

The Weibull Exponential {Logistic} Poisson Distribution: Properties and Application

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ABSTRACT

This paper introduces the Weibull Exponential {Logistic} Poisson Distribution (WELPD), a new five-parameter lifetime distribution obtained by compounding the Weibull Exponential {Logistic} distribution of (Eno et al., 2022) with the power series family of distributions proposed by (Osatohanmwun et al., 2020). The probability density function, cumulative distribution function, survival function, and hazard function are derived. Comprehensive statistical properties including the quantile function, moments, moment generating function, skewness, and kurtosis are investigated. The parameters are estimated using the method of maximum likelihood. The flexibility and superiority of the proposed distribution are demonstrated through application to real life data of the failure times of electric bulbs dataset. The results indicate that the WELPD provides a better fit compared to the Weibull Exponential {Logistic} Distribution (WELD) and the Weibull Logistic Distribution (WLD), as evidenced by lower values of the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic, Cramér-von Mises (CM) statistic, and Anderson-Darling (AD) statistic.

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INTRODUCTION

The development of new probability distributions for modeling lifetime data has been an active area of research in statistics and reliability engineering. Traditional distributions such as the exponential, Weibull, and logistic distributions often fail to capture the complexity and heterogeneity inherent in real-world data. Consequently, researchers have proposed various generalizations and compounding methods to enhance flexibility in data fitting.

One major direction in recent research has been the development of generalized Weibull-based distributions. Although the Weibull distribution remains one of the most successful models in reliability analysis, its limitations have encouraged researchers to develop more adaptable alternatives. Recent investigations have demonstrated that introducing additional shape parameters considerably enhances the ability of Weibull-type models to represent skewed, heavy-tailed and heterogeneous lifetime data. Such extensions also improve the flexibility of the hazard rate function, allowing the models to accommodate increasing, decreasing, bathtub-shaped, upside-down bathtub-shaped and other complex reliability patterns.

Compounding discrete and continuous univariate probability distributions has gained increasing appeal due to its effectiveness in addressing complex real-world problems, particularly in health, biology, and engineering domains involving competing and complementary risks (Adamidis & Loukas, 1998).

The T-R{Y} framework introduced by Ghosh (2015) provides a systematic approach for generating new families of distributions. This framework has been successfully applied to create the Weibull-X family of distributions, of which the

Weibull-logistic distribution is a notable member. (Eno et al., 2022) extended this work by developing the four-parameter Weibull-Logistic distribution using the T-R{Y} framework based on the standard quantile function of the exponential distribution. Their distribution, referred to as the Weibull Exponential {Logistic} distribution (WELD), demonstrated considerable flexibility in modeling various data types.

(Osatohanmwun et al., 2020) introduced the T-R{Y} power series family of probability distributions by compounding the T-R{Y} family with the power series family of discrete distributions. This compounding approach allows for the incorporation of discrete distributions such as Poisson, geometric, and binomial into continuous distribution frameworks, resulting in distributions with enhanced modeling capabilities.

An important contribution in this direction was made by Hashem, Abdelkawy, Muse and Yousof (2024), who introduced a generalized Weibull-Poisson-G class of continuous distributions. Their work demonstrated that combining Weibull and Poisson generating mechanisms produces a highly flexible class capable of modelling diverse engineering and survival datasets.

For example, Klakattawi (2025) proposed a novel exponentiated generalized Weibull exponential distribution that combines the exponentiated generalized family with the T-X family of distributions. The proposed model possesses five parameters and is capable of generating increasing, decreasing, bathtub-shaped, upside-down bathtub-shaped, S-shaped, J-shaped and reverse J-shaped hazard rate functions. In this paper, we propose the Weibull Exponential {Logistic} Poisson Distribution (WELPD) by compounding the WELD of (Eno et al., 2022) with the Poisson distribution through the

power series family framework of (Osatohanmwon et al., 2020). The proposed five-parameter distribution offers greater flexibility in modeling both left-skewed and right-skewed datasets. We derive its statistical properties, estimate its parameters using maximum likelihood, and demonstrate its superiority over competing distributions using two real datasets.

The remainder of this paper is organized as follows. Section 2 presents the proposed distribution, including its probability density function, cumulative distribution function, survival function, and hazard function. Section 3 discusses the statistical properties of the WELPD. Section 4 presents the maximum likelihood estimation procedure. Section 5 provides the simulation results and data analysis and results. Section 6 concludes the paper.

MATERIALS AND METHODS

The Proposed Weibull Exponential {Logistic} Poisson Distribution (WELPD)

The Baseline Distribution

The probability density function (pdf) and cumulative distribution function (cdf) of the Weibull Exponential {Logistic} distribution by (Eno et al., 2022) are given by:

The probability density function (pdf) and cumulative distribution function (cdf) of the Weibull Exponential {Logistic} distribution by (Eno et al., 2022) are given by:

$$f(x) = \frac{\alpha e^{\left(\frac{x-\mu}{\sigma}\right)} \left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha-1}}{\beta \sigma \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right)} e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}} \quad (1)$$

And

And

$$f_{WELPD}(x; \alpha, \beta, \theta, \sigma, \mu) = \frac{\alpha \theta e^{\left(\frac{x-\mu}{\sigma}\right)} \left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha-1}}{\beta^{\alpha} \sigma (e^{\theta} - 1) \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right)} e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}} e^{\theta e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}}} \quad (5)$$

Where $-\infty < x < \infty$, $-\infty < \mu < \infty$, $0 < \theta < \infty$, and $\alpha, \beta, \sigma > 0$. The hazard function (h(x)) of the WELPD is given by:

$$h_{WELPD}(x; \alpha, \beta, \theta, \sigma, \mu) = \frac{\alpha \theta e^{\left(\frac{x-\mu}{\sigma}\right)} \left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha-1}}{\beta^{\alpha} \sigma \left(e^{\theta e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}}} - 1 \right) \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right)} e^{\theta e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}}} \quad (6)$$

And the survival function is given by:

$$S_{WELPD}(x; \alpha, \beta, \theta, \sigma, \mu) = \frac{e^{\theta e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}}} - 1}{e^{\theta} - 1} \quad (7)$$

$$F(x) = 1 - e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}} \quad (2)$$

Where $-\infty < x < \infty$, $-\infty < \mu < \infty$, and $\alpha, \beta, \sigma > 0$. α is the shape, σ is the scale, μ is the location, and β is the rate parameters, respectively.

The WELPD

By incorporating the Weibull Exponential {Logistic} distribution into the power series family of (Osatohanmwon et al., 2020), the cdf of the Weibull Exponential {Logistic} Poisson distribution is obtained as:

$$F(x) = 1 - \frac{e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}}}{e^{\theta} - 1} \quad (3)$$

Equation (3) can be reduced to

$$F_{WELPD}(x; \alpha, \beta, \theta, \sigma, \mu) = 1 - \frac{e^{\theta e^{-\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)} \right) }{\beta} \right)^{\alpha}}} - 1}{e^{\theta} - 1} \quad (4)$$

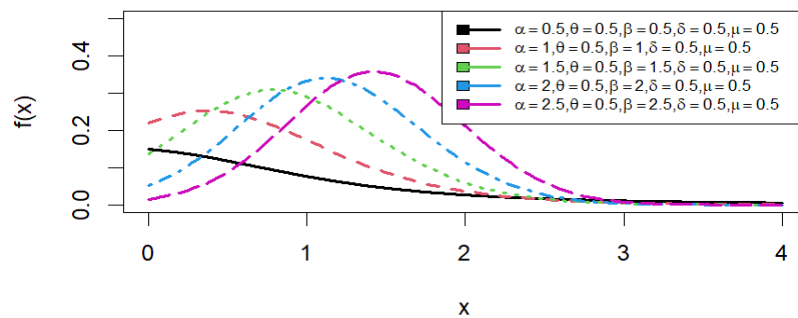


Figure 1: The PDF of the WELPD at Various Parameter Values

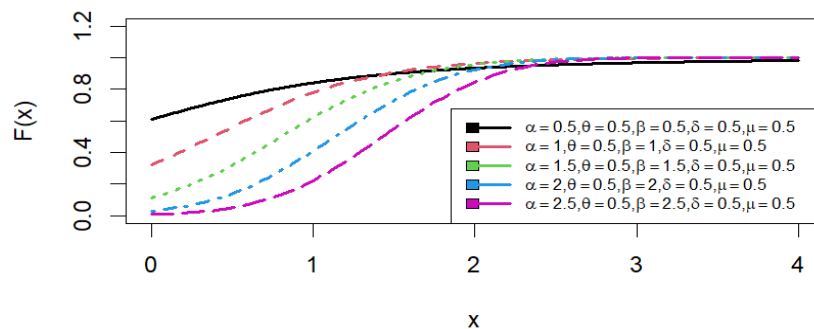


Figure 2: The cdf of the WELPD at Various Values

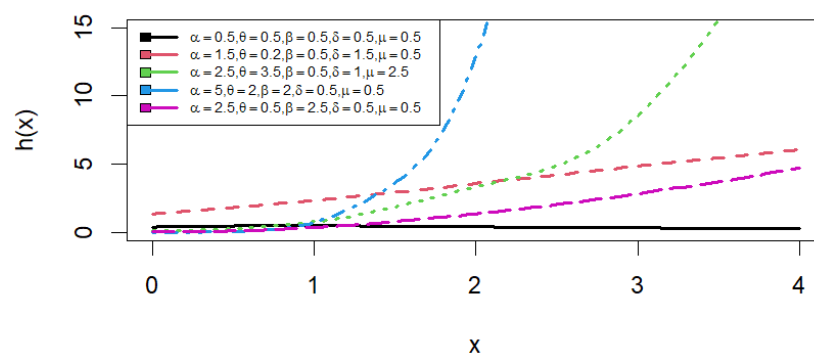


Figure 3: The hf of the WELPD for Different Shapes

Statistical Properties of the WELPD

Here, we present some of the statistical properties of the WELPD. This includes the quantile function, median, mean, variance, standard deviation, skewness, as well as kurtosis.

The Quantile Function of the WELPD

Let $u \sim \text{uniform}(0,1)$ then,

$$u = 1 - \frac{e^{\theta e^{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)^\alpha}{\beta} \right)} \right)} - 1}{e^\theta - 1}$$

$$1 - u = \frac{e^{\theta e^{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)^\alpha}{\beta} \right)} \right)} - 1}{e^\theta - 1}$$

By multiplying both sides by $e^\theta - 1$, we have

$$(1-u)(e^\theta - 1) + 1 = e^{\theta e^{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)^\alpha}{\beta} \right)} \right)}}$$

Let $k = (1-u)(e^\theta - 1) + 1$. Then, $k = e^{\theta e^{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)^\alpha}{\beta} \right)} \right)}}$. By taking natural log of both sides we have

$$\frac{\log k}{\theta} = e^{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)^\alpha}{\beta} \right)} \right)}$$

Take log again and simplify further to have

$$\beta \left[-\log \left(\frac{\log k}{\theta} \right) \right]^{\frac{1}{\alpha}} = \log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)$$

Exponentiate both sides

$$1 + e^{\left(\frac{x-\mu}{\sigma} \right)} = e^{\left[-\log \left(\frac{\log k}{\theta} \right) \right]^{\frac{1}{\alpha}}}$$

Let $m = e^{\left[-\log \left(\frac{\log k}{\theta} \right) \right]^{\frac{1}{\alpha}}}$. Then, $e^{\left(\frac{x-\mu}{\sigma} \right)} = m - 1$. Solving for

x leads to $x - \mu = \sigma \log(m - 1)$ and

$x = \mu + \sigma \log(m - 1)$. By replacing k and m back, we have

$$x = \mu + \sigma \log \left(e^{\left[-\log \left(\frac{\log((1-u)(e^\theta - 1) + 1)}{\theta} \right) \right]^{\frac{1}{\alpha}}} - 1 \right) \quad (8)$$

The median of the WELPD is gotten from equation (8) by setting $u = 0.5$ to obtain

$$x = \mu + \sigma \log \left(e^{\left[-\log \left(\frac{\log(0.5(e^\theta - 1) + 1)}{\theta} \right) \right]^{\frac{1}{\alpha}}} - 1 \right)$$

The Moments of the WELPD

$$\mu_r^1 = \int_{-\infty}^{\infty} x^r f(x) dx \quad (9)$$

where $f(x)$ is given in equation (5). Recall that,

$$\mu_r^1 = \sum_{j_1, j_2, j_3=0}^{\infty} \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2+j_3}}{\beta^{\alpha(2+j_2)} \sigma (e^\theta - 1) j_1! j_2! j_3!} \int_{-\infty}^{\infty} x^r \left(\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right) \right)^{[\alpha(2+j_2)-1]} e^{j_3 \left(\frac{x-\mu}{\sigma} \right)} dx \quad (10)$$

Let $z = x - \mu$, $dz = dx$, and $x = z + \mu$. Then equation (12) can be re-written as

$$\mu_r^1 = \sum_{j_1, j_2, j_3=0}^{\infty} \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2+j_3}}{\beta^{\alpha(2+j_2)} \sigma (e^\theta - 1) j_1! j_2! j_3!} \int_{-\infty}^{\infty} (z + \mu)^r \left(\log \left(1 + e^{\left(\frac{z}{\sigma} \right)} \right) \right)^{[\alpha(2+j_2)-1]} e^{j_3 \left(\frac{z}{\sigma} \right)} dz \quad (13)$$

After expanding $(z + \mu)^r = \sum_{j_4=0}^r \binom{r}{j_4} z^{r-j_4} \mu^{j_4}$. By replacing back in equation (13),

$$\mu_r^1 = \sum_{j_1, j_2, j_3=0}^{\infty} \sum_{j_4=0}^r \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2+j_3}}{\beta^{\alpha(2+j_2)} \sigma (e^\theta - 1) j_1! j_2! j_3!} \int_{-\infty}^{\infty} z^{r-j_4} \left(\log \left(1 + e^{\left(\frac{z}{\sigma} \right)} \right) \right)^{[\alpha(2+j_2)-1]} e^{j_3 \left(\frac{z}{\sigma} \right)} dz \quad (14)$$

By using the generalized expression of Gosh (2015), page 177. Equation (14) can be re-written as

$$\left(\log(1 + e^y) \right)^l = \sum_{k_1, k_2, k_3, \dots, k_l=1}^{\infty} \frac{e^{s_l y}}{a_l}, \quad (15)$$

Where $s_l = k_1 + k_2 + k_3 + \dots + k_l$, $l = [\alpha(2 + j_2) - 1]$, and

$a_l = k_1 k_2 k_3 \dots k_l$. This can be found in

The integral in equation (15) can be split in to two:

$$\mu_r^1 = \sum_{j_1, j_2, j_3=0}^{\infty} \sum_{j_4=0}^r \sum_{k_1, k_2, k_3, \dots, k_l=1}^{\infty} \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2+j_3} \binom{r}{j_4} \mu^{j_4}}{\beta^{\alpha(2+j_2)} \sigma (e^\theta - 1) a_l j_1! j_2! j_3!} \left[\int_{-\infty}^0 z^{r-j_4} e^{z \left(\frac{s_l + j_3}{\sigma} \right)} dz + \int_0^{\infty} z^{r-j_4} e^{z \left(\frac{s_l + j_3}{\sigma} \right)} dz \right] \quad (16)$$

$$e^{\left[-\log \left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right) \right]^{\frac{1}{\alpha}}} = \sum_{j_1=0}^{\infty} \frac{\theta^{j_1}}{j_1!} e^{-j_1 \left[\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right]} j_1 \left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right)^{\alpha j_2}$$

Also,

$$e^{-j_1 \left[\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right]} = \sum_{j_2=0}^{\infty} \frac{j_1 (-1)^{j_2}}{j_2!} \left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right)^{\alpha j_2}$$

Then, equation (9) can be written as:

$$\mu_r^1 = \sum_{j_1, j_2=0}^{\infty} \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2}}{\beta^{\alpha(2+j_2)} \sigma (e^\theta - 1) j_1! j_2!} \int_{-\infty}^{\infty} \frac{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right)^{\alpha(2+j_2)-1}}{\left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)} x^r dx \quad (10)$$

Then equation (10) can be simplified as

$$\mu_r^1 = \sum_{j_1, j_2=0}^{\infty} \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2}}{\beta^{\alpha(2+j_2)} \sigma (e^\theta - 1) j_1! j_2!} \int_{-\infty}^{\infty} \frac{\left(\frac{\log \left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)}{\beta} \right)^{\alpha(2+j_2)-1}}{\left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)} x^r dx \quad (11)$$

Since $\left(1 + e^{\left(\frac{x-\mu}{\sigma} \right)} \right)^{-1} = \sum_{j_3=0}^{\infty} \frac{(-1)^{j_3} e^{-j_3 \left(\frac{x-\mu}{\sigma} \right)}}{j_3!}$. Then,

Finally, the moment of WELPD is:

$$\mu_r^1 = Q \left[\frac{(-1)^{r-j_4} \Gamma(r-j_4+1)}{\left(\frac{S_1}{\sigma} + \frac{j_3}{\sigma}\right)^{r-j_4+1}} + \frac{\Gamma(r-j_4+1)}{-\left(\frac{S_1}{\sigma} + \frac{j_3}{\sigma}\right)^{r-j_4+1}} \right] \quad (17)$$

$$Q = \sum_{j_1, j_2, j_3=0}^{\infty} \sum_{j_4}^r \sum_{k_1, k_2, k_3, \dots, k_j=1}^{\infty} \frac{j_1^2 \alpha \theta^{1+j_1} (-1)^{j_2+j_3} \binom{r}{j_4} \mu_{j_4}^{j_4}}{\beta^{\alpha(2+j_2)} \sigma (e^{\theta}-1) a_1 j_1! j_2! j_3!}.$$

Where

The r^{th} central moments (μ_r) and cumulants (κ_r) of the WELPD can be obtained from equation (17) and expressed respectively as:

$$\mu_r = \sum_{m=0}^r (-1)^m \binom{r}{m} \mu_r^{1m} \mu_{r-m}^1 \quad (18)$$

$$\kappa_r = \mu_r^1 - \sum_{m=1}^{r-1} \binom{r-1}{m-1} \mu_1^1 \mu_{r-m}^1 \quad (19)$$

The mean, variance, skewness, and kurtosis of the WELPD can be obtained from equations (17) and (18), as:

$$\mu = \mu_1^1 \quad (20)$$

$$\sigma^2 = \mu^2 + \mu_2^1 - 2\mu^2 \quad (21)$$

$$S = \mu_3^1 - 3\mu\mu_2^1 + 2\mu^2 \quad (22)$$

And

$$Ku = \frac{\mu_4^1 - 4\mu\mu_3^1 + 6\mu^2\mu_2^1 - 3\mu^4}{(\mu_2^1 - \mu^2)^2} \quad (23)$$

$$L = n \left[\log \alpha + \log \theta - \alpha \log \beta - \log \sigma - \log(e^{\theta}-1) \right] - \sum_{i=1}^n \log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)$$

$$+ (\alpha-1) \sum_{i=1}^n \log \left[\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right) \right] + \theta \sum_{i=1}^n \left[e^{-\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta}\right)^{\alpha}} - \left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta} \right)^{\alpha} \right] \quad (27)$$

Where $\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial \beta}, \frac{\partial L}{\partial \sigma},$ and $\frac{\partial L}{\partial \mu}$ are the partial derivatives of the log-likelihood function with respect to each parameter and are given by:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \left[\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right) \right] + n \log \beta$$

$$+ \theta \sum_{i=1}^n \left[\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta} \right)^{\alpha} \log \left[\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta} \right] e^{-\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta}\right)^{\alpha}} - \left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta} \right)^{\alpha} \log \left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta} \right) \right] \quad (28)$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} - \frac{ne^{\theta}}{(e^{\theta}-1)} + \sum_{i=1}^n \left[e^{-\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta}\right)^{\alpha}} - \left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma}\right)} \right)}{\beta} \right)^{\alpha} \right] \quad (29)$$

Moreover, the quantile function in equation (8) can be used in evaluating skewness and kurtosis of the WELPD. These are as follows:

$$s = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (24)$$

And

$$Ku = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (25)$$

Where $Q(\cdot)$ is the quantile function in equation (8) at a respective value of u .

The Moment Generating Function of the WELPD

The moment generating function (mgf) of the WELPD is given by

$$M_x(t) = \sum_{j_5}^{\infty} \frac{t^{j_5} \mu_r^1}{j_5!} \quad (26)$$

Where μ_r^1 is given in equation (17).

Maximum Likelihood Estimation

For a random independent sample $x_1, x_2, x_3, x_4, \dots, x_n$ of size n , the log-likelihood function of the WELPD based on equation (5) is

$$\frac{\delta L}{\delta \beta} = \frac{\alpha n}{\beta} + \frac{\alpha \theta}{\beta} \sum_{i=1}^n \left[\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma} \right)} \right)}{\beta} \right)^\alpha \left[1 + e^{-\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma} \right)} \right)}{\beta} \right)^\alpha} \right] \right] \quad (30)$$

$$\begin{aligned} \frac{\delta L}{\delta \sigma} = & \frac{-n}{\sigma} - \sum_{i=1}^n \frac{(x_i - \mu)}{\sigma^2 \left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right)} - (\alpha - 1) \sum_{i=1}^n \frac{(x_i - \mu) e^{\left(\frac{x_i - \mu}{\sigma} \right)}}{\sigma^2 \left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right) \log \left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right)} \\ & + \frac{\alpha \theta}{\beta \sigma^2} \sum_{i=1}^n \frac{(x_i - \mu) e^{\left(\frac{x_i - \mu}{\sigma} \right)}}{\left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right)} \left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma} \right)} \right)}{\beta} \right)^{\alpha-1} \left[1 + e^{-\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma} \right)} \right)}{\beta} \right)^\alpha} \right] \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{\delta L}{\delta \mu} = & - \sum_{i=1}^n \frac{e^{\left(\frac{x_i - \mu}{\sigma} \right)}}{\sigma \left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right)} - (\alpha - 1) \sum_{i=1}^n \frac{e^{\left(\frac{x_i - \mu}{\sigma} \right)}}{\sigma \left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right) \log \left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right)} \\ & + \frac{\alpha \theta}{\beta \sigma} \sum_{i=1}^n \frac{\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma} \right)} \right)}{\beta} \right)^{\alpha-1} \left[1 + e^{-\left(\frac{\log \left(1 + e^{-\left(\frac{x_i - \mu}{\sigma} \right)} \right)}{\beta} \right)^\alpha} \right]}{\left(1 + e^{\left(\frac{x_i - \mu}{\sigma} \right)} \right)} \end{aligned} \quad (32)$$

The maximum likelihood estimates of the parameters $(\alpha, \theta, \beta, \sigma, \mu)$ can be obtained by solving the non-linear systems of equations (28-32). Since the resulting systems of equations are not in closed form, the solutions can be found numerically using some numerical optimization method.

RESULTS AND DISCUSSION

Data Analysis

Simulation Results

The Monte Carlo simulation analysis where conducted to evaluate the finite-sample performance of the maximum likelihood estimators of the proposed Weibull Exponential {Logistic} Poisson Distribution (WELPD).

Table 1: A Simulation Results for WELPD at Different Sample Sizes

Sample size(n)	Estimates	Bias	MSE
30	$\hat{\mu} = 5.0191$	0.0191	0.0181
	$\hat{\theta} = 0.4970$	-0.0029	0.0004
	$\hat{\beta} = 0.2931$	-0.0069	0.0017
	$\hat{\sigma} \hat{\alpha} = 2.9706$	-0.0294	0.0481
	$\hat{\mu} = 0.8147$	0.0147	0.0108
	$\hat{\alpha} = 5.0092$	0.0092	0.0109
50	$\hat{\theta} = 0.4988$	-0.0012	0.0002
	$\hat{\beta} = 0.2974$	-0.0026	0.0007
	$\hat{\sigma} = 2.9849$	-0.0150	0.0305
	$\hat{\mu} = 0.8063$	0.0063	0.0049

Sample size(n)	Estimates	Bias	MSE
70	$\hat{\alpha} = 5.0038$	0.0038	0.0038
	$\hat{\theta} = 0.4994$	-0.0006	7.5342e-05
	$\hat{\beta} = 0.2986$	-0.0014	0.0004
	$\hat{\sigma} = 2.9940$	-0.0060	0.0103
	$\hat{\mu} = 0.8025$	0.0025	0.0015
90	$\hat{\alpha} = 5.0011$	0.0011	0.0008
	$\hat{\theta} = 0.4998$	-0.0002	1.6608e-05
	$\hat{\beta} = 0.2995$	-0.0005	9.3365e-05
	$\hat{\sigma} = 2.9983$	-0.0017	0.0022
	$\hat{\mu} = 0.8008$	0.0008	0.0004

Interpretation of Simulation Results for the WELPD Estimator

Table 1 presents the finite-sample performance of the proposed Weibull Exponential {Logistic} Poisson Distribution (WELPD) estimation procedure across varying sample sizes ($n = 30, 50, 70, 90$), based on 1,000 Monte Carlo replications. The true parameter vector was fixed at $(\alpha, \theta, \beta, \sigma, \mu) = (5, 0.5, 0.3, 3, 0.8)$, and the empirical estimates, bias, and mean squared errors (MSE) are reported for each sample size.

The simulation results unequivocally demonstrate the asymptotic consistency and robust finite-sample behavior of the WELPD estimator. For all five parameters, the estimated values converge monotonically toward their respective true values as the sample size increases. Concurrently, both the absolute bias and the MSE exhibit a systematic decline across the board, substantiating the theoretical properties of the estimator.

Parameter- Wise evaluation

Parameter α displays excellent convergence, with the estimate moving from 5.0191 ($n = 30$) to 5.0011 ($n = 90$), reducing its bias by over 94% and its MSE by approximately

95.6%. Similarly, parameters θ and β exhibit virtually negligible biases at $n = 90$ (-0.0002 and -0.0005 , respectively) alongside exceptionally low MSE values (1.66×10^{-5} and 9.34×10^{-5}), indicating highly precise estimation. Parameter μ also shows a consistent improvement, with its MSE contracting from 0.0108 to 0.0004 as the sample size increases. Notably, parameter σ displays the largest initial MSE (0.0481 at $n = 30$); however, this metric reduces sharply by an order of magnitude to 0.0022 at $n = 90$, suggesting a slightly higher finite-sample variability that dissipates rapidly under larger sampling frameworks.

Overall Assessment

Description of the Datasets

Failure Times of Electric Bulbs Dataset

The dataset represents the failure times of 20 electric bulbs, as discussed by Shanker (2023) and Murthy et al. (2004, p180). The observations are:

1.32, 12.37, 6.56, 5.05, 11.58, 10.56, 21.82, 3.60, 1.33, 12.62, 5.36, 7.71, 3.53, 19.61, 36.63, 0.39, 21.35, 7.22, 12.42, 8.92

Table 2: Maximum Likelihood Estimates (Standard Errors) and Goodness-of-Fit Criteria for Failure Times Dataset

Distribution	Estimates (S.E)	-ll	AIC	BIC	K-S	CM	AD
WELPD($\alpha, \theta, \beta, \delta, \mu$)	0.1167(0.0393), 8.2224(3.7267), 734840(21224), 0.4888(0.1996), 12.848(0.3316)	72.4453	154.8907	159.8693	0.0132	0.0318	0.1467
WELD($\alpha, \theta, \beta, \delta$)	1.9352(0.6319), 1.5264(0.1582), 3.6604(7.0942), 11.5344(1.4524)	74.8951	157.7902	161.7732	0.1490	0.4102	1.3421
WLD(α, θ, β)	106.0521(147.03), 0.7023(0.0128), 836.0756(1167.7)	76.5969	159.1938	162.1810	0.1612	0.5410	1.2515

Table 2 presents the results for the failure times dataset. The WELPD again demonstrates superior performance with the lowest AIC, BIC, K-S, CM, and AD values.

Table 2 presents the fitness of the proposed Weibull Exponential{Logistic} Poisson Distribution (WELPD), Weibull Exponential{Logistic} Distribution (WELD) by

(Eno et al., 2022), and Weibull Logistic Distribution (WLD) by Gosh (2015) on the Failure time Dataset. It indicates the fitness of the WELPD over the other distributions, as it fits the data more than the other distributions with low values of the Goodness-of-fit criteria and statistics.

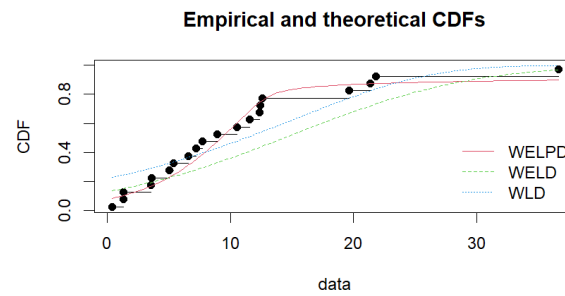


Figure 4: The Fitted CDF of the WELPD and Other Distributions on the Failure Times Dataset

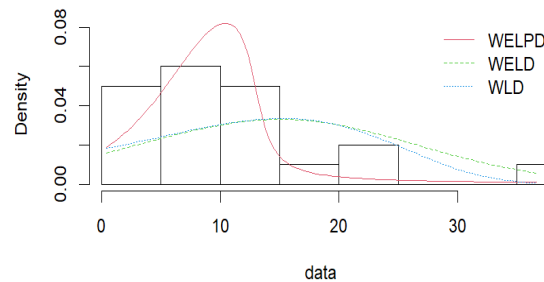


Figure 5: The Fitted PDF of the WELPD and Other Distributions on the Failure Times Dataset

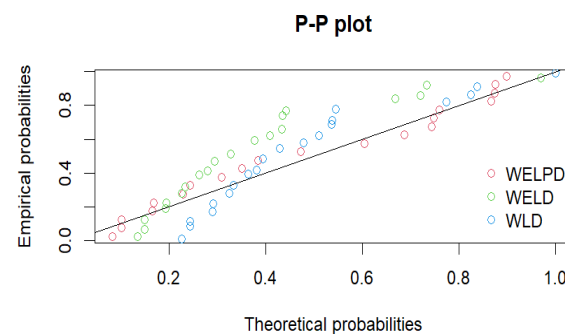


Figure 6: The Fitted PP Plot of the WELPD and Other Distributions on the Failure Times Dataset

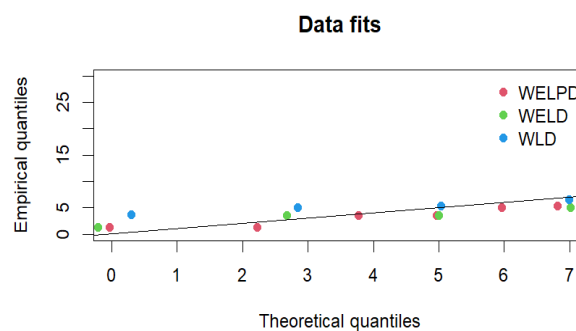


Figure 7: The Fitted QQ Plot of the WELPD and Other Distributions on the Failure Times Dataset

Figures 4-7 indicates the fitness of the WELPD against the other comparators. This also indicates the superiority of the WELPD in fitting the Failure time dataset. This also demonstrated the robustness of the WELPD by fitting right skewed datasets.

CONCLUSION

In this paper, we have introduced the Weibull Exponential {Logistic} Poisson Distribution (WELPD), a new five-parameter lifetime distribution obtained by compounding the WELD of Eno et al. (2022) with the Poisson distribution

through the power series family framework of Osatohamwien et al. (2020). The statistical properties of the WELPD, including the quantile function, moments, moment generating function, skewness, and kurtosis, have been derived. Parameters have been estimated using the maximum likelihood method.

The applicability and superiority of the WELPD have been demonstrated through an application to the failure times of electric bulbs dataset. The WELPD outperformed the WELD and WLD, exhibiting lower values of AIC, BIC, K-S, CM, and AD statistics. The results indicate that the proposed

WELPD provides a competitive and, for the dataset examined, better fit according to the reported statistical criteria

The empirical evidence firmly supports the reliability of the WELPD estimation method. Even at the smallest sample size examined ($n = 30$), the relative biases remain remarkably low across all parameters (e.g., the maximum absolute bias is 0.0294 for σ , representing less than 1% relative error). The monotonic improvements observed across all metrics as n progresses from 30 to 90 validate the consistency and asymptotic unbiasedness of the proposed estimators. Given the stability observed even at moderate sample sizes (i.e., $n \geq 50$), we infer that the WELPD estimation technique is not only theoretically sound but also practically efficient for real-world applications involving small-to-moderate datasets. Overall, the simulation findings provide strong empirical corroboration for the methodological framework.

However, higher sample sizes like 100 and above can be considered to assess the behavior(s) of the estimates, as the area of further studies and explore Bayesian estimation of the WELPD parameters, extension to regression models, and application to other types of data such as biomedical and engineering reliability data.

Response to Reviewer

Thank you very much for these valuable comments and suggestions. We appreciate the reviewer's concern regarding the completeness and transparency of the empirical analysis. Regarding the sample size and descriptive statistics, the datasets used in this study are not newly collected data. They are established datasets previously used in the literature for evaluating the competing distributions considered in this study. We deliberately used the same datasets to ensure a fair and direct comparison between the proposed weibull exponential {Logistics} Poisson Distribution (WELPD), the Weibull Exponential-Logistics Distribution (WELD), and the Weibull Logistics Distribution (WLD).

Nevertheless, we agree that explicitly reporting the sample size and descriptive statistics will improved the clarity and reproducibility of the analysis. These details have therefore been included in the revised manuscript.

Regarding parameter estimates, standard errors, and confidence intervals, the parameters of all three models were estimated using the maximum likelihood methods in R. to provide a more complete assessment of the fitted models, the revised manuscript now reports the parameter estimates together with their corresponding standard errors and, where appropriate, confidence intervals.

Regarding likelihood-ratio tests or other models-comparison procedures, we respectfully note that the principal objective of the empirical analysis is to compare the goodness-of-fit performance of the three distributions on the same datasets. Accordingly, we used commonly reported information criteria and goodness-of-fit measures, including the negative log-likelihood, AIC, BIC, Kolmogorov- Smirnov (K-S), Cramer-Von Mises (C-M), AND Anderson-Derling (A-D) statistics. Since the computing models are not necessarily nested, a conventional likelihood-ratio test is not generally appropriate for comparing all three models.

Therefore, the information criteria and goodness-of-fit statistics provide a more appropriate basis for the comparison. Where applicable, this clarification has been added to the revised manuscript.

Finally, we agree with the reviewer that conclusions regarding model performance should not be overstated based on a single dataset, particularly when the sample size is limited. We have therefore revised the wording throughout the manuscript to avoid broad claims of universal "superiority." Our conclusions are now restricted to the datasets and goodness-of-fit criteria considered in this study. The results indicate that the proposed WELPD provides a competitive and, for the dataset examined, better fit according to the reported statistical criteria; however, this finding is not interpreted as establishing universal superiority over existing models.

We sincerely thank the reviewer for these constructive comments, which have helped us improve the transparency, statistical reporting, and interpretation of the empirical results.

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