



Fixed-Point Theorems for Multivalued Weak-Type Contractions in Metric Spaces

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ABSTRACT

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This paper introduces the notion of weak multi-valued contraction mapping in the context of metric spaces and examines whether fixed points exist for such contractions under certain appropriate assumptions. A few implications in single-valued mappings are found. New multi-valued analogs of the fixed point theorems presented here are deduced as corollaries because the fixed-point of a single-valued mapping satisfying the weak contractive condition is not always unique, making the concepts more appropriate for fixed point theorems of multi-valued functions. Furthermore, non-trivial instances are given to demonstrate the validity of our findings.

Keywords: metric spaces; contraction mappings; weak contraction; multi-valued contraction; fixed point

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INTRODUCTION

One of the most notable contraction mapping results was published by Banach (1922) in his 1922 thesis; this well-known result is referred to as the Banach Contraction mapping principle. It asserts that each contraction mapping has a unique fixed point on a complete metric space. Because of its use in mathematics and other fields of study, fixed point theory gains a lot of attention. The fixed point theorem in metric spaces is crucial to the development of problem-solving techniques in the sciences and mathematics. Numerous academics have studied in this field and have extended the results by taking into account a more general space, putting certain restrictions on the contraction mapping domain, or taking into consideration more general contractive circumstances. In the meantime, various modifications and applications have been made to the main result in Banach (1922). Some generalizations of the contraction mapping principle diminish the inequality (see, for example, Ćirić (1974)), while others weaken the topology of the underlying space (see Jleli and Samet (2018) and its references). A mapping is continuous if it meets the Banach condition. Therefore, it makes sense to wonder if Banach's fixed point theorem can be applied to mappings that aren't always continuous. Kannan (1968) is the first work in this direction. Several fixed point theorems have been derived from Kannan's result without requiring that the mapping be continuous. In this regard, Chatterjea (1972) work contains significant modifications to Banach-type contractive definitions. Chatterjea's formulation relaxes the traditional Lipschitz-type criterion by modifying the image's distance in terms of the distances between points and their images. Furthermore, Reich(1971) used the Banach's and Kannan's contractions in metric space to verify fixed point findings. Accordingly, Hardy and Rogers(1973) introduced a notable enhancement of the Banach fixed point theorem. Recently, Roldán et al. (2020) identified a large number of Hardy-Roger's type contractions in the setting of both metric and b -metric spaces and established some new fixed point theorems for a family of contractions depending on two functions and some parameters under the name multiparametric contractions. Ćirić (1974), Reich(1971), and Rus (2001)

independently presented further significant variants of the Banach contraction mapping concept. Many researchers have generalized the Banach contraction mapping principle in different ways (see, for instance, [Michael (1966), Reich (1972), Karapinar et. al (2018), Karapinar (2018), Mohammad et. al (2023), Muhammed et. al (2022)] and the references therein). In accordance with the Banach Contraction mapping principle, Nadler (1969) established the notion of multivalued contraction and suggested a related fixed point result. Furthermore, the notion of weak KKM set-valued mapping from topological vector spaces to hyperconvex metric spaces is extended by Ravi (2021). The concept of α_L -compatible mappings of a L -fuzzy map is introduced by Shagari et. al (2019). By combining this concept with the Meir-Keeler ($M - K$) contraction approach, certain common fixed point theorems for L -fuzzy compatible maps are established. Furthermore, Yahaya et. al (2024) the concepts of Jaggi, Dass-Gupta and Caristi-Ciric-type bilateral multivalued contractions are introduced in the framework of metric spaces and prove its corresponding fixed point results, while the result is extended by Kanwal et. al (2025) in investigating Nonlinear Activation Mechanism Through the fixed-circle and fixed-disc problems of the modified multivalued bilateral Jaggi-type and Dass-Gupta-type contractions introduced by Yahaya et. al (2024). Recently, Jibo and Shagari (2026) presented the notion of multi-valued paired contraction (MVPC) in the ambience of metric space theory and established the Nadler's version of fixed point result. And Omidire and Momure (2026) extends the classical Hardy-Rogers contraction by incorporating an enrichment parameter together with a modified arrangement of the associated distance terms, thereby established the existence and uniqueness results for fixed points of modified enriched Hardy-Rogers contractions in Banach Space.

In this paper, the concept of weak contraction from the case of single-valued mappings is extended to multi-valued mappings. For this purpose, we introduced the notion of multivalued weak Kannan-type contraction, multivalued weak Chatterjea-type contraction, multi-valued weak Reich-type contraction, and multi-

valued weak Hardy-Rogers type contraction and then corresponding fixed point theorems for the picard iteration associated to a multi-valued contractions are obtained.

PRELIMINARY

We recall some basic definitions and preliminaries that will be needed in this paper. Let (X, d) be a metric space, $CB(X)$ be a collection of non-empty closed and bounded subset of X and $A, B \in CB(X)$. The Hausdorff metric H on $CB(X)$ induced by the metric d is given by

$$H(A, B) = \max \left\{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \right\}$$

$$D(a, B) = \left\{ \inf_{b \in B} d(a, b) \right\}.$$

$$D(A, B) = \inf \{ d(a, b) : a \in A, b \in B \}$$

It is known that H is a metric on $CB(X)$ and H is called the Hausdorff metric or Pompeiu-Hausdorff metric induced by d . It is also known that $(CB(X), H)$ is a complete metric space whenever (X, d) is a complete metric space.

Definition 1. Let (X, d) be a complete metric space and $T : X \rightarrow X$.

1. (Kannan (1968)) There exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \lambda [d(x, Tx) + d(y, Ty)], \text{ for all } x, y \in X.$$

2. (Charterjea (1972)) There exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \lambda [d(x, Ty) + d(y, Tx)], \text{ for all } x, y \in X.$$

3. (Reich (1971)) There exists $\lambda \in [0, \frac{1}{3})$ such that

$$d(Tx, Ty) \leq \lambda [d(x, y) + d(x, Tx) + d(y, Ty)], \text{ for all } x, y \in X.$$

4. (Hardy and Rogers (1973)) There exists $\lambda \in [0, \frac{1}{5})$ such that

$$d(Tx, Ty) \leq \lambda [d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)], \text{ for all } x, y \in X.$$

5. (Berinde (2004)) There exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(y, Tx), \text{ for all } x, y \in X.$$

Lemma 1. (Berinde and Berinde (2007)) Let (X, d) be a metric space. Let $A, B \in X$ and $\alpha > 1$. Then, for every $a \in A$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + \alpha.$$

MULTI-VALUED WEAK CONTRACTION MAPPINGS

In this section we introduce the notion of multivalued weak Kannan-type contraction, multivalued weak Chatterjea-type contraction, multivalued weak Reich-type contraction, and multivalued weak Hardy-Rogers-type contraction. In each case, we establish the corresponding fixed point theorem in the setting of metric spaces.

Definition 2. Let (X, d) be a metric space. A multivalued mapping $T : X \rightarrow CB(X)$ is called a multivalued weak Kannan-type contraction if there exists $\mu \in [0, \frac{1}{2})$ such that

$$H(Tx, Ty) \leq \mu [D(x, Tx) + D(y, Ty)] + (1 - 2\mu)D(y, Tx), \quad (1)$$

for all distinct $x, y \in X$.

Theorem 1. Suppose that (X, d) be a complete metric space such that $T : X \rightarrow CB(X)$ is a multivalued weak Kannan-type contraction. Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_0 \in X$ be arbitrary. Since $T : X \rightarrow CB(X)$, then $Tx_0 \in CB(X)$. This implies that $Tx_0 \neq \emptyset$. Therefore, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. For this $x_1 \in X$, $Tx_1 \in CB(X)$ implies that $Tx_1 \neq \emptyset$. Hence, there exists $x_2 \in X$ such that $x_2 \in Tx_1$. Similarly, $Tx_2 \in CB(X)$ implies that $Tx_2 \neq \emptyset$. It follows that, there exists $x_3 \in X$ such that $x_3 \in Tx_2$. Continuing in this manner, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for $n = 0, 1, 2, \dots$. Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. Hence, we presume that $x_n \neq x_{n+1}$ for all n . By Lemma 1, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + \alpha \\ &\leq \mu [D(x_0, Tx_0) + D(x_1, Tx_1)] + (1 - 2\mu)D(x_1, Tx_0) + \alpha \\ &= \mu [d(x_0, x_1) + d(x_1, x_2)] + (1 - 2\mu)d(x_1, x_1) + \alpha \\ &= \mu [d(x_0, x_1) + d(x_1, x_2)] + \alpha \\ &= \mu d(x_0, x_1) + \mu d(x_1, x_2) + \alpha, \end{aligned} \quad (2)$$

thus inequality (2) becomes

$$d(x_1, x_2)(1 - \mu) \leq \mu d(x_0, x_1) + \alpha, \quad (3)$$

therefore inequality (3) gives

$$d(x_1, x_2) \leq \left(\frac{\mu}{1 - \mu} \right) d(x_0, x_1) + \frac{\alpha}{(1 - \mu)}. \quad (4)$$

Again, By Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + \alpha^2 \\ &\leq \mu [D(x_1, Tx_1) + D(x_2, Tx_2)] + (1 - 2\mu)D(x_2, Tx_1) + \alpha^2 \\ &= \mu [d(x_1, x_2) + d(x_2, x_3)] + (1 - 2\mu)d(x_2, x_2) + \alpha^2 \\ &= \mu [d(x_1, x_2) + d(x_2, x_3)] + \alpha^2 \\ &= \mu d(x_1, x_2) + \mu d(x_2, x_3) + \alpha^2, \end{aligned} \quad (5)$$

thus inequality (5) becomes

$$d(x_2, x_3)(1 - \mu) \leq \mu d(x_1, x_2) + \alpha^2, \quad (6)$$

therefore inequality (6) gives

$$\begin{aligned} d(x_2, x_3) &\leq \left(\frac{\mu}{1 - \mu} \right) d(x_1, x_2) + \frac{\alpha^2}{(1 - \mu)} \\ &\leq \left(\frac{\mu}{1 - \mu} \right) \left[\left(\frac{\mu}{1 - \mu} \right) d(x_0, x_1) + \frac{\alpha}{(1 - \mu)} \right] + \frac{\alpha^2}{(1 - \mu)} \\ &\leq \left(\frac{\mu}{1 - \mu} \right)^2 d(x_0, x_1) + \frac{\mu\alpha}{(1 - \mu)^2} + \frac{\alpha^2}{(1 - \mu)}. \end{aligned} \quad (7)$$

Similarly, by Lemma 1, for $x_3 \in Tx_2$, we can find $x_4 \in Tx_3$ such that

$$\begin{aligned} d(x_3, x_4) &\leq H(Tx_2, Tx_3) + \alpha^3 \\ &\leq \mu [D(x_2, Tx_2) + D(x_3, Tx_3)] + (1 - 2\mu)D(x_3, Tx_2) + \alpha^3 \\ &= \mu [d(x_2, x_3) + d(x_3, x_4)] + (1 - 2\mu)d(x_3, x_3) + \alpha^3 \\ &= \mu [d(x_2, x_3) + d(x_3, x_4)] + \alpha^3 \\ &= \mu d(x_2, x_3) + \mu d(x_3, x_4) + \alpha^3, \end{aligned} \quad (8)$$

thus inequality (8) becomes

$$d(x_3, x_4)(1 - \mu) \leq \mu d(x_2, x_3) + \alpha^3, \quad (9)$$

therefore inequality (9) gives

$$\begin{aligned} d(x_3, x_4) &\leq \left(\frac{\mu}{1-\mu} \right) d(x_2, x_3) + \frac{\alpha^3}{(1-\mu)} \\ &\leq \left(\frac{\mu}{1-\mu} \right) \left[\left(\frac{\mu}{1-\mu} \right)^2 d(x_0, x_1) + \frac{\mu\alpha}{(1-\mu)^2} + \frac{\alpha^2}{(1-\mu)} \right] + \frac{\alpha^3}{(1-\mu)} \\ &\leq \left(\frac{\mu}{1-\mu} \right)^3 d(x_0, x_1) + \frac{\mu^2\alpha}{(1-\mu)^3} + \frac{\mu\alpha^2}{(1-\mu)^2} + \frac{\alpha^3}{(1-\mu)}. \end{aligned} \quad (10)$$

From inequalities (4),(7),(10) inductively we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{\mu^{n-i} \alpha^i}{(1-\mu)^{n-i+1}},$$

where $\lambda = \frac{\mu}{1-\mu}$. Now, we show that, the sequence $\{x_n\}$ in X is Cauchy. Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{\mu^{n-i} \alpha^i}{(1-\mu)^{n-i+1}} \\ &\quad + \lambda^{n+1} d(x_0, x_1) + \sum_{i=1}^n \frac{\mu^{n+1-i} \alpha^i}{(1-\mu)^{n-i+2}} \\ &\quad + \dots + \lambda^{m-1} d(x_0, x_1) + \sum_{i=1}^{m-2} \frac{\mu^{m-1-i} \alpha^i}{(1-\mu)^{m-i}} \\ &= (\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}) d(x_0, x_1) \\ &\quad + \sum_{i=1}^{n-1} \frac{\mu^{n-i} \alpha^i}{(1-\mu)^{n-i+1}} \\ &\quad + \sum_{i=1}^n \frac{\mu^{n+1-i} \alpha^i}{(1-\mu)^{n-i+2}} \\ &\quad + \sum_{i=1}^{m-2} \frac{\mu^{m-1-i} \alpha^i}{(1-\mu)^{m-i}} \\ &= \sum_{i=n}^{n+i-1} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{\mu^{m-1-i} \alpha^i}{(1-\mu)^{m-i}} \\ &\leq \sum_{i=n}^{\infty} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{\mu^{m-1-i} \alpha^i}{(1-\mu)^{m-i}}. \end{aligned}$$

$\rightarrow 0$ as $n, m \rightarrow \infty$. Hence, (x_n) in X is a Cauchy sequence. The completeness of (X, d) guarantees that we can find $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Now,

$$\begin{aligned} D(u, Tu) &\leq d(u, x_{n+1}) + D(x_{n+1}, Tu) \\ &\leq d(u, x_{n+1}) + H(Tx_n, Tu) \\ &\leq d(u, x_{n+1}) + \mu \left[D(x_n, Tx_n) + D(u, Tu) \right] \\ &\quad + (1-2\mu)D(u, Tx_n) \\ &\leq d(u, x_{n+1}) + \mu \left[D(x_n, x_{n+1}) + D(u, Tu) \right] \\ &\quad + (1-2\mu)d(u, x_{n+1}). \end{aligned} \quad (11)$$

Letting $n \rightarrow \infty$ in inequality (11) gives $D(u, Tu) \leq \mu D(u, Tu)$.

Since $1-\mu \neq 0$, then $D(u, Tu) \leq 0$, this implies that $u \in Tu$. $\square$

Corollary 1. Suppose that (X, d) be a complete metric space and there exists $\mu \in [0, \frac{1}{2})$ such that $T : X \rightarrow X$ satisfy

$$d(Tx, Ty) \leq \mu[d(x, Tx) + d(y, Ty)] + (1-2\mu)d(y, Tx),$$

for all distinct $x, y \in X$. Then there exists $u \in X$ such that $Tu = u$.

Proof. By defining the multivalued mapping $T : X \rightarrow CB(X)$ as $Tx = \{gx\}$, for all $x \in X$, where $g : X \rightarrow X$ is a single valued mapping the result follows. $\square$

Remark 2. By setting $\mu = \frac{1}{2}$ in Corollary 1 we have Kannan contraction of (Kannan (1968)).

Definition 3. Let (X, d) be a metric space. A multivalued mapping $T : X \rightarrow CB(X)$ is called a multivalued weak Chatterjea-type contraction if there exists $\mu \in [0, \frac{1}{2})$ such that

$$H(Tx, Ty) \leq \mu[D(x, Ty) + D(y, Tx)] + (1-2\mu)D(y, Tx), \quad (12)$$

for all distinct $x, y \in X$.

Theorem 3. Suppose that (X, d) be a complete metric space such that $T : X \rightarrow CB(X)$ is a multivalued weak Chatterjea-type contraction. Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_0 \in X$ be arbitrary. Since $T : X \rightarrow CB(X)$, then $Tx_0 \in CB(X)$. This implies that $Tx_0 \neq \emptyset$. Therefore, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. For this $x_1 \in X$, $Tx_1 \in CB(X)$ implies that $Tx_1 \neq \emptyset$. Hence, there exists $x_2 \in X$ such that $x_2 \in Tx_1$. Similarly, $Tx_2 \in CB(X)$ implies that $Tx_2 \neq \emptyset$. It follows that, there exists $x_3 \in X$ such that $x_3 \in Tx_2$. Continuing in this manner, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for $n = 0, 1, 2, \dots$. Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. Hence, we presume that $x_n \neq x_{n+1}$ for all n . By Lemma 1, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + \alpha \\ &\leq \mu[D(x_0, Tx_1) + D(x_1, Tx_0)] + (1-2\mu)D(x_1, Tx_0) + \alpha \\ &= \mu[d(x_0, x_2) + d(x_1, x_1)] + (1-2\mu)d(x_1, x_1) + \alpha \\ &= \mu[d(x_0, x_2)] + \alpha \\ &\leq \mu[d(x_0, x_1) + d(x_1, x_2)] + \alpha \\ &= \mu d(x_0, x_1) + \mu d(x_1, x_2) + \alpha, \end{aligned} \quad (13)$$

thus inequality (13) becomes

$$d(x_1, x_2)(1-\mu) \leq \mu d(x_0, x_1) + \alpha, \quad (14)$$

therefore inequality (14) gives

$$d(x_1, x_2) \leq \left(\frac{\mu}{1-\mu} \right) d(x_0, x_1) + \frac{\alpha}{(1-\mu)}. \quad (15)$$

Again, By Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + \alpha^2 \\ &\leq \mu[D(x_1, Tx_2) + D(x_2, Tx_1)] + (1-2\mu)D(x_2, Tx_1) + \alpha^2 \\ &= \mu[d(x_1, x_3) + d(x_2, x_2)] + (1-2\mu)d(x_2, x_2) + \alpha^2 \\ &= \mu[d(x_1, x_3)] + \alpha^2 \\ &\leq \mu[d(x_1, x_2) + d(x_2, x_3)] \\ &= \mu d(x_1, x_2) + \mu d(x_2, x_3) + \alpha^2, \end{aligned} \quad (16)$$

thus inequality (16) becomes

$$d(x_2, x_3)(1-\mu) \leq \mu d(x_1, x_2) + \alpha^2, \quad (17)$$

therefore inequality (17) gives

$$\begin{aligned} d(x_2, x_3) &\leq \left(\frac{\mu}{1-\mu} \right) d(x_1, x_2) + \frac{\alpha^2}{(1-\mu)} \\ &\leq \left(\frac{\mu}{1-\mu} \right) \left[\left(\frac{\mu}{1-\mu} \right) d(x_0, x_1) + \frac{\alpha}{(1-\mu)} \right] + \frac{\alpha^2}{(1-\mu)} \\ &\leq \left(\frac{\mu}{1-\mu} \right)^2 d(x_0, x_1) + \frac{\mu\alpha}{(1-\mu)^2} + \frac{\alpha^2}{(1-\mu)}. \end{aligned} \quad (18)$$

Similarly, by Lemma 1, for $x_3 \in Tx_2$, we can find $x_4 \in Tx_3$ such that

$$\begin{aligned} d(x_3, x_4) &\leq H(Tx_2, Tx_3) + \alpha^3 \\ &\leq \mu[D(x_2, Tx_3) + D(x_3, Tx_2)] + (1 - 2\mu)D(x_3, Tx_2) + \alpha^3 \\ &= \mu[d(x_2, x_4) + d(x_3, x_3)] + (1 - 2\mu)d(x_3, x_3) + \alpha^3 \\ &= \mu[d(x_2, x_4)] + \alpha^3 \\ &\leq \mu[d(x_2, x_4) + d(x_3, x_3)] \\ &= \mu d(x_2, x_3) + \mu d(x_3, x_4) + \alpha^3, \end{aligned} \quad (19)$$

thus inequality (19) becomes

$$d(x_3, x_4)(1 - \mu) \leq \mu d(x_2, x_3) + \alpha^3, \quad (20)$$

therefore inequality (20) gives

$$\begin{aligned} d(x_3, x_4) &\leq \left(\frac{\mu}{1 - \mu} \right) d(x_2, x_3) + \frac{\alpha^3}{1 - \mu} \\ &\leq \left(\frac{\mu}{1 - \mu} \right) \left[\left(\frac{\mu}{1 - \mu} \right)^2 d(x_0, x_1) + \frac{\mu\alpha}{(1 - \mu)^2} + \frac{\alpha^2}{1 - \mu} \right] \\ &\leq \left(\frac{\mu}{1 - \mu} \right)^3 d(x_0, x_1) + \frac{\mu^2\alpha}{(1 - \mu)^3} + \frac{\mu\alpha^2}{(1 - \mu)^2} + \frac{\alpha^3}{1 - \mu}. \end{aligned} \quad (21)$$

From inequalities (15), (18), (21) inductively we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{\mu^{n-i}\alpha^i}{(1 - \mu)^{n-i+1}},$$

where $\lambda = \frac{\mu}{1 - \mu}$. Now, we show that, the sequence $\{x_n\}$ in X is Cauchy. Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{\mu^{n-i}\alpha^i}{(1 - \mu)^{n-i+1}} \\ &\quad + \lambda^{n+1} d(x_0, x_1) + \sum_{i=1}^n \frac{\mu^{n+1-i}\alpha^i}{(1 - \mu)^{n-i+2}} \\ &\quad + \dots + \lambda^{m-1} d(x_0, x_1) + \sum_{i=1}^{m-2} \frac{\mu^{m-1-i}\alpha^i}{(1 - \mu)^{m-i}} \\ &= (\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}) d(x_0, x_1) \\ &\quad + \sum_{i=1}^{n-1} \frac{\mu^{n-i}\alpha^i}{(1 - \mu)^{n-i+1}} \\ &\quad + \sum_{i=1}^n \frac{\mu^{n+1-i}\alpha^i}{(1 - \mu)^{n-i+2}} \\ &\quad + \sum_{i=1}^{m-2} \frac{\mu^{m-1-i}\alpha^i}{(1 - \mu)^{m-i}} \\ &= \sum_{i=n}^{n+i-1} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{\mu^{m-1-i}\alpha^i}{(1 - \mu)^{m-i}} \\ &\leq \sum_{i=n}^{\infty} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{\mu^{m-1-i}\alpha^i}{(1 - \mu)^{m-i}}. \end{aligned}$$

$\rightarrow 0$ as $n, m \rightarrow \infty$. Hence, (x_n) in X is a Cauchy sequence. The completeness of (X, d) guarantees that we can

find $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Now,

$$\begin{aligned} D(u, Tu) &\leq d(u, x_{n+1}) + D(x_{n+1}, Tu) \\ &\leq d(u, x_{n+1}) + H(Tx_n, Tu) \\ &\leq d(u, x_{n+1}) + \mu \left[D(x_n, Tu) + D(u, Tx_n) \right] \\ &\quad + (1 - 2\mu)D(u, Tx_n) \\ &\leq d(u, x_{n+1}) + \mu \left[D(x_n, Tu) + d(u, x_{n+1}) \right] \\ &\quad + (1 - 2\mu)d(u, x_{n+1}). \end{aligned} \quad (22)$$

Letting $n \rightarrow \infty$ in inequality (22) gives $D(u, Tu) \leq \mu D(u, Tu)$.

Since $1 - \mu \neq 0$, then $D(u, Tu) \leq 0$, this implies that $u \in Tu$. $\square$

Corollary 2. Suppose that (X, d) be a complete metric space and there exists $\mu \in [0, \frac{1}{2})$ such that $T : X \rightarrow X$ satisfy

$$d(Tx, Ty) \leq \mu[d(x, Ty) + d(y, Tx)] + (1 - 2\mu)d(y, Tx),$$

for all distinct $x, y \in X$. Then there exists $u \in X$ such that $Tu = u$.

Remark 4. By setting $\mu = \frac{1}{2}$ in Corollary 2 we have Chatterjea contraction of (Chatterjea (1972)).

Definition 4. Let (X, d) be a metric space. A multivalued mapping $T : X \rightarrow CB(X)$ is called a multivalued weak Reich-type contraction if there exists $\mu \in [0, \frac{1}{3})$ such that

$$H(Tx, Ty) \leq \mu[d(x, y) + D(x, Tx) + D(y, Ty)] + (1 - 3\mu)D(y, Tx), \quad (23)$$

for all distinct $x, y \in X$.

Theorem 5. Suppose that (X, d) be a complete metric space such that $T : X \rightarrow CB(X)$ is a multivalued weak Reich-type contraction. Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_0 \in X$ be arbitrary. Since $T : X \rightarrow CB(X)$, then $Tx_0 \in CB(X)$. This implies that $Tx_0 \neq \emptyset$. Therefore, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. For this $x_1 \in X$, $Tx_1 \in CB(X)$ implies that $Tx_1 \neq \emptyset$. Hence, there exists $x_2 \in X$ such that $x_2 \in Tx_1$. Similarly, $Tx_2 \in CB(X)$ implies that $Tx_2 \neq \emptyset$. It follows that, there exists $x_3 \in X$ such that $x_3 \in Tx_2$. Continuing in this manner, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for $n = 0, 1, 2, \dots$. Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. Hence, we presume that $x_n \neq x_{n+1}$ for all n . By Lemma 1, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + \alpha \\ &\leq \mu[d(x_0, x_1) + D(x_0, Tx_0) + D(x_1, Tx_1)] \\ &\quad + (1 - 3\mu)D(x_1, Tx_0) + \alpha \\ &= \mu[d(x_0, x_1) + d(x_0, x_1) + d(x_1, x_2)] \\ &\quad + (1 - 3\mu)d(x_1, x_1) + \alpha \\ &= \mu[2d(x_0, x_1) + d(x_1, x_2)] + \alpha \\ &= 2\mu d(x_0, x_1) + \mu d(x_1, x_2) + \alpha, \end{aligned} \quad (24)$$

thus inequality (24) becomes

$$d(x_1, x_2)(1 - \mu) \leq 2\mu d(x_0, x_1) + \alpha, \quad (25)$$

therefore inequality (25) gives

$$d(x_1, x_2) \leq \left(\frac{2\mu}{1 - \mu} \right) d(x_0, x_1) + \frac{\alpha}{(1 - \mu)}. \quad (26)$$

Again, By Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + \alpha^2 \\ &\leq \mu[d(x_1, x_2) + D(x_1, Tx_1) + D(x_2, Tx_2)] \\ &\quad + (1 - 3\mu)D(x_2, Tx_1) + \alpha^2 \\ &= \mu[d(x_1, x_2) + d(x_1, x_2) + d(x_2, x_3)] \\ &\quad + (1 - 3\mu)d(x_2, x_2) + \alpha^2 \\ &= \mu[2d(x_1, x_2) + d(x_2, x_3)] + \alpha^2 \\ &= 2\mu d(x_1, x_2) + \mu d(x_2, x_3) + \alpha^2, \end{aligned} \quad (27)$$

thus inequality (27) becomes

$$d(x_2, x_3)(1 - \mu) \leq 2\mu d(x_1, x_2) + \alpha^2, \quad (28)$$

therefore inequality (28) gives

$$\begin{aligned} d(x_2, x_3) &\leq \left(\frac{2\mu}{1-\mu}\right) d(x_1, x_2) + \frac{\alpha^2}{(1-\mu)} \\ &\leq \left(\frac{2\mu}{1-\mu}\right) \left[\left(\frac{2\mu}{1-\mu}\right) d(x_0, x_1) + \frac{\alpha}{(1-\mu)} \right] + \frac{\alpha^2}{(1-\mu)} \\ &\leq \left(\frac{2\mu}{1-\mu}\right)^2 d(x_0, x_1) + \frac{2\mu\alpha}{(1-\mu)^2} + \frac{\alpha^2}{(1-\mu)}. \end{aligned} \quad (29)$$

Similarly, by Lemma 1, for $x_3 \in Tx_2$, we can find $x_4 \in Tx_3$ such that

$$\begin{aligned} d(x_3, x_4) &\leq H(Tx_2, Tx_3) + \alpha^3 \\ &\leq \mu[d(x_2, x_3) + D(x_2, Tx_2) + D(x_3, Tx_3)] \\ &\quad + (1 - 3\mu)D(x_3, Tx_2) + \alpha^3 \\ &= \mu[d(x_2, x_3) + d(x_2, x_3) + d(x_3, x_4)] \\ &\quad + (1 - 3\mu)d(x_3, x_3) + \alpha^3 \\ &= \mu[2d(x_2, x_3) + d(x_3, x_4)] + \alpha^3 \\ &= 2\mu d(x_2, x_3) + \mu d(x_3, x_4) + \alpha^3, \end{aligned} \quad (30)$$

thus inequality (30) becomes

$$d(x_3, x_4)(1 - \mu) \leq 2\mu d(x_2, x_3) + \alpha^3, \quad (31)$$

therefore inequality (31) gives

$$\begin{aligned} d(x_3, x_4) &\leq \left(\frac{2\mu}{1-\mu}\right) d(x_2, x_3) + \frac{\alpha^3}{(1-\mu)} \\ &\leq \left(\frac{2\mu}{1-\mu}\right) \left[\left(\frac{2\mu}{1-\mu}\right)^2 d(x_0, x_1) + \frac{2\mu\alpha}{(1-\mu)^2} + \frac{\alpha^2}{(1-\mu)} \right] \\ &\quad + \frac{\alpha^3}{(1-\mu)} \\ &\leq \left(\frac{2\mu}{1-\mu}\right)^3 d(x_0, x_1) + \frac{(2\mu)^2\alpha}{(1-\mu)^3} + \frac{2\mu\alpha^2}{(1-\mu)^2} + \frac{\alpha^3}{(1-\mu)}. \end{aligned} \quad (32)$$

From inequalities (26),(29),(32) inductively we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{(2\mu)^{n-i} \alpha^i}{(1-\mu)^{n-i+1}},$$

where $\lambda = \frac{\mu}{1-\mu}$. Now, we show that, the sequence $\{x_n\}$ in

X is Cauchy. Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{(2\mu)^{n-i} \alpha^i}{(1-\mu)^{n-i+1}} \\ &\quad + \lambda^{n+1} d(x_0, x_1) + \sum_{i=1}^n \frac{(2\mu)^{n+1-i} \alpha^i}{(1-\mu)^{n-i+2}} \\ &\quad + \dots + \lambda^{m-1} d(x_0, x_1) + \sum_{i=1}^{m-2} \frac{(2\mu)^{m-1-i} \alpha^i}{(1-\mu)^{m-i}} \\ &= (\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}) d(x_0, x_1) \\ &\quad + \sum_{i=1}^{n-1} \frac{(2\mu)^{n-i} \alpha^i}{(1-\mu)^{n-i+1}} \\ &\quad + \sum_{i=1}^n \frac{(2\mu)^{n+1-i} \alpha^i}{(1-\mu)^{n-i+2}} \\ &\quad + \sum_{i=1}^{m-2} \frac{(2\mu)^{m-1-i} \alpha^i}{(1-\mu)^{m-i}} \\ &= \sum_{i=n}^{n+i-1} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{(2\mu)^{m-1-i} \alpha^i}{(1-\mu)^{m-i}} \\ &\leq \sum_{i=n}^{\infty} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{(2\mu)^{m-1-i} \alpha^i}{(1-\mu)^{m-i}}. \end{aligned}$$

$\rightarrow 0$ as $n, m \rightarrow \infty$. Hence, (x_n) in X is a Cauchy sequence. The completeness of (X, d) guarantees that we can find $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Now,

$$\begin{aligned} D(u, Tu) &\leq d(u, x_{n+1}) + D(x_{n+1}, Tu) \\ &\leq d(u, x_{n+1}) + H(Tx_n, Tu) \\ &\leq d(u, x_{n+1}) + \mu \left[d(x_n, u) + D(x_n, Tx_n) + D(u, Tu) \right] \\ &\quad + (1 - 3\mu)D(u, Tx_n) \\ &\leq d(u, x_{n+1}) + \mu \left[d(x_n, u) + d(x_n, x_{n+1}) + D(u, Tu) \right] \\ &\quad + (1 - 3\mu)d(u, x_{n+1}). \end{aligned} \quad (33)$$

Letting $n \rightarrow \infty$ in inequality (33) gives $D(u, Tu) \leq \mu D(u, Tu)$.

Since $1 - \mu \neq 0$, then $D(u, Tu) \leq 0$, this implies that $u \in Tu$. $\square$

Corollary 3. Suppose that (X, d) be a complete metric space and there exists $\mu \in [0, \frac{1}{3})$ such that $T : X \rightarrow X$ satisfy

$$d(Tx, Ty) \leq \mu[d(x, y) + d(x, Tx) + d(y, Ty)] + (1 - 3\mu)d(y, Tx),$$

for all distinct $x, y \in X$. Then there exists $u \in X$ such that $Tu = u$.

Remark 6. By setting $\mu = \frac{1}{3}$ in Corollary 3 we have Reich contraction of (Reich (1971)).

Definition 5. Let (X, d) be a metric space. A multivalued mapping $T : X \rightarrow CB(X)$ is called a multivalued weak Hardy-Rogers-type contraction if there exists $\mu \in [0, \frac{1}{5})$ such that

$$\begin{aligned} H(Tx, Ty) &\leq \mu[d(x, y) + D(x, Tx) + D(y, Ty) \\ &\quad + D(x, Ty) + D(y, Tx)] + (1 - 5\mu)D(y, Tx), \end{aligned} \quad (34)$$

for all distinct $x, y \in X$.

Theorem 7. Suppose that (X, d) be a complete metric space such that $T : X \rightarrow CB(X)$ is a multivalued weak Hardy-Rogers-type contraction. Then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $x_0 \in X$ be arbitrary. Since $T : X \rightarrow CB(X)$, then $Tx_0 \in CB(X)$. This implies that $Tx_0 \neq \emptyset$. Therefore, there exists $x_1 \in X$ such that $x_1 \in Tx_0$. For this $x_1 \in X$, $Tx_1 \in CB(X)$ implies that $Tx_1 \neq \emptyset$. Hence, there exists $x_2 \in X$ such that $x_2 \in Tx_1$. Similarly, $Tx_2 \in CB(X)$ implies that $Tx_2 \neq \emptyset$. It follows that, there exists $x_3 \in X$ such that $x_3 \in Tx_2$. Continuing in this manner, we construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for $n = 0, 1, 2, \dots$. Note that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} \in Tx_{n_0+1}$ and the proof is finished. Hence, we presume that $x_n \neq x_{n+1}$ for all n . By Lemma 1, for $x_1 \in Tx_0$, we can find $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + \alpha \\ &\leq \mu[d(x_0, x_1) + D(x_0, Tx_0) + D(x_1, Tx_1) + D(x_0, Tx_1) + D(x_1, Tx_0)] \\ &\quad + (1 - 5\mu)D(x_1, Tx_0) + \alpha \\ &= \mu[d(x_0, x_1) + d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_2) + d(x_1, x_1)] \\ &\quad + (1 - 5\mu)d(x_1, x_1) + \alpha \\ &= \mu[2d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_2)] + \alpha \\ &\leq \mu[2d(x_0, x_1) + d(x_1, x_2) + d(x_0, x_1) + d(x_1, x_2)] + \alpha \\ &= 3\mu d(x_0, x_1) + 2\mu d(x_1, x_2) + \alpha \end{aligned} \quad (35)$$

thus inequality (35) becomes

$$d(x_1, x_2)(1 - 2\mu) \leq 3\mu d(x_0, x_1) + \alpha, \quad (36)$$

therefore inequality (36) gives

$$d(x_1, x_2) \leq \left(\frac{3\mu}{1 - 2\mu} \right) d(x_0, x_1) + \frac{\alpha}{(1 - 2\mu)}. \quad (1)$$

Again, By Lemma 1, for $x_2 \in Tx_1$, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + \alpha^2 \\ &\leq \mu[d(x_1, x_2) + D(x_1, Tx_1) + D(x_2, Tx_2) + D(x_1, Tx_2) + D(x_2, Tx_1)] \\ &\quad + (1 - 5\mu)D(x_2, Tx_1) + \alpha^2 \\ &= \mu[d(x_1, x_2) + d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3) + d(x_2, x_2)] \\ &\quad + (1 - 5\mu)d(x_2, x_2) + \alpha^2 \\ &= \mu[2d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_3)] + \alpha^2 \\ &\leq \mu[2d(x_1, x_2) + d(x_2, x_3) + d(x_1, x_2) + d(x_2, x_3)] + \alpha^2 \\ &= 3\mu d(x_1, x_2) + 2\mu d(x_2, x_3) + \alpha^2, \end{aligned} \quad (37)$$

thus inequality (37) becomes

$$d(x_2, x_3)(1 - 2\mu) \leq 3\mu d(x_1, x_2) + \alpha^2, \quad (38)$$

therefore inequality (38) gives

$$\begin{aligned} d(x_2, x_3) &\leq \left(\frac{3\mu}{1 - 2\mu} \right) d(x_1, x_2) + \frac{\alpha^2}{(1 - 2\mu)} \\ &\leq \left(\frac{3\mu}{1 - 2\mu} \right) \left[\left(\frac{3\mu}{1 - 2\mu} \right) d(x_0, x_1) + \frac{\alpha}{(1 - 2\mu)} \right] + \frac{\alpha^2}{(1 - 2\mu)} \\ &\leq \left(\frac{3\mu}{1 - 2\mu} \right)^2 d(x_0, x_1) + \frac{3\mu\alpha}{(1 - 2\mu)^2} + \frac{\alpha^2}{(1 - 2\mu)}. \end{aligned} \quad (39)$$

Similarly, by Lemma 1, for $x_3 \in Tx_2$, we can find $x_4 \in Tx_3$

such that

$$\begin{aligned} d(x_3, x_4) &\leq H(Tx_2, Tx_3) + \alpha^3 \\ &\leq \mu[d(x_2, x_3) + D(x_2, Tx_2) + D(x_3, Tx_3) \\ &\quad + D(x_2, Tx_3) + D(x_3, Tx_2)] \\ &\quad + (1 - 5\mu)D(x_3, Tx_2) + \alpha^3 \\ &= \mu[d(x_2, x_3) + d(x_2, x_3) + d(x_3, x_4) \\ &\quad + d(x_2, x_4) + d(x_3, x_3)] \\ &\quad + (1 - 5\mu)d(x_3, x_3) + \alpha^3 \\ &= \mu[2d(x_2, x_3) + d(x_3, x_4) + d(x_2, x_4)] + \alpha^3 \\ &\leq \mu[2d(x_2, x_3) + d(x_3, x_4) + d(x_2, x_3) \\ &\quad + d(x_3, x_4)] + \alpha^3 \\ &= 3\mu d(x_2, x_3) + 2\mu d(x_3, x_4) + \alpha^3, \end{aligned} \quad (40)$$

thus inequality (40) becomes

$$d(x_3, x_4)(1 - 2\mu) \leq 3\mu d(x_2, x_3) + \alpha^3, \quad (41)$$

therefore inequality (41) gives

$$\begin{aligned} d(x_3, x_4) &\leq \left(\frac{3\mu}{1 - 2\mu} \right) d(x_2, x_3) + \frac{\alpha^3}{(1 - 2\mu)} \\ &\leq \left(\frac{3\mu}{1 - 2\mu} \right) \left[\left(\frac{3\mu}{1 - 2\mu} \right)^2 d(x_0, x_1) \right. \\ &\quad \left. + \frac{3\mu\alpha}{(1 - 2\mu)^2} + \frac{\alpha^2}{(1 - 2\mu)} \right] + \frac{\alpha^3}{(1 - 2\mu)} \\ &\leq \left(\frac{3\mu}{1 - 2\mu} \right)^3 d(x_0, x_1) + \frac{(3\mu)^2\alpha}{(1 - 2\mu)^3} \\ &\quad + \frac{3\mu\alpha^2}{(1 - 2\mu)^2} + \frac{\alpha^3}{(1 - 2\mu)}. \end{aligned} \quad (42)$$

From inequalities (1),(39),(42) inductively we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{(3\mu)^{n-i}\alpha^i}{(1 - 2\mu)^{n-i+1}},$$

where $\lambda = \frac{3\mu}{1 - 2\mu}$. Now, we show that, the sequence $\{x_n\}$ in

X is Cauchy. Let $m, n \in \mathbb{N}$ with $n \leq m$, then

$$\begin{aligned}
 d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
 &\quad + \dots + d(x_{m-1}, x_m) \\
 &\leq \lambda^n d(x_0, x_1) + \sum_{i=1}^{n-1} \frac{(3\mu)^{n-i} \alpha^i}{(1-2\mu)^{n-i+1}} \\
 &\quad + \lambda^{n+1} d(x_0, x_1) + \sum_{i=1}^n \frac{(3\mu)^{n+1-i} \alpha^i}{(1-2\mu)^{n-i+2}} \\
 &\quad + \dots + \lambda^{m-1} d(x_0, x_1) + \sum_{i=1}^{m-2} \frac{(3\mu)^{m-1-i} \alpha^i}{(1-2\mu)^{m-i}} \\
 &= (\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}) d(x_0, x_1) \\
 &\quad + \sum_{i=1}^{n-1} \frac{(3\mu)^{n-i} \alpha^i}{(1-2\mu)^{n-i+1}} \\
 &\quad + \sum_{i=1}^n \frac{(3\mu)^{n+1-i} \alpha^i}{(1-2\mu)^{n-i+2}} \\
 &\quad + \sum_{i=1}^{m-2} \frac{(3\mu)^{m-1-i} \alpha^i}{(1-2\mu)^{m-i}} \\
 &= \sum_{i=n}^{n+i-1} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{(3\mu)^{m-1-i} \alpha^i}{(1-2\mu)^{m-i}} \\
 &\leq \sum_{i=n}^{\infty} \lambda^i d(x_0, x_1) + \sum_{i=n}^{n+i-2} \frac{(3\mu)^{m-1-i} \alpha^i}{(1-2\mu)^{m-i}}.
 \end{aligned}$$

$\rightarrow 0$ as $n, m \rightarrow \infty$. Hence, (x_n) in X is a Cauchy sequence. The completeness of (X, d) guarantees that we can find $u \in X$ such that $x_n \rightarrow u$ as $n \rightarrow \infty$. Now,

$$\begin{aligned}
 D(u, Tu) &\leq d(u, x_{n+1}) + D(x_{n+1}, Tu) \\
 &\leq d(u, x_{n+1}) + H(Tx_n, Tu) \\
 &\leq d(u, x_{n+1}) + \mu \left[d(x_n, u) + D(x_n, Tx_n) \right. \\
 &\quad \left. + D(u, Tu) + D(x_n, Tu) + D(u, Tx_n) \right] \\
 &\quad + (1-5\mu)D(u, Tx_n) \\
 &\leq d(u, x_{n+1}) + \mu \left[d(x_n, u) + d(x_n, x_{n+1}) \right. \\
 &\quad \left. + D(u, Tu) + D(x_n, Tu) + d(u, x_{n+1}) \right] \\
 &\quad + (1-5\mu)d(u, x_{n+1}). \tag{43}
 \end{aligned}$$

Letting $n \rightarrow \infty$ in inequality (43) gives $D(u, Tu) \leq 2\mu D(u, Tu)$.

Since $1 - 2\mu \neq 0$, then $D(u, Tu) \leq 0$, this implies that $u \in Tu$. $\square$

Corollary 4. Suppose that (X, d) be a complete metric space and there exists $\mu \in [0, \frac{1}{5})$ such that $T : X \rightarrow X$ satisfy

$$\begin{aligned}
 d(Tx, Ty) &\leq \mu[d(x, y) + d(x, Tx) + d(y, Ty) \\
 &\quad + d(x, Ty) + d(y, Tx)] + (1-5\mu)d(y, Tx),
 \end{aligned}$$

for all distinct $x, y \in X$. Then there exists $u \in X$ such that $Tu = u$.

Remark 8. By setting $\mu = \frac{1}{5}$ in Corollary 4 we have Hardy Rogers contraction of (Hardy and Rogers (1973)).

To verify Theorem 1, we provide the following example.

Example 1. Let $X = \{(3, 3), (3, 4), (5, 5)\}$ be equipped with the taxicab metric $d : X * X \rightarrow \mathbb{R}$, given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Obviously, (X, d) is a complete metric space.

Consider a multivalued mapping $T : X \rightarrow CB(X)$ defined as follows:

$$Tx = \begin{cases} (3, 4) & \text{if } x = (3, 3), \\ (5, 5) & \text{if } x \neq (3, 3). \end{cases}$$

Now, we examine the following cases:

Case I: for $x = (3, 3)$, we have

$$\begin{aligned}
 D((3, 3), T(3, 3)) &= \inf\{d((3, 3), y) : y \in T(3, 3)\} \\
 &= d((3, 3), (3, 4)) \\
 &= 1
 \end{aligned}$$

Case II: for $x = (3, 4)$, we have

$$\begin{aligned}
 D((3, 4), T(3, 4)) &= \inf\{d((3, 4), y) : y \in T(3, 4)\} \\
 &= d((3, 4), (5, 5)) \\
 &= 2
 \end{aligned}$$

Case III: for $x = (5, 5)$, we have

$$\begin{aligned}
 D((5, 5), T(5, 5)) &= \inf\{d((5, 5), y) : y \in T(5, 5)\} \\
 &= d((5, 5), (5, 5)) \\
 &= 0
 \end{aligned}$$

Now, for $x \in X$ with $D(x, Tx) > 0$ i.e

$x \in \{(3, 3), (3, 4)\}$, we have

$$\begin{aligned}
 H(T(3, 3), T(3, 4)) &= H(\{(3, 3)\}, \{(3, 4)\}) = 1 \\
 &\leq \mu \left[D((3, 3), T(3, 3)) + D((3, 4), T(3, 4)) \right] \\
 &\quad + (1-2\mu)D((3, 4), T(3, 3)) \\
 &= \mu \left[D((3, 3), (3, 4)) + D((3, 4), (5, 5)) \right] \\
 &\quad + (1-2\mu)D((3, 4), (3, 4)) \\
 &= \mu \left[D((3, 3), (3, 4)) + D((3, 4), (5, 5)) \right] \\
 &= \mu \left[1 + 2 \right] \\
 &= 3\mu.
 \end{aligned}$$

It follows that, $\mu \geq \frac{1}{3} \in (0, \frac{1}{2})$, hence all the hypotheses of Theorem 1 are satisfied. We see that T has a fixed point $(5, 5) \in X$.

CONCLUSION

This study has established forms of multivalued contractions and accompanying fixed point theorems in the context of metric spaces. Theorem 1 addressed the prerequisites for the existence of fixed point of a multivalued weak Kannan-type contraction, while Theorem 3 demonstrated the existence of fixed point of the multivalued weak Chatterjea-type contraction. Furthermore, Theorem 5 examined the possibility and rationale for the existence of fixed point for a class of multivalued weak Reich-type contraction, and Theorem 7 established the existence of fixed point for multivalued weak Hardy-Rogers-type contraction. An example 1 supports the hypothesis of the main finding. It has been noted that the concept in these study enhances several related findings in the literature. This results may be extended in metric-like space, modular metric space and some more general spaces.

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