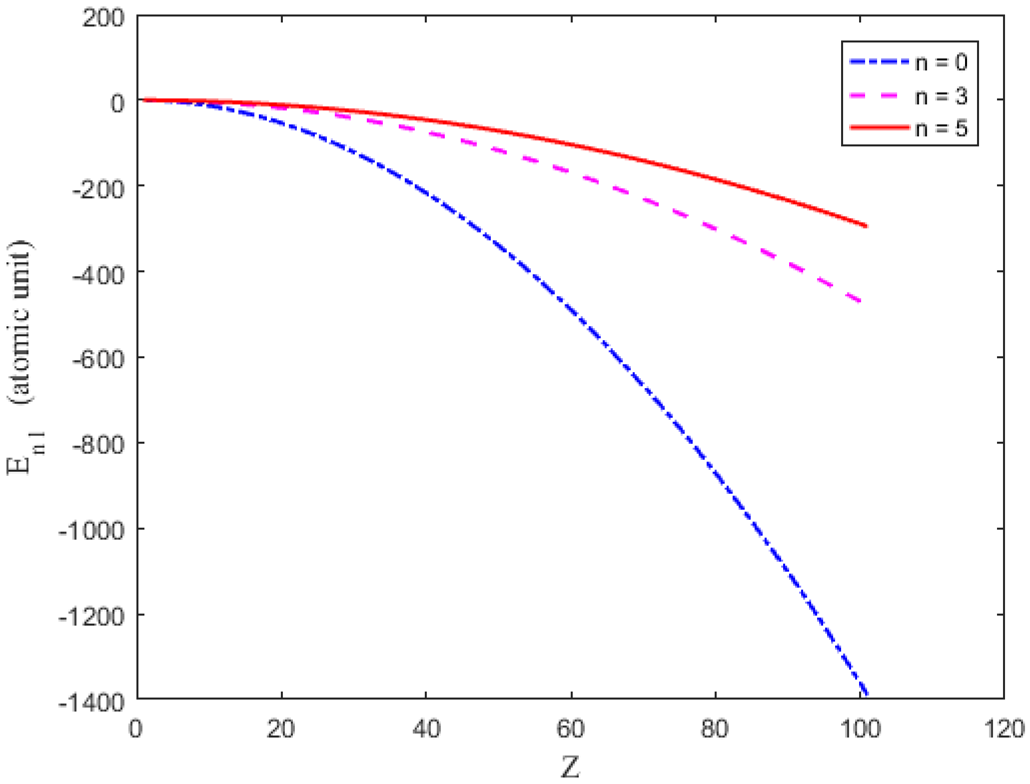

Figure 1: Graphical representation of the variation of bound state energy eigenvalues as a function of *Z* for different principal quantum numbers

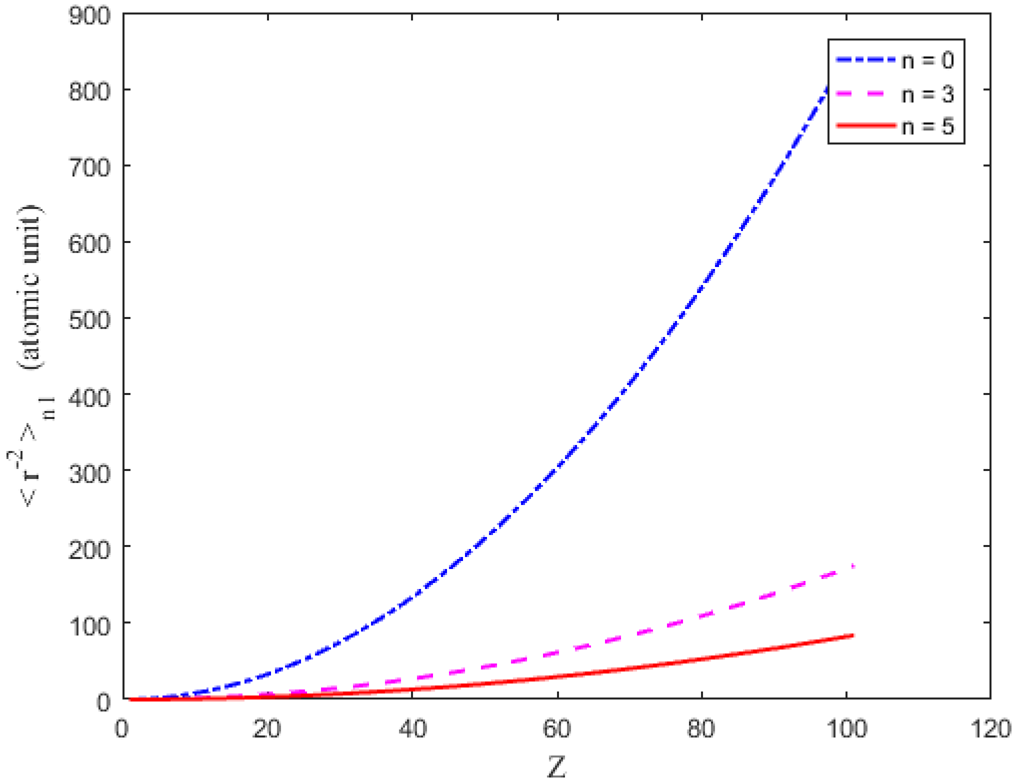

Figure 2: Variation of expectation value of inverse separation-squared as a function of *Z* for different principal quantum numbers

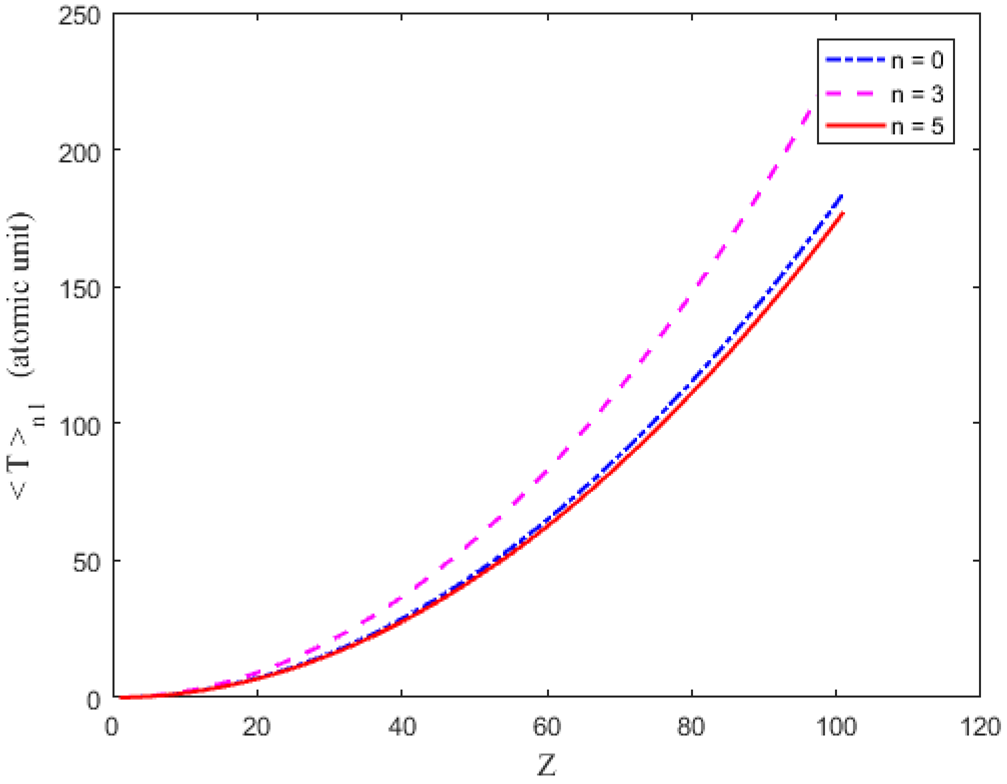

Figure 3: Variation of expectation value of kinetic energy as a function of *Z* for different principal quantum numbers

CONCLUSION

In the present work, the ideas of supersymmetric quantum mechanics and ansatz solution approach have been used to deduce expressions for bound state energy eigenvalues and normalized radial wave functions of the Tietz-Hulthén potential. In modeling the centrifugal term of the effective potential of the Schrödinger equation, a Pekeris-like approximation scheme is employed. With the help of Hellmann-Feynman theorem and the expression for bound state eigen energies, formulas for expectation values of inverse separation-squared and kinetic energy of the Tietz-Hulthén potential were obtained. Numerical values of bound state energy eigenvalues and expectation values were calculated at arbitrary principal and angular momentum quantum numbers. Results obtained for computed energy eigenvalues of Tietz-Hulthén potential for $Z = 0$ and $V_0 = 0$ are in excellent agreement with available data in the literature for Tietz and Hulthén potentials respectively. Studies have also shown that increase in parameter *Z* results in monotonic increase in the mean kinetic energy of the system

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