

Fibrewise Fuzzy Soft Topological Spaces

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ABSTRACT

Fibrewise topology provides a framework for studying topological structures associated with a projection onto a fixed base space, while fuzzy soft topology offers a flexible mathematical setting for handling uncertainty through fuzzy soft sets and their topological properties. However, the interaction between fibrewise structures and fuzzy soft topological spaces has not been sufficiently developed. This paper introduces the concept of fibrewise fuzzy soft topological spaces by considering a fibrewise set over a fuzzy soft topological base space and requiring the associated projection to be fuzzy soft continuous. The induced fibrewise fuzzy soft topology is investigated, and fundamental properties involving fuzzy soft continuous and fuzzy soft open fibrewise mappings are established. The concepts of fibrewise closed fuzzy soft topological spaces and fibrewise open fuzzy soft topological spaces are also introduced through fuzzy soft closed and fuzzy soft open projection mappings, respectively. Several propositions concerning these classes are obtained, including preservation properties under suitable fibrewise mappings, finite coverings, finite products, and subspace constructions. Characterisations associated with fibrewise closed and fibrewise open structures are also examined. These results extend the interaction between classical fibrewise topology and fuzzy soft topology and provide a theoretical framework for further investigation of fibrewise structures in fuzzy soft topological settings.

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INTRODUCTION

Fibrewise topology provides a systematic framework for studying spaces and mappings relative to a fixed base space, with the projection map serving as the central structural object (James, 1989). Fuzzy set theory was introduced by Zadeh (1965), and Chang (1968) subsequently developed the notion of fuzzy topological spaces. Soft set theory was proposed by Molodtsov (1999) as a parameterised approach to uncertainty, while fuzzy soft sets combined soft parameters with fuzzy membership (Maji et al., 2001). Fuzzy soft mappings and fuzzy soft topological structures were then developed further, including continuity, openness, closedness, products, and projection mappings (Kharal & Ahmad, 2009; Tanay & Kandemir, 2011; Varol & Aygün, 2012; Aygünoglu et al., 2014). Related work has considered fibrewise topology in fuzzy and soft settings (Hussain & Yousif, 2021; Yousif, 2021). However, a unified treatment in which the structural projection is defined directly over a fuzzy soft topological base, together with a systematic analysis of the closed and open forms generated by that projection, remains insufficiently developed. The present study addresses this gap by placing fuzzy soft topology within the common-base architecture of fibrewise topology. The contribution is therefore not merely a reformulation of fuzzy soft topology: the common base and the structural projection impose additional compatibility conditions on mappings, products, restrictions, and preservation results. The paper is organised as follows. Section 2 recalls the cited preliminary notions. Section 3 introduces the proposed fibrewise fuzzy soft topological structure and its induced topology, products, and coproducts. Section 4 introduces the fibrewise closed and

fibrewise open forms and develops their principal results, proofs, and illustrative examples in one continuous section. Section 5 summarises the mathematical contribution and indicates directions for further work. This study aims to introduce fibrewise fuzzy soft topological spaces and establish fundamental properties and characterisations of their fibrewise open and fibrewise closed forms.

MATERIALS AND METHODS

This section recalls the basic concepts of fibrewise topology, fuzzy sets, fuzzy soft sets, fuzzy soft mappings, and fuzzy soft topological spaces required for the development of the proposed framework. These definitions are taken from the cited literature and establish the notation used in the subsequent sections.

Definition 2.1 (James, 1989)

A mapping $\vartheta : M \rightarrow N$, in which M and N are FW sets over B , with projection $p_M : M \rightarrow B$ and $p_N : N \rightarrow B$, is said to be fibrewise mapping if $p_N \circ \vartheta = p_M$, or $\vartheta(M_b) \subseteq N_b$, for each point $b \in B$.

Be aware a fibrewise mapping $\vartheta : M \rightarrow N$ over B is constrained by restriction, a fibrewise mapping $\vartheta : M_{B^*} \rightarrow N_{B^*}$ over B^* for all subset $B^* \subseteq B$.

Definition 2.2 (James, 1989)

Let M be a fibrewise set over a topological space B , with projection $p : M \rightarrow B$. A topology on M is called a fibrewise topology over B whenever p is continuous.

Definition 2.3 (James, 1989)

Let M and N be fibrewise topological spaces over the same base space B , and let $f: M \rightarrow N$ be a fibrewise mapping. Then:

- f is fibrewise continuous if it is continuous with respect to the topologies on M and N . $\partial(m)b \in B, m \in M_b$
- f is fibrewise open if $f(U)$ is open in N for every open set U in M . $\partial(m)b \in B, m \in M_b$
- f is fibrewise closed if $f(C)$ is closed in N for every closed set C in M . $\partial(m)b \in B, m \in M_b$

Definition 2.4 (Yousif, 2021)

Let M be a fibrewise topological space over B , with projection $p: M \rightarrow B$.

- M is called fibrewise closed if the projection p is a closed mapping.
- M is called fibrewise open if the projection p is an open mapping.

Definition 2.5 (Zadeh, 1965)

Let X be a nonempty set. A fuzzy set A in X is characterised by a membership function $\mu_A: X \rightarrow [0,1]$, where $\mu_A(x)$ is the degree of membership of x in A .

$$\mu = \{(m, \mu(m)): m \in M, 0 \leq \mu(m) \leq 1\},$$

The collection of all fuzzy subsets in M will be denoted by I^M that is $I^M = \{\mu: \mu \text{ is a fuzzy subset of } M\}$ and $\mu(M)$ is said to be the membership function.

Example 2.6 (Negoița & Ralescu, 1975)

Consider a fuzzy subset of the real numbers representing the concept of “real numbers close to zero.” Its membership function assigns larger values to numbers near zero and smaller values as the distance from zero increases. The original membership function and its diagram are retained below. $\mu_A: \mathbb{R} \rightarrow [0,1]$

$$\mu_A(x) = \frac{1}{1+(x-10)^2}, \forall x \in \mathbb{R} \quad (1)$$

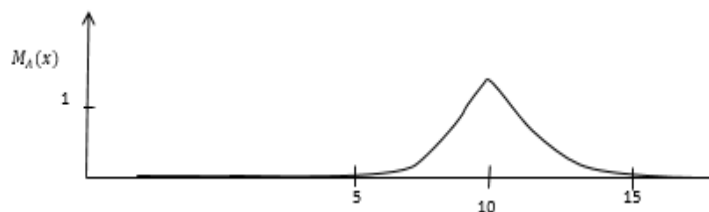


Figure1: Diagram of Example 1.6

Definition 2.7 (Maji et al., 2001)

Let X be a universe and E a set of parameters. A pair (F, E) is called a fuzzy soft set over X iff $F: E \rightarrow I^X$, where I^X denotes the family of all fuzzy subsets of X .

Thus, for every parameter $e \in E$, $F(e)$ is a fuzzy subset of X . $(f, E) = \{(e, f(e)) \mid e \in E, f(e) \in I^M\}$ can be used to represent a fuzzy soft set f over X . $(I^M)^E$ represents the family of all fuzzy soft sets over M .

Definition 2.8 (Kharal & Ahmad, 2009)

Let X, Y be universes with parameter sets E, K , respectively. If $u: X \rightarrow Y$ and $p: E \rightarrow K$ are mappings, then the pair (u, p)

induces a fuzzy soft mapping between the corresponding families of fuzzy soft sets.

Definition 2.9 (Kharal & Ahmad, 2009)

Let (F, E) and (G, K) be fuzzy soft sets over X and Y , respectively, and let (u, p) be a fuzzy soft mapping induced by $u: X \rightarrow Y$ and $p: E \rightarrow K$.

(a) The image of (F, E) under (u, p) , denoted by $(u, p)(F, E)$, is the fuzzy soft set over Y obtained by applying the underlying mappings to the membership values and parameters, as specified by the image formula below.

$$u(f)_k(y) = \begin{cases} \bigvee_{u(x)=y} \bigvee_{p(e)=k} f_e(x), & \text{if } x \in u^{-1}(y); \\ 0, & \text{otherwise} \end{cases} \quad \forall k \in p(E), \forall y \in Y. \quad (2)$$

Otherwise,

(b) The inverse image of (G, K) under (u, p) , denoted by $(u, p)^{-1}(G, K)$, is the fuzzy soft set over X defined by the corresponding inverse-image operation.

If u and p are injective (respectively, surjective), the induced fuzzy soft mapping is called injective (respectively, surjective).

Definition 2.10 (Tanay & Kandemir, 2011)

Let (H, E) be a fuzzy soft set over X , and let $\tilde{\tau}$ be a family of fuzzy soft subsets of (H, E) . then $\tilde{\tau}$ is called a fuzzy soft topology on (H, E) if the following conditions hold:

- the null fuzzy soft set and the absolute fuzzy soft set (H, E) belong to $\tilde{\tau}$;
- if (F, E) and (G, E) belong to $\tilde{\tau}$, then their fuzzy soft intersection belongs to $\tilde{\tau}$;
- For every indexed family $\{(F_i, E): i \in I\}$ contained in $\tilde{\tau}$, the fuzzy soft union $\bigcup_i (F_i, E)$ belongs to $\tilde{\tau}$.

The pair $(H, \tilde{\tau}, E)$ is called a fuzzy soft topological space. Members of $\tilde{\tau}$ are called fuzzy soft open sets, and a fuzzy soft set is called fuzzy soft closed if its complement is fuzzy soft open.

As in ordinary topology, the discrete fuzzy soft topology contains all fuzzy soft subsets, whereas the indiscrete fuzzy soft topology contains only the null and absolute fuzzy soft sets.

Definition 2.11 (Aygünöğlu et al., 2014; Osmanoğlu & Tokat, 2014)

Let $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ be fuzzy soft topological spaces, and let $P_{(f,u)}: (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a fuzzy soft mapping. Then:

- P is fuzzy soft continuous if $P^{-1}(G, L) \in \tilde{\tau}$ whenever $(G, L) \in \tilde{\sigma}$;
- P is fuzzy soft open if $P(F, E) \in \tilde{\sigma}$ whenever $(F, E) \in \tilde{\tau}$;

- iii. P is fuzzy soft closed if $P(C, E)$ is fuzzy soft closed in $(K, \tilde{\sigma}, L)$ whenever (C, E) is fuzzy soft closed in $(H, \tilde{\tau}, E)$.

Definition 2.12 (Aygünöğlu et al., 2014)

Let (F, A) and (G, B) be fuzzy soft sets over nonempty universes X_1 and X_2 , respectively. Their Cartesian product is the fuzzy soft set $(F \times G, A \times B)$ over $X_1 \times X_2$, where, for $(a, b) \in A \times B$ and $(x, y) \in X_1 \times X_2$, the membership value is defined by $(F \times G)_{(a, b)}(x, y) = \min\{F_a(x), G_b(y)\}$. Thus, $(F \times G, A \times B)$ is a fuzzy soft set on $X_1 \times X_2$ with parameter set $A \times B$.

According to this definition the fuzzy soft set $(f \times g)_{A \times B}$ is a fuzzy soft set on $X_1 \times X_2$ and the universal parameter set is $E_1 \times E_2$.

Definition 2.13 (Aygünöğlu et al., 2014)

Let (F, A) and (G, B) be fuzzy soft sets over X_1 and X_2 . The projection fuzzy soft mappings $P_i, i \in \{1, 2\}$, are induced by the ordinary projections $p_i: X_1 \times X_2 \rightarrow X_i$ and $q_i: A \times B \rightarrow A$ (for $i = 1$) or B (for $i = 2$).

These projections send the Cartesian-product fuzzy soft structure to its corresponding first or second component.

Definition 2.14 (Varol & Aygün, 2012; Aygünöğlu et al., 2014)

Let $\{(X_i, E_i), \tilde{\tau}_i: i \in I\}$ be a family of fuzzy soft topological spaces. Put $X = \prod_{i \in I} X_i$ and $E = \prod_{i \in I} E_i$, and let P_i denote the corresponding projection fuzzy soft mappings. The product fuzzy soft topology on X with parameter set E is the coarsest fuzzy soft topology for which every projection P_i is fuzzy soft continuous.

RESULTS AND DISCUSSION

Fibrewise Fuzzy Soft Topological Spaces

This section introduces the concept of a fibrewise fuzzy soft topological space and establishes some of its fundamental characterizations and mapping properties. Particular attention is given to induced fibrewise fuzzy soft topologies, continuity and openness of fibrewise mappings, and the behaviour of these notions under fibrewise products and coproducts.

Definition 3.1

Let H be a fibrewise set over a fuzzy soft topological space (B, Ω, G) , and let $p: H \rightarrow B$ be the associated projection. A fuzzy soft topology $\tilde{\tau}$ on H is called a fibrewise fuzzy soft topology over B if p is fuzzy soft continuous. In this case, $(H, \tilde{\tau}, E)$ is called a fibrewise fuzzy soft topological space over (B, Ω, G) .

Let H be a fibrewise set over B , and let $(K, \tilde{\sigma}, L)$ be a fibrewise fuzzy soft topological space over (B, Ω, G) . Suppose that $\theta: H \rightarrow (K, \tilde{\sigma}, L)$

is a fibrewise mapping. The fuzzy soft topology induced on H by θ is the smallest fibrewise fuzzy soft topology for which θ is fuzzy soft continuous. This topology is called the induced fibrewise fuzzy soft topology on H .

Proposition 3.2

Let $(H, \tilde{\tau}, E)$ carry the fibrewise fuzzy soft topology induced by the fibrewise mapping $\theta: (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$, where $(K, \tilde{\sigma}, L)$ is a fibrewise fuzzy soft topological space over (B, Ω, G) . Let $(Z, \tilde{\gamma}, M)$ be any fibrewise fuzzy soft topological space over the same base space and let $\psi: (Z, \tilde{\gamma}, M) \rightarrow (H, \tilde{\tau}, E)$ be a fibrewise mapping. Then ψ is fuzzy soft continuous $\Leftrightarrow \theta \circ \psi$ is fuzzy soft continuous.

Proof

Suppose first that ψ is fuzzy soft continuous. Since $\tilde{\tau}$ is the topology induced by θ , the mapping θ is fuzzy soft continuous. Hence, the composition $\theta \circ \psi: (Z, \tilde{\gamma}, M) \rightarrow (K, \tilde{\sigma}, L)$ is fuzzy soft continuous. Conversely, suppose that $\theta \circ \psi$ is fuzzy soft continuous. By the definition of the topology induced by θ , the fuzzy soft open sets generating $\tilde{\tau}$ are inverse images $\theta^{-1}(V, L)$, where (V, L) is fuzzy soft open in $(K, \tilde{\sigma}, L)$. For every such set, $\psi^{-1}(\theta^{-1}(V, L)) = (\theta \circ \psi)^{-1}(V, L)$.

Since $\theta \circ \psi$ is fuzzy soft continuous, the right-hand side is fuzzy soft open in $(Z, \tilde{\gamma}, M)$. Therefore, the inverse image under ψ of every generator of the induced topology $\tilde{\tau}$ is fuzzy soft open. Hence ψ is fuzzy soft continuous. Thus, ψ is fuzzy soft continuous $\Leftrightarrow \theta \circ \psi$ is fuzzy soft continuous.

Proposition 3.3

Let $\theta: (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a fibrewise fuzzy soft homeomorphism onto its image, where $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ are fibrewise fuzzy soft topological spaces over (B, Ω, G) . Let $\psi: (Z, \tilde{\gamma}, M) \rightarrow (H, \tilde{\tau}, E)$ be a fibrewise mapping. Then ψ is fuzzy soft open $\Leftrightarrow \theta \circ \psi$ is fuzzy soft open onto $\theta(H)$.

Proof

Suppose that ψ is fuzzy soft open. If (T, M) is fuzzy soft open in $(Z, \tilde{\gamma}, M)$, then $\psi(T, M)$ is fuzzy soft open in $(H, \tilde{\tau}, E)$. Since θ is fuzzy soft open onto $\theta(H)$, $(\theta \circ \psi)(T, M) = \theta(\psi(T, M))$ is fuzzy soft open in $\theta(H)$. Thus $\theta \circ \psi$ is fuzzy soft open onto $\theta(H)$.

Conversely, suppose that $\theta \circ \psi$ is fuzzy soft open onto $\theta(H)$. For every fuzzy soft open set (T, M) in $(Z, \tilde{\gamma}, M)$, the set $(\theta \circ \psi)(T, M)$ is fuzzy soft open in $\theta(H)$. Since $\theta^{-1}: \theta(H) \rightarrow H$ is fuzzy soft continuous and open, $\psi(T, M) = \theta^{-1}((\theta \circ \psi)(T, M))$ is fuzzy soft open in $(H, \tilde{\tau}, E)$. Hence ψ is fuzzy soft open.

Fibrewise Products

Let $\{(H_r, \tilde{\tau}_r, E_r)\}_{r \in I}$ be a family of fibrewise fuzzy soft topological spaces over the common fuzzy soft base space (B, Ω, G) . Their fibrewise product is denoted by $\prod_B(H_r, \tilde{\tau}_r, E_r)$ and is equipped with the fibrewise fuzzy soft topology induced by the canonical projections $\pi_r: \prod_B(H_r, \tilde{\tau}_r, E_r) \rightarrow (H_r, \tilde{\tau}_r, E_r)$, $r \in I$.

Accordingly, a fibrewise mapping $\psi: (Z, \tilde{\gamma}, M) \rightarrow \prod_B(H_r, \tilde{\tau}_r, E_r)$ is fuzzy soft continuous if and only if each component mapping $\pi_r \circ \psi: (Z, \tilde{\gamma}, M) \rightarrow (H_r, \tilde{\tau}_r, E_r)$ is fuzzy soft continuous for every $r \in I$. In particular, if $(H_r, \tilde{\tau}_r, E_r) = (H, \tilde{\tau}, E)$ for every $r \in I$, the diagonal mapping $\Delta: (H, \tilde{\tau}, E) \rightarrow \prod_B(H_r, \tilde{\tau}_r, E_r)$ satisfies $\pi_r \circ \Delta = id_M$ for every $r \in I$. Since the identity mapping is fuzzy soft continuous, the universal property of the fibrewise product implies that Δ is fuzzy soft continuous.

Fibrewise Coproducts

Let $\{(H_r, \tilde{\tau}_r, E_r)\}_r$ be a family of fibrewise fuzzy soft topological spaces over (B, Ω, G) . Their fibrewise coproduct is denoted by $\coprod_B(H_r, \tilde{\tau}_r, E_r)$ and is equipped with the fuzzy soft coproduct topology determined by the canonical fibrewise injections $\omega_r: (H_r, \tilde{\tau}_r, E_r) \rightarrow \coprod_B(H_r, \tilde{\tau}_r, E_r)$.

For a fibrewise mapping $\vartheta: \coprod_B(H_r, \tilde{\tau}_r, E_r) \rightarrow (H, \tilde{\tau}, E)$, the mapping ϑ is fuzzy soft continuous if and only if every component $\vartheta \circ \omega_r: (H_r, \tilde{\tau}_r, E_r) \rightarrow (H, \tilde{\tau}, E)$ is fuzzy soft continuous. If $(H_r, \tilde{\tau}_r, E_r) = (H, \tilde{\tau}, E)$ for every $r \in I$, the codiagonal mapping $\nabla: \coprod_B(H_r, \tilde{\tau}_r, E_r) \rightarrow (H, \tilde{\tau}, E)$ is determined by $\nabla \circ \omega_r = id_M, r \in I$, and is fuzzy soft

continuous by the universal property of the fibrewise coproduct.

Example 3.4

Let (B, Ω, G) be a fuzzy soft topological space, and let $(H_1, \tilde{\tau}_1, E_1)$ and $(H_2, \tilde{\tau}_2, E_2)$ be fibrewise fuzzy soft topological spaces over B . Their fibrewise product is $H_1 \times_B H_2 = \{(h_1, h_2) \in H_1 \times H_2 : p_1(h_1) = p_2(h_2)\}$.

Let $\pi_1 : H_1 \times_B H_2 \rightarrow H_1$, $\pi_2 : H_1 \times_B H_2 \rightarrow H_2$ be the canonical projections. Equip $H_1 \times_B H_2$ with the fibrewise fuzzy soft topology generated by inverse images $\pi_1^{-1}(U_1)$ and $\pi_2^{-1}(U_2)$, where $U_1 \in \tilde{\tau}_1$ and $U_2 \in \tilde{\tau}_2$. Then both projections are fuzzy soft continuous.

Consequently, if $(Z, \tilde{\gamma}, M)$ is a fibrewise fuzzy soft topological space and $f_1 : Z \rightarrow H_1$, $f_2 : Z \rightarrow H_2$ are fibrewise fuzzy soft continuous mappings satisfying $p_1 \circ f_1 = p_2 \circ f_2$, then the induced mapping $f : Z \rightarrow H_1 \times_B H_2$, $f(z) = (f_1(z), f_2(z))$ is fuzzy soft continuous. This example illustrates how the fibrewise fuzzy soft product preserves the common base-space structure while extending the usual product construction to the fibrewise fuzzy soft setting.

Fibrewise Closed and Fibrewise Open Fuzzy Soft Topological Spaces

This section introduces two classes of fibrewise fuzzy soft topological spaces defined through the behaviour of the structural projection. The terminology “fibrewise” is used consistently throughout. To improve readability, the full expressions “fibrewise closed fuzzy soft topological space” and “fibrewise open fuzzy soft topological space” are used instead of repeated abbreviations. The definitions introduced below are followed by preservation results, local characterizations, product properties, restriction results, and illustrative examples.

Definition 4.1 (Proposed)

A fibrewise fuzzy soft topological space $(H, \tilde{\tau}, E)$ over (B, Ω, G) is called a fibrewise closed fuzzy soft topological space if its structural projection $P_{fu} : (H, \tilde{\tau}, E) \rightarrow (B, \Omega, G)$ is a fuzzy soft closed mapping; that is, for every fuzzy soft closed set (F, C) in $(H, \tilde{\tau}, E)$, the image $P_{fu}(F, C)$ is fuzzy soft closed in (B, Ω, G) .

Example 4.2

Let (B, Ω, G) be a fuzzy soft topological space and let $D = \{d_1, d_2\}$ be a finite discrete space. Consider the product $H = B \times D$ with the corresponding fuzzy soft product topology and structural projection $P_{fu}(b, d) = b$. Let (F, C) be a fuzzy soft closed subset of $(H, \tilde{\tau}, E)$. Because D is finite and discrete, the intersections of (F, C) with $B \times \{d_1\}$ and $B \times \{d_2\}$ are fuzzy soft closed in the corresponding slices. Moreover, $P_{fu}(F, C) = P_{fu}((F, C) \cap (B \times \{d_1\})) \cup P_{fu}((F, C) \cap (B \times \{d_2\}))$. Each projected slice is fuzzy soft closed in (B, Ω, G) , and a finite fuzzy soft union of fuzzy soft closed sets is fuzzy soft closed. Therefore P_{fu} is a fuzzy soft closed mapping. Hence $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

Proposition 4.3

Let $\theta : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a surjective fibrewise fuzzy soft closed mapping over (B, Ω, G) . If $(K, \tilde{\sigma}, L)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) , then $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

Proof

Since θ is fibrewise, the structural projections satisfy $P_{K(fu)} \circ \theta = P_{fu}$. Let (F, C) be a fuzzy soft closed subset of $(H, \tilde{\tau}, E)$. Since θ is a fuzzy soft closed mapping, $\theta(F, C)$ is fuzzy soft closed in $(K, \tilde{\sigma}, L)$. Because $(K, \tilde{\sigma}, L)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) , the set $P_{fu}(\theta(F, C))$ is fuzzy soft closed in (B, Ω, G) . By the fibrewise identity, $P_{fu}(F, C) = (P_{K(fu)} \circ \theta)(F, C) = P_{K(fu)}(\theta(F, C))$. Thus $P_{fu}(F, C)$ is fuzzy soft closed in (B, Ω, G) . Hence P_{fu} is a fuzzy soft closed mapping, and therefore $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

Proposition 4.4

Let $\{(H_i, \tilde{\tau}_i, E_i)\}_{i=1}^n$ be a finite fuzzy soft closed covering of $(H, \tilde{\tau}, E)$. Assume that, for every i , the restricted space $(H_i, \tilde{\tau}_i, E_i)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) . Then $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

Proof

Let $P_{fu} : (H, \tilde{\tau}, E) \rightarrow (B, \Omega, G)$ be the structural projection and let (F, C) be fuzzy soft closed in $(H, \tilde{\tau}, E)$. Since each H_i is fuzzy soft closed in H , $(F, C) \cap (H_i, E_i)$ is fuzzy soft closed in $(H_i, \tilde{\tau}_i, E_i)$. Let $P_{i(fu)} = P_{fu}(H_i)$. By hypothesis, $P_{i(fu)}$ is fuzzy soft closed. Therefore each set $P_{i(fu)}((F, C) \cap (H_i, E_i))$ is fuzzy soft closed in (B, Ω, G) . Since the covering is finite, $P_{fu}(F, C) = \bigcup_{i=1}^n P_{i(fu)}((F, C) \cap (H_i, E_i))$. The right-hand side is a finite fuzzy soft union of fuzzy soft closed sets. Hence $P_{fu}(F, C)$ is fuzzy soft closed in (B, Ω, G) , so P_{fu} is a fuzzy soft closed mapping. Therefore $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

Proposition 4.5

Let $(H, \tilde{\tau}, E)$ be a fibrewise fuzzy soft topological space over (B, Ω, G) . Then $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) if and only if, for every fuzzy soft point $\tilde{b}$ of the base and every fuzzy soft open set (F, E) containing the whole fibre $H_{\tilde{b}}$, there exists a fuzzy soft open neighbourhood (V, G) of $\tilde{b}$ such that $P_{fu}^{-1}(V, G) \subset (F, E)$.

Proof

($\Rightarrow$) Assume that $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) . Then P_{fu} is a fuzzy soft closed mapping. Let $\tilde{b}$ be a fuzzy soft point of (B, Ω, G) , and let (F, E) be fuzzy soft open in $(H, \tilde{\tau}, E)$ with $H_{\tilde{b}} \subset (F, E)$. The complement $(H, E) \setminus (F, E)$ is fuzzy soft closed, so $P_{fu}((H, E) \setminus (F, E))$ is fuzzy soft closed in (B, Ω, G) . Since $H_{\tilde{b}} \subset (F, E)$, $\tilde{b}$ does not belong to this image. Define $(V, G) = (B, G) \setminus P_{fu}((H, E) \setminus (F, E))$. Then (V, G) is a fuzzy soft open neighbourhood of $\tilde{b}$. If $x \in P_{fu}^{-1}(V, G)$ but $x \notin (F, E)$, then $P_{fu}(x) \in P_{fu}((H, E) \setminus (F, E))$, contradicting $P_{fu}(x) \in (V, G)$. Therefore $P_{fu}^{-1}(V, G) \subset (F, E)$.

($\Leftarrow$) Conversely, assume the stated neighborhood condition. Let (C, E) be fuzzy soft closed in $(H, \tilde{\tau}, E)$, and let $\tilde{b} \notin P_{fu}(C, E)$. Then $H_{\tilde{b}} \subset (H, E) \setminus (C, E)$. By hypothesis, there exists a fuzzy soft open neighbourhood (V, G) of $\tilde{b}$ such that $P_{fu}^{-1}(V, G) \subset (H, E) \setminus (C, E)$. Consequently, $(V, G) \cap P_{fu}(C, E) = \emptyset$. Hence every fuzzy soft point outside $P_{fu}(C, E)$ has a fuzzy soft open neighborhood disjoint from $P_{fu}(C, E)$. Thus $P_{fu}(C, E)$ is fuzzy soft closed in (B, Ω, G) . Therefore P_{fu} is a fuzzy soft closed mapping, and

$(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

Example 4.6

Let $D = \{d_1, d_2, d_3\}$ be finite and discrete, and let $H = B \times D$ with structural projection $P_{fu}(b, d) = b$. Suppose (F, E) is a fuzzy soft open set containing the entire fibre $H_{\tilde{b}} = \{\tilde{b}\} \times D$. For each $d_j \in D$, choose a fuzzy soft open neighbourhood (V_j, G) of $\tilde{b}$ such that $(V_j, G) \times \{d_j\} \subset (F, E)$. Since D is finite, $(V, G) = (V_1, G) \cap (V_2, G) \cap (V_3, G)$ is a fuzzy soft open neighbourhood of $\tilde{b}$ and $P_{fu}^{-1}(V, G) \subset (F, E)$. This example demonstrates explicitly the local characterization in Proposition 4.5.

Definition 4.7 (Proposed)

A fibrewise fuzzy soft topological space $(H, \tilde{\tau}, E)$ over (B, Ω, G) is called a fibrewise open fuzzy soft topological space if its structural projection $P_{fu} : (H, \tilde{\tau}, E) \rightarrow (B, \Omega, G)$ is a fuzzy soft open mapping; that is, for every fuzzy soft open set (F, C) in $(H, \tilde{\tau}, E)$, the image $P_{fu}(F, C)$ is fuzzy soft open in (B, Ω, G) .

Example 4.8

Let (B, Ω, G) be a fuzzy soft topological space and let (D, δ, A) be a non-null fuzzy soft topological space. Consider $H = B \times D$ with the fuzzy soft product topology and structural projection $P_{fu}(b, d) = b$. For a basic fuzzy soft open set $(V, G) \times (W, A)$, where (W, A) is non-null, $P_{fu}((V, G) \times (W, A)) = (V, G)$. Therefore the image of each basic fuzzy soft open set is fuzzy soft open in (B, Ω, G) . Since every fuzzy soft open set in the product is a fuzzy soft union of basic open sets and P_{fu} preserves unions, P_{fu} is a fuzzy soft open mapping. Hence $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Proposition 4.9

Let $\theta : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a surjective fibrewise fuzzy soft open mapping over (B, Ω, G) . If $(K, \tilde{\sigma}, L)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) , then $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Proof

Let (F, E) be fuzzy soft open in $(H, \tilde{\tau}, E)$. Since θ is fuzzy soft open, $\theta(F, E)$ is fuzzy soft open in $(K, \tilde{\sigma}, L)$. Because $(K, \tilde{\sigma}, L)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) , $P_{K(fu)}(\theta(F, E))$ is fuzzy soft open in (B, Ω, G) . Since θ is fibrewise, $P_{K(fu)} \circ \theta = P_{fu}$. Thus $P_{fu}(F, E) = P_{K(fu)}(\theta(F, E))$ is fuzzy soft open in (B, Ω, G) . Therefore P_{fu}

is a fuzzy soft open mapping, and $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Proposition 4.10

Let $\{(H_r, \tilde{\tau}_r, E_r)\}_{r=1}^n$ be a finite family of fibrewise open fuzzy soft topological spaces over the same base (B, Ω, G) . Assume that the fibrewise product is formed with compatible parameter sets and the standard fibrewise fuzzy soft product topology. Then $(H, \tilde{\tau}, E) = \prod_B \{(H_r, \tilde{\tau}_r, E_r)\}_{r=1}^n$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Proof

Let P_{fu} denote the structural projection of the fibrewise product. A basic fuzzy soft open set has the form $(U, E) = \bigcap_{r=1}^n \pi_r^{-1}(U_r, E_r)$, where each (U_r, E_r) is fuzzy soft open in $(H_r, \tilde{\tau}_r, E_r)$. By compatibility of the fibrewise product, a base point belongs to $P_{fu}(U, E)$ exactly when its common fibre meets every coordinate condition. Hence $P_{fu}(U, E) = \bigcap_{r=1}^n P_{r(fu)}(U_r, E_r)$. Since each $(H_r, \tilde{\tau}_r, E_r)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) , every $p_{(r(fu))}(U_r, E_r)$ is fuzzy soft open in (B, Ω, G) . Their finite fuzzy soft intersection is fuzzy soft open. Since arbitrary fuzzy soft open sets in the product are fuzzy soft unions of basic open sets and P_{fu} preserves unions, P_{fu} maps every fuzzy soft open set to a fuzzy soft open set. Therefore $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Remark 4.11

Let $(H, \tilde{\tau}, E)$ and $(K, \tilde{\sigma}, L)$ be fibrewise fuzzy soft topological spaces over (B, Ω, G) . If $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) , then under the standard fibrewise product topology the second projection $\pi_2 : (H, \tilde{\tau}, E) \times_B (K, \tilde{\sigma}, L) \rightarrow (K, \tilde{\sigma}, L)$ is fuzzy soft open whenever the relevant fibre components are non-null. Indeed, for a basic fuzzy soft open set $(U, E) \times_B (V, L)$, the image under π_2 is the portion of (V, L) lying over $P_{fu}(U, E)$, and $P_{fu}(U, E)$ is fuzzy soft open in the base.

Proposition 4.12

Let $\theta : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a fibrewise fuzzy soft mapping. Assume that the relevant second projections in the fibrewise products are surjective and fuzzy soft open, and that $id_H \times \theta : H \times_B H \rightarrow H \times_B K$ is fuzzy soft open. Then θ is a fuzzy soft open mapping.

Proof

The fibrewise product maps satisfy the commutative relation

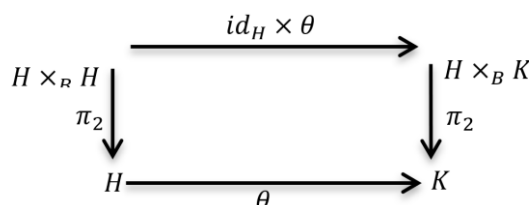


Figure 2: Diagram of Proposition 4.12

Let (U, E) be fuzzy soft open in $(H, \tilde{\tau}, E)$. By surjectivity and openness of the left second projection, (U, E) is obtained as the image of a suitable fuzzy soft open set in $H \times_B H$. Applying $id_H \times \theta$ preserves fuzzy soft openness by hypothesis, and applying the right second projection again

preserves fuzzy soft openness. The commutative relation then shows that $\theta(U, E)$ is fuzzy soft open in $(K, \tilde{\sigma}, L)$. Therefore θ is a fuzzy soft open mapping.

Proposition 4.13

Let $\theta : (H, \tilde{\tau}, E) \rightarrow (K, \tilde{\sigma}, L)$ be a continuous surjective fibrewise fuzzy soft mapping over (B, Ω, G) . Then:

- (a) If $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) , then $(K, \tilde{\sigma}, L)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) ;
 (b) if $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) , then $(K, \tilde{\sigma}, L)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Proof

Since θ is fibrewise, the structural projections satisfy $P_{fu} = P_{K(fu)} \circ \theta$.

For (a), let (C, L) be fuzzy soft closed in $(K, \tilde{\sigma}, L)$. Since θ is continuous, $\theta^{-1}(C, L)$ is fuzzy soft closed in $(H, \tilde{\tau}, E)$. Because $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) , $P_{fu}(\theta^{-1}(C, L))$ is fuzzy soft closed in the base. Surjectivity of θ gives $P_{K(fu)}(C, L) = P_{fu}(\theta^{-1}(C, L))$. Hence $P_{K(fu)}(C, L)$ is fuzzy soft closed in (B, Ω, G) , proving that $(K, \tilde{\sigma}, L)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) .

For (b), let (U, L) be fuzzy soft open in $(K, \tilde{\sigma}, L)$. Continuity of θ gives that $\theta^{-1}(U, L)$ is fuzzy soft open in $(H, \tilde{\tau}, E)$. Because $(H, \tilde{\tau}, E)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) , $P_{fu}(\theta^{-1}(U, L))$ is fuzzy soft open in the base. Again, surjectivity yields $P_{K(fu)}(U, L) = P_{fu}(\theta^{-1}(U, L))$. Therefore $P_{K(fu)}(U, L)$ is fuzzy soft open in (B, Ω, G) , and $(K, \tilde{\sigma}, L)$ is a fibrewise open fuzzy soft topological space over (B, Ω, G) .

Proposition 4.14

Let $(H, \tilde{\tau}, E)$ be a fibrewise closed fuzzy soft topological space over (B, Ω, G) , respectively a fibrewise open fuzzy soft topological space over (B, Ω, G) , and let $(B^*, \tilde{\Omega}^*, G^*)$ be a fuzzy soft subspace of the base. Put $H_{B^*} = P_{fu}^{-1}(B^*)$ and equip H_{B^*} with the induced fuzzy soft subspace topology. Then the restricted projection $P_{B^*(fu)} = P_{fu|_{H_{B^*}}} : (H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*}) \rightarrow (B^*, \tilde{\Omega}^*, G^*)$ is fuzzy soft closed, respectively fuzzy soft open. Consequently, $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$ is a fibrewise closed fuzzy soft topological space over $(B^*, \tilde{\Omega}^*, G^*)$, respectively a fibrewise open fuzzy soft topological space over $(B^*, \tilde{\Omega}^*, G^*)$.

Proof

Suppose first that $(H, \tilde{\tau}, E)$ is a fibrewise closed fuzzy soft topological space over (B, Ω, G) . Let $(C, E)_{B^*}$ be fuzzy soft closed in $(H_{B^*}, \tilde{\tau}_{B^*}, E_{B^*})$. By the subspace construction, there exists a fuzzy soft closed set (D, E) in $(H, \tilde{\tau}, E)$ such that $(C, E)_{B^*} = (D, E) \cap (H_{B^*}, E_{B^*})$. Therefore $P_{B^*(fu)}((C, E)_{B^*}) = P_{fu}(D, E) \cap (B^*, G^*)$. Since $P_{fu}(D, E)$ is fuzzy soft closed in (B, Ω, G) , its intersection with the fuzzy soft subspace $(B^*, \tilde{\Omega}^*, G^*)$ is fuzzy soft closed. Hence the restricted projection is fuzzy soft closed. The proof of the open case is identical, with “closed” replaced by “open.”

Example 4.15

Let B^* be a proper fuzzy soft subspace of B and let $H = B \times D$ be as in Example 4.8. Then $H_{B^*} = B^* \times D$ and the restricted structural projection is simply $P_{B^*(fu)}(b, d) = b$ for $b \in B^*$. Thus the restriction preserves the same projection behaviour on the smaller base, providing a concrete instance of Proposition 4.14.

Remark 4.16

The preceding results show that fibrewise closedness and fibrewise openness are determined by the behaviour of the structural projection onto the fuzzy soft base space. Consequently, these properties are preserved or transferred only when the relevant fibrewise mapping, product construction, or base restriction is compatible with that projection. This observation explains why the projection identities used in Propositions 4.3, 4.9, 4.13, and 4.14 are essential rather than merely notational.

CONCLUSION

This study introduced fibrewise fuzzy soft topological spaces by combining the common-base structure of fibrewise topology with fuzzy soft topology. The framework is defined through fuzzy soft continuity of the structural projection, while its fibrewise closed and fibrewise open forms are determined by closedness and openness of that projection. The principal results establish an induced-topology criterion, mapping properties, a finite closed-cover property, a local neighbourhood characterisation of fibrewise closedness, an open-product result under explicit compatibility assumptions, preservation under continuous surjective fibrewise fuzzy soft mappings, and inheritance under restriction to fuzzy soft subspaces of the base. The examples demonstrate that the common base is an active part of the construction rather than auxiliary notation. These results provide a foundation for investigating separation axioms, compactness, connectedness, ideal structures, and related invariants in the fibrewise fuzzy soft setting. A systematic comparison with classical fibrewise and fuzzy soft results remains important for isolating those properties that are specific to the combined framework.

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