

Structural Properties of Full Contraction Transformation Semigroup on a Finite Chain

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ABSTRACT

Let X_n finite chain, and let CT_n denote the semigroup of full contraction transformations on X_n . In this study, we investigate structural properties of CT_n , with particular emphasis on characterization, rank and defect, and Idempotent elements of CT_n . Also, principal theorems were proved. A contraction transformation is a mapping $\varphi: X_n \rightarrow X_n$ satisfying $|x\varphi - y\varphi| \leq |x - y|$ for all $x, y \in X_n$. We establish the foundational structure for studying these properties and identify the results that require independent characterization for CT_n .

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INTRODUCTION

The algebraic theory of semigroups has received extensive scholarly attention, primarily because of its close relationship with other areas of abstract algebra, such as group and ring theory. Several key concepts in semigroup theory, such as Green's relations, idempotent elements, generating sets, and ranks, have evolved into fundamental tools for studying the structure of algebraic systems. The study of transformation semigroups gives a natural and important framework for understanding these concepts. (Howie, 1995; Ganyushkin & Mazorchuk, 2009).

A transformation semigroup is a semigroup of mappings from a set X_n into itself under the operation of composition. Transformation semigroups play a significant role in abstract algebra and theoretical computer science, where algebraic properties of mappings are served as a tool to study computational processes and structures. Specifically, finite transformation semigroups have been widely investigated by (Ganyushkin & Mazorchuk, 2009, Umar, 1996) because of their rich algebraic and combinatorial properties.

Let $X_n = \{1, 2, \dots, n\}$ be a finite chain. The full contraction transformation semigroup, denoted by CT_n , consists of all mappings $\varphi: X_n \rightarrow X_n$ satisfying

$$|x\varphi - y\varphi| \leq |x - y|$$

$\forall x, y \in X_n$. Hence, every element of CT_n is a contraction transformation, although it is not necessarily order-preserving. This distinction enables CT_n a broader class of transformations than the order-preserving full contraction transformation semigroup.

Contraction transformation semigroups have received interest in the study of finite semigroups. Their fundamental properties provide a better understanding into the algebraic behaviour of these semigroups. The removal of the order-preserving condition introduces additional transformations and makes it important to investigate how the algebraic structure changes when all contraction mappings on a finite chain are considered (Adeshola, 2013; Garba, Ibrahim, & Imam, 2017; Ibrahim and Makanjuola, 2018; Ibrahim et al., 2019).

In the analysis of a finite semigroup, it is sometimes important to establish whether the semigroup is regular, inverse, or generated by a particular collection of elements. Another

important aspect is the characterization of Green's relations, which explain the relationships between elements through their principal ideals. These relations are essential in understanding the internal structure and classification of semigroups. (Green, 1951; Howie, 1995).

Motivated by these considerations, this study investigates selected algebraic properties of the full contraction transformation semigroup CT_n on a finite chain. Specifically, we focus on its characterization, rank and defect, and idempotent. The aim is to determine structural results for CT_n and examine the extent to which its algebraic properties differ from other semigroups.

Basic Definitions

Let X_n be an ordered chain with the usual metric $d(x, y) = |x - y|$, where $n \geq 2$. Let T_n be the full transformation semigroup on X_n , consisting of all mappings from X_n into itself, under the operation of composition (Ganyushkin & Mazorchuk, 2009).

The following definitions form the foundation for the study of the full contraction transformation semigroup.

Definition 1.1

The full contraction transformation semi group on X_n , denoted by CT_n , is defined as (Adeshola, 2013; Garba, et al., 2017, Ibrahim et al., 2018).

$$CT_n = \{\varphi \in T_n : |x\varphi - y\varphi| \leq |x - y| \forall x, y \in X_n\}$$

Therefore, CT_n consists of all full contraction transformations on X_n .

Definition 1.2

Let $\varphi \in CT_n$. The image of φ , denoted by $\text{Im}(\varphi)$, is the set $\text{Im}(\varphi) = \{x\varphi : x \in X_n\}$.

The rank of φ , denoted by $\text{rank}(\varphi)$, is the order of its image: $\text{rank}(\varphi) = |\text{Im}(\varphi)|$.

Definition 1.3

The defect of a transformation $\varphi \in CT_n$, denoted by $\text{def}(\varphi)$, is the number of elements of X_n that do not belong to the image of φ . Thus,

$$\text{def}(\varphi) = n - |\text{Im}(\varphi)| = n - \text{rank}(\varphi).$$

MATERIALS AND METHODS

Preliminaries

The materials used in this study consist of the finite chain X_n where n a positive integer is $\{1, 2, \dots, n\}$, and the full transformation semi group T_n , consisting of all mappings from set into itself. The main object of this study is the full contraction transformation semi group CT_n , defined as the set of all transformations $\varphi \in T_n$ satisfying

$$|x\varphi - y\varphi| \leq |x - y|$$

$$\forall x, y \in X_n.$$

The study also employs the basic concepts of semigroup theory, including composition of transformations, rank, defect, idempotent elements, and constant transformations. These concepts give the mathematical structure for studying the fundamental algebraic properties of CT_n .

Methods

An algebraic and deductive method is adopted to determine the fundamental properties of the full contraction transformation semigroup. First, the closure of CT_n under composition is used to determine whether it forms a subsemigroup of T_n . The characterization of contraction transformations is then obtained by associating the contraction condition for arbitrary pairs of elements to the corresponding condition on adjacent elements of the finite chain.

Also, the relationship between the rank of a composition and the ranks of its constituent transformations is studied to determine the rank and defect property. The characterization of idempotent elements is established directly from the defining equation $\varphi^2 = \varphi$, while the idempotency of constant transformations is investigated using their defining mapping property.

The results are obtained through mathematical definitions, properties of transformation composition, and direct proofs. This method gives a structural foundation for the fundamental algebraic results presented in this work. The terminology and framework used throughout this research follow standard treatments of finite transformation semigroups and semigroup theory (Howie, 1995; Ganyushkin & Mazorchuk, 2009).

RESULTS AND DISCUSSION

Fundamental Properties of CT_n

Theorem 3.1

Let $X_n = \{1, 2, \dots, n\}$ be a finite chain. The set CT_n of all full contraction transformations on X_n forms a subsemigroup of the full transformation semigroup T_n .

Proof

Let $\varphi, \omega \in CT_n$. By definition,

$$|x\varphi - y\varphi| \leq |x - y|$$

And

$$|x\omega - y\omega| \leq |x - y|$$

$$\forall x, y \in X_n.$$

Consider the composition $\varphi\omega$. Then

$$|x(\varphi\omega) - y(\varphi\omega)| = |(x\varphi)\omega - (y\varphi)\omega|$$

$$\leq |x\varphi - y\varphi|$$

$$\leq |x - y|.$$

Thus, $\varphi\omega \in CT_n$. Since composition of transformations is associative and CT_n is non-empty, hence, CT_n is a subsemigroup of T_n .

Characterization of Contraction Transformations

Theorem 3.2

A transformation $\varphi: X_n \rightarrow X_n$ belongs to CT_n if and only if $|\varphi(i+1) - \varphi(i)| \leq 1$

$$\forall 1 \leq i < n.$$

Proof

Suppose that $\varphi \in CT_n$. By the contraction property, $|\varphi(i+1) - \varphi(i)| \leq |(i+1) - i| = 1$.

Conversely, suppose that

$$|\varphi(i+1) - \varphi(i)| \leq 1$$

$$\forall 1 \leq i < n.$$

For any $x < y$ in X_n , the triangle inequality leads to

$$|\varphi(y) - \varphi(x)| \leq \sum_{i=x}^{y-1} |\varphi(i+1) - \varphi(i)|$$

$$\leq \sum_{i=x}^{y-1} 1$$

$$= y - x.$$

Hence,

$$|\varphi(y) - \varphi(x)| \leq |y - x|,$$

And thus, $\varphi \in CT_n$.

Rank and Defect Of CT_n

Theorem 3.3

Let $\varphi, \omega \in CT_n$. Then

$$\text{rank}(\varphi\omega) \leq \min\{\text{rank}(\varphi)\text{rank}(\omega)\}.$$

Proof

Since,

$$\text{Im}(\varphi\omega) = (\text{Im } \varphi)\omega,$$

Then, we have

$$|\text{Im}(\varphi\omega)| \leq |\text{Im } \varphi|.$$

Also, we observe that $\varphi\omega$ is obtained by applying ω to the elements of $\text{Im } \varphi$. Therefore,

$$|\text{Im}(\varphi\omega)| \leq |\text{Im } \omega|.$$

Consequently,

$$\text{rank}(\varphi\omega) \leq \min\{\text{rank}(\varphi)\text{rank}(\omega)\}.$$

Idempotent Elements

Theorem 3.4

A transformation $\varphi \in CT_n$ is an idempotent element if and only if

$$(x\varphi)\varphi = x\varphi$$

$$\forall x \in X_n.$$

Equivalently, φ is idempotent if and only if the order $\text{Im}(\varphi)$ is equal to the order of fixed elements by φ .

Proof

By definition, φ is idempotent if and only if

$$\varphi^2 = \varphi.$$

This is similar to

$$(x\varphi)\varphi = x\varphi$$

$$\forall x \in X_n.$$

Every element of $\text{Im}(\varphi)$ has the form $x\varphi$ for some $x \in X_n$.

Hence, the stated condition holds precisely when the number of every image point is equal to number fixed point by φ .

Constant Transformations Of CT_n

Theorem 3.5

All constant transformation in CT_n is idempotent.

Proof

Let $\varphi \in CT_n$ be a constant transformation such that

$$x\varphi = c$$

$$\forall x \in X_n, \text{ where } c \in X_n.$$

Then

$$(x\varphi)\varphi = c\varphi = c = x\varphi.$$

Hence,

$$\varphi^2 = \varphi.$$

Therefore, every constant transformation in CT_n is idempotent.

Example

Let $X_4 = \{1, 2, 3, 4\}$ and consider the contraction full transformation

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{pmatrix}.$$

Its image is $\text{Im}(\varphi) = \{1, 2, 3\}$, and therefore $\text{rank}(\varphi) = 3$ and $\text{def}(\varphi) = 1$. The example shows how rank and defect describe the structural properties of a transformation. Also, if

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix},$$

Then their composition gives

$$\varphi \circ \omega = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix},$$

Which is also a contraction. This demonstrates the closure of the set under composition and therefore its semigroup structure. An idempotent transformation is given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix},$$

$$\sigma^2 = \sigma \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{pmatrix} = \sigma.$$

Likewise, the transformation

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

Is a constant transformation of rank 1 and defect 3 and it is also an idempotent element. These examples show concretely how composition, rank, defect, idempotent elements, and constant transformations arise naturally in the study of the contraction full transformation semigroup.

CONCLUSION

In conclusion, the study demonstrates the essential algebraic properties of the semigroup under consideration by determine composition of transformations, rank, defect, idempotent elements, and constant transformations. The results provide a better understanding of the structure and behaviour of contraction full transformations semigroup and add to the broader study of transformation semigroups and their algebraic properties.

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