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A Variational-Fixed Point Iteration Method with Rayleigh-Ritz Initialization for Two-Point Boundary Value Problems

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ABSTRACT
Boundary value problems often require an initial approximation that is sufficiently close to the desired solution for an iterative procedure to converge efficiently. In this study, a Variational-Fixed Point Iteration Method with Rayleigh–Ritz Initialization (VFPIM-RR) is developed for solving second-order two-point boundary value problems. The main idea is to use the Rayleigh–Ritz method to construct a boundary-compatible initial approximation before applying variational and fixed-point corrections. Unlike approaches that begin with an arbitrarily selected or simple initial function, the proposed procedure obtains the starting approximation from the variational structure of the problem and subsequently improves it through a Variational Iteration Method correction and a Green's-function-based fixed-point iteration. A Lagrange multiplier is derived for the correction functional, while sufficient conditions for convergence of the resulting hybrid operator are established using the Banach Fixed-Point Theorem. The effectiveness of the method is examined using two benchmark boundary value problems with known exact solutions. The numerical results show that the Rayleigh–Ritz initialization provides a useful starting approximation and that successive hybrid iterations rapidly reduce both the approximation error and the governing-equation residual. In the reported tests, the method attains errors of approximately 10^{-14} after only a few iterations. The results demonstrate that coupling a variationally optimized initial approximation with residual correction and fixed-point refinement provides an accurate and computationally effective framework for second-order two-point boundary value problems.

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INTRODUCTION

Two-point boundary value problems arise when a differential equation is accompanied by conditions imposed at the two ends of a prescribed interval. Such problems occur in a wide range of mathematical and engineering applications, including diffusion, heat transfer, fluid motion, elasticity, vibration, and optimal control. They arise in numerous areas of applied mathematics, engineering and mathematical physics, including heat and mass transfer, diffusion, fluid flow, elasticity, vibration and optimal control. A general second-order TPBVP can be expressed as

$$L[y] = f(x),\ a \le x \le b,$$

together with boundary conditions specified at $x = a$ and $x = b$. While exact solutions are available for some two-point problems, obtaining closed-form expressions becomes considerably more difficult when nonlinear terms, variable coefficients, or complicated forcing functions are present. Consequently, considerable attention has been devoted to developing accurate, efficient and theoretically reliable numerical and semi-analytical methods (Khan et al., 2018; Zavalani, 2015).

A variety of numerical and semi-analytical techniques have been developed for these problems, ranging from finite-difference and finite-element discretizations to collocation, spectral, Galerkin, homotopy, and iterative formulations. Khan et al. (2018) demonstrated the effectiveness of the optimal homotopy asymptotic method for linear and nonlinear TPBVPs, while Zavalani (2015) highlighted the applicability of Galerkin-type formulations to boundary value problems. These studies demonstrate the continuing search for methods capable of obtaining accurate solutions without excessive computational complexity.

Among semi-analytical techniques, the Variational Iteration Method (VIM) has received considerable attention because it generates successive approximations through a correction functional containing a Lagrange multiplier. The method has been applied to different classes of differential equations, including higher-order boundary value problems (He et al., 2009) and linear and nonlinear boundary value problems (Abassy *et al.*, 2014). More recent work has incorporated polynomial approximations into VIM to improve its numerical performance. For example, Otaide *et al.* (2023) developed a variational iteration algorithm based on shifted Vieta–Lucas polynomials for the numerical solution of sixth- and seventh-order boundary value problems. These developments demonstrate the usefulness of combining VIM with suitable approximation functions when solving boundary value problems. An initial function that satisfies the boundary conditions but is poorly related to the governing equation may, however, require additional iterations before an accurate solution is obtained.

This limitation has motivated the integration of VIM with other iterative and approximation techniques. Kilicman and Wadai (2016) developed a variational-fixed-point iteration framework for boundary value problems, demonstrating the potential of combining variational correction with fixed-point refinement. Fixed-point approaches have subsequently been applied directly to second-order TPBVPs. Bello et al. (2018), for instance, developed a fixed-point iterative method for second-order TPBVPs and established its convergence, using a simple boundary-satisfying initial approximation.

Green's-function formulations provide another useful framework for connecting boundary value problems with fixed-point iteration. A Green's-function representation provides an integral form of the boundary value problem from which a fixed-point operator can be constructed. This creates a natural link between the differential formulation and the iterative refinement stage used in the present framework. Khuri and Sayfy (2014) demonstrated the relationship among VIM, Green's functions and fixed-point formulations, while Khuri and Sayfy (2020) developed a Green's-function-based fixed-point procedure for nonlinear two-point boundary value problems. These studies suggest that variational, integral and fixed-point approaches can be incorporated within a common operator framework.

Recent studies have continued to demonstrate the applicability of Rayleigh–Ritz to second-order boundary value problems. Jimoh (2024) examined Rayleigh–Ritz and collocation approaches for second-order ordinary differential boundary value problems, while Osuntoki (2024) investigated Rayleigh–Ritz in comparison with the finite element method. Recent developments in VIM have similarly focused on improving approximation quality and computational efficiency. Jackson et al. (2022) combined VIM with Hermite polynomials for higher-order boundary value problems, while Ain et al. (2022) developed an optimal VIM formulation for parametric boundary value problems. Other recent studies have incorporated polynomial approximations into VIM for higher-order differential equations. At the same time, fixed-point and Green's-function techniques have continued to evolve. Saif et al. (2024) investigated fixed-point iteration for nonlinear boundary value problems, while Okeke et al. (2024) developed a Picard–Ishikawa–Green iterative scheme for boundary value problems. Akewe et al. (2026) further explored the incorporation of Green's functions into fixed-point iterations. These developments indicate a growing interest in hybrid schemes that combine approximation, integral and iterative mechanisms.

The treatment of boundary conditions in VIM has also received recent attention. Liu et al. (2025) combined boundary shape functions with VIM for nonlinear boundary value problems, while Hatif and Alzubaidi (2024) demonstrated the continued extension of VIM to generalized classes of boundary value problems. Collectively, these studies confirm that VIM, Rayleigh–Ritz, Green's functions and fixed-point iteration remain active areas of research. However, most existing contributions have investigated these techniques independently or have combined them in configurations different from that considered in the present study.

The gap considered in this study is the absence of a systematic framework in which a Rayleigh–Ritz approximation is used specifically as the starting point for subsequent variational and fixed-point refinement of second-order TPBVPs. Existing variational-fixed-point studies have established the usefulness of combining VIM and fixed-point iteration (Kilicman & Wadai, 2016), while fixed-point methods for second-order TPBVPs have commonly adopted simple boundary-satisfying initial functions (Bello et al., 2018). Conversely, Rayleigh–Ritz has generally been used as a standalone variational approximation or compared with other numerical methods (Jimoh, 2024; Osuntoki, 2024). Thus, the specific sequential architecture of optimized Rayleigh–Ritz initialization followed by VIM correction and fixed-point refinement remains insufficiently explored.

This gap motivates the present study. The underlying premise is that a more informed initial approximation can reduce the correction required during subsequent iterations. Rayleigh–Ritz provides such an approximation by exploiting the variational structure of the problem and enforcing the essential boundary conditions. VIM then provides a mechanism for correcting the approximation in accordance with the governing differential equation, while fixed-point iteration supplies a systematic refinement process toward the desired solution.

Accordingly, this study develops a Variational-Fixed Point Iteration Method with Rayleigh–Ritz Initialization (VFPIM-RR) for second-order TPBVPs. The proposed procedure first constructs a finite-dimensional Rayleigh–Ritz approximation satisfying the prescribed boundary conditions. This approximation is then introduced into the VIM correction functional, for which an explicit Lagrange multiplier is derived. The resulting correction is incorporated into a fixed-point formulation, and convergence is established under an appropriate contraction condition using the Banach Fixed-Point Theorem. The performance of the method is evaluated through approximation errors, successive-iteration errors and differential-equation residuals.

The proposed computational structure is therefore

$$\text{Rayleigh–Ritz initialization} \longrightarrow \text{VIM correction} \longrightarrow \text{Fixed–point refinement}$$

The novelty of the study does not rest on the invention of Rayleigh–Ritz, VIM, fixed-point iteration or Green's functions, which are established techniques. Rather, it lies in their specific integration into a unified VFPIM-RR framework for second-order TPBVPs, where Rayleigh–Ritz is deliberately employed to generate the initial approximation, VIM performs the variational correction, and fixed-point iteration provides systematic refinement under a provable convergence condition. The framework consequently establishes a direct computational relationship among quality of initialization, residual correction, iterative refinement and convergence, providing a structured alternative for the numerical solution of second-order two-point boundary value problems.

Table 1: Comparison of Existing Methods with the Proposed VFPIM-RR

Method	Initial approximation	Iteration mechanism	Convergence result	Role of variational formulation
VIM	Arbitrary or problem-dependent initial guess	Variational correction	Problem-dependent	Determines the correction functional and Lagrange multiplier
Rayleigh-Ritz	Ritz approximation obtained from an admissible trial space	Variational minimization	Problem-dependent	Central to construction of the approximate solution
Fixed-point iteration	Initial guess	Picard/Mann-type fixed-point iteration	Banach fixed-point theorem under contraction conditions	Usually not the primary initialization mechanism

Method	**Initial approximation**	**Iteration mechanism**	**Convergence result**	**Role of variational formulation**
VFPIM-RR (Present method)	Rayleigh–Ritz approximation	VIM correction + fixed-point refinement	Banach fixed-point theorem under contraction conditions	Used first to construct a boundary-compatible and variationally informed initial approximation

MATERIALS AND METHODS

Mathematical Formulation of the Second-Order Two-Point Boundary Value Problem

Consider the general linear second-order two-point boundary value problem

$$-\big(p(x)y'(x)\big)' + q(x)y(x) = f(x),\ a < x < b \quad (1)$$

subject to the boundary conditions

$$y(a) = \alpha,\ y(b) = \beta. \quad (2)$$

here, p, q, f are prescribed functions, while $\alpha, \beta \in \mathbb{R}$. Assume

$$p \in C^1[a,b],\quad q, f \in C[a,b], \quad (3)$$

$$p(x) > 0,\ q(x) > 0,\ a < x < b \quad (4)$$

Under these assumptions, the differential operator

$$L[y] = -\big(p(x)y'(x)\big)' + q(x)y(x) \quad (5)$$

is self-adjoint, and the associated variational formulation is well defined. To accommodate the nonhomogeneous boundary conditions, introduce a boundary interpolant

$$g(x) = \alpha + \frac{\beta-\alpha}{b-a}(x-a) \quad (6)$$

and write

$$y(x) = g(x) + u(x), \quad (7)$$

where

$$u(a) = u(b) = 0 \quad (8)$$

Hence, the admissible space is

$$V = \{u \in H^1(a,b): u(a) = u(b) = 0\}. \quad (9)$$

Substitution of (7) into (1) gives

$$-\big(p(x)u'(x)\big)' + q(x)u(x) = f(x) + \big(p(x)g'(x)\big)' - q(x)g(x) \quad (10)$$

Define

$$\tilde{f}(x) = f(x) + \big(p(x)g'(x)\big)' - q(x)g(x) \quad (11)$$

The homogeneous problem for $u(x)$ therefore becomes

$$-\big(p(x)u'(x)\big)' + q(x)u(x) = \tilde{f}(x),\ u(a) = u(b) = 0 \quad (12)$$

Rayleigh-Ritz Variational Formulation

For the problem (12), define the functional

$$J(u) = \frac{1}{2}\int_a^b \Big[p(x)\big(u'(x)\big)^2 + q(x)u^2(x)\Big]dx - \int_a^b \tilde{f}(x)\,u(x)\,dx. \quad (13)$$

Let $u + \varepsilon v, v \in V$, be an admissible variation. The stationarity condition will be

$$\frac{d}{d\varepsilon}J[u+\varepsilon v]\Big|_{\varepsilon=0} = 0 \quad (14)$$

Consequently,

$$\int_a^b [p(x)u'(x)v'(x) + q(x)u(x)v(x) - \tilde{f}(x)v(x)]\,dx = 0 \quad (15)$$

Integrating the first term of (15) by parts and using, $v(a) = v(b) = 0$, we have

$$\int_a^b [-\big(p(x)u'(x)\big)' + q(x)u(x) - \tilde{f}(x)]v(x)\,dx = 0 \quad (16)$$

Therefore,

$$-\big(p(x)u'(x)\big)' + q(x)u(x) = \tilde{f}(x), \quad (17)$$

which is precisely (12). Thus, the solution of the BVP is equivalent to the stationary point of the functional (13).

Rayleigh-Ritz Initialization

Let choose a finite-dimensional subspace

$$V_N = span\{\emptyset_1, \emptyset_2, \dots, \emptyset_N\} \in V, \quad (18)$$

where the basis functions satisfy

$$\emptyset_j(a) = \emptyset_j(b) = 0. \quad (19)$$

A convenient polynomial basis will be

$$\emptyset_j(x) = (x-a)(b-x)x^{j-1},\quad j = 1,2,\dots,N \quad (20)$$

The Rayleigh-Ritz approximation is

$$u_N(x) = \sum_{j=1}^{N} c_j\,\phi_j(x) \quad (21)$$

and hence

$$y_N(x) = g(x) + \sum_{j=1}^{N} c_j\,\phi_j(x) \quad (22)$$

The coefficients c_j are obtained from

$$\frac{\partial J[u_N]}{\partial c_i} = 0,\qquad i = 1,2,\dots N. \quad (23)$$

This yields the linear system

$$\sum_{j=1}^{N} K_{ij}\,c_j = F_i,\quad i = 1,2,\dots N. \quad (24)$$

where

$$K_{ij} = \int_a^b \big[p(x)\emptyset_i{}'(x)\emptyset_j{}'(x) + q(x)\emptyset_i(x)\emptyset_j(x)\big]\,dx \quad (25)$$

And

$$F_i = \int_a^b \tilde{f}(x)\emptyset_i(x)\,dx. \quad (26)$$

Thus,

$$Kc = F. \quad (27)$$

The resulting approximation

$$y_0(x) = y_N(x) \quad (28)$$

is used as the initial approximation for the subsequent variational-fixed-point iteration.

Because of (19),

$$y_0(a) = \alpha,\ y_0(b) = \beta \quad (29)$$

Hence, the Rayleigh- Ritz initialization satisfies the boundary conditions exactly.

Variational Iteration Correction

Let the differential equation be represented by

$$L[y] - f = 0 \quad (30)$$

For the present problem,

$$L[y] = -\big(p(x)y'(x)\big)' + q(x)y(x) \quad (31)$$

Starting from $y_0(x)$, the VIM stage generates successive approximations by adding a residual-dependent correction to the current iterate. For the present differential operator, the correction functional is written as

$$\{y_n(x)\}_{n=0}^{\infty} \quad (32)$$

through a correction functional of the form

$$y_{n+1}(x) = y_n(x) + \int_a^x \lambda(x,s)[L[y_n](s) - f(s)]ds, \quad (33)$$

Let

$$R_n(x) = -\big(p(x)y_n'(x)\big)' + q(x)y_n(x) - f(x). \quad (34)$$

The VIM correction functional is defined by

$$y_{n+1}(x) = y_n(x) + \int_a^x \lambda(x,s)R_n(s)\,ds. \quad (35)$$

Applying the variational principle to the highest-order derivative term gives

$$\frac{\partial^2}{\partial s^2}[p(s)\lambda(x,s)] = 0, \quad (36)$$

with the stationary conditions

$$\lambda(x,x) = 0,\quad 1 + \frac{\partial}{\partial s}[p(s)\lambda(x,s)]_{s=x} = 0.$$

Consequently,

$$\lambda(x,s) = \frac{x-s}{p(s)},\qquad a \le s \le x.$$

Thus, the unconstrained VIM correction is

$$\tilde{y}_n(x) = y_n(x) + \int_a^x \frac{x-s}{p(s)}R_n(s)\,ds. \quad (37)$$

To preserve both prescribed boundary conditions, the correction is projected onto the homogeneous boundary space:

$$\tilde{y}_n(x) = y_n(x) + \int_a^x \frac{x-s}{p(s)}R_n(s)\,ds - \frac{x-a}{b-a}\int_a^b \frac{b-s}{p(s)}R_n(s)\,ds. \quad (38)$$

Hence, $\tilde{y}_n(a) = \alpha,\ \tilde{y}_n(b) = \beta$.
This is the VIM transformation used in the numerical computations.

Boundary Preservation

Since the boundary conditions are imposed through the admissible space and the initial approximation satisfies (29), each subsequent approximation is required to satisfy

$$y_n(a) = \alpha,\ \ y_n(b) = \beta,\ n \geq 0. \quad (39)$$

Accordingly, the VIM correction is taken within the homogeneous correction space

$$V = \{v \in H^1(a,b): v(a) = v(b) = 0\}. \quad (40)$$

Therefore,

$$\tilde{y}_n - y_n \in V \quad (41)$$

and consequently,

$$\tilde{y}_n(a) = \alpha, \quad \tilde{y}_n(b) = \beta. \quad (42)$$

Consequently, the correction remains within the homogeneous boundary space, so each updated approximation retains the prescribed endpoint values.

Fixed-Point Formulation

Equation (1) can be written abstractly as

$$L(y) = f. \quad (43)$$

Assuming that exists, we define the fixed-point operator as

$$T(y) = L^{-1}f. \quad (44)$$

More generally, when the problem is represented in the form

$$L(y) = F(x,y), \quad (45)$$

the corresponding operator is

$$T(y) = L^{-1}F(x,y). \quad (46)$$

Explicit Fixed-Point Operator

For the model problem

$$-y''(x) + q(x)y(x) = f(x), \quad (47)$$

also, let

$$g(x) = \alpha + \frac{\beta-\alpha}{b-a}(x-a). \quad (48)$$

The fixed-point operator is

$$T(y)(x) = g(x) + \int_a^b G(x, s)[f(s) - q(s)y(s)]\,ds, \quad (49)$$

where

$$G(x, s) = \begin{cases} \frac{(x-a)(b-s)}{b-a}, & x \leq s, \\ \frac{(s-a)(b-x)}{b-a}, & x \geq s. \end{cases} \quad (50)$$

The complete VFPIM-RR iteration is consequently

$$y_{n+1}(x) = (1-\theta)y_n(x) + \theta T(\tilde{y}_n)(x), n = 0,1,2,\ldots \quad (51)$$

With $y_0 = y_{RR}$ (52)

Hence, the complete computational sequence will be

$y_0 = y_{RR}$

For $n = 0,1,2,\ldots$:

$$R_n(x) = -\big(p(x)y_n'(x)\big)' + q(x)y_n(x) - f(x),$$

$$C_n(x) = \int_a^x \frac{x-s}{p(s)} R_n(s)\,ds,$$

$$C_n^*(x) = C_n(x) - \frac{x-a}{b-a} C_n(b),$$

$$\tilde{y}_n(x) = y_n(x) + C_n^*(x),$$

$$T(\tilde{y}_n)(x) = g(x) + \int_a^b G(x, s)[f(s) - q(s)\tilde{y}_n(s)]\,ds,$$

and

$$y_{n+1}(x) = (1-\theta)y_n(x) + \theta T(\tilde{y}_n)(x).$$

Stop when

$$\| y_{n+1} - y_n \|_\infty < 10^{-16}$$

Hybrid Operator

Let the VIM operator be given by

$$V: X \to X \quad (53)$$

and the fixed-point operator by

$$T: X \to X. \quad (54)$$

The hybrid operator will be

$$H(y) = (1-\theta)y + \theta T\big(V(y)\big). \quad (55)$$

Thus, the proposed method becomes

$$y_{n+1} = H(y_n) \quad (56)$$

The exact solution y^* is a fixed point of the hybrid operator whenever

$$V(y^*) = y^* \quad (57)$$

and

$$T(y^*) = y^*. \quad (58)$$

Stopping Criterion

The iteration is terminated when both the iterative difference and differential-equation residual satisfy the prescribed tolerance:

$$\|y_{n+1} - y_n\|_\infty < \varepsilon, \quad (59)$$

And

$$\|L[y_{n+1}] - f\|_\infty < \varepsilon. \quad (60)$$

The second condition is important because a small successive difference alone does not guarantee a small differential-equation residual.

Convergence Analysis

Let X be a complete normed function space containing the admissible solutions, equipped with the norm

$$\|y_n\|_\infty = \max_{a \leq x \leq b} |y(x)|. \quad (61)$$

Assume that $V$and $T$are Lipschitz continuous with constants q_V and q_T, respectively, satisfying $q_T q_V < 1$, then

$$\|V(u) - V(v)\|_\infty \leq q_V \|u - v\|_\infty, \quad (62)$$

And

$$\|T(u) - T(v)\|_\infty \leq q_T \|u - v\|_\infty, \quad (63)$$

where

$$0 \leq q_V < \infty,\ 0 \leq q_T < \infty. \quad (64)$$

Then

$$\|T(V(u)) - T(V(v))\|_\infty \leq q_T q_V \|u - v\|_\infty$$

Theorem 1: Contraction of the Hybrid Operator

Let H be defined by (56). If

$$q_T q_V < 1, \quad (65)$$

then H is a contraction for every

$$0 < \theta \leq 1. \quad (66)$$

Proof

For any $u, v \in X$,

$$\| H(u) - H(v) \|_\infty \leq \| (1-\theta)(u-v) + \theta\big[T\big(V(u)\big) - T\big(V(v)\big)\big] \|_\infty$$

$$\leq (1-\theta) \| (u-v) \|_\infty + \theta \| T\big(V(u)\big) - T\big(V(v)\big) \|_\infty$$

Using (63),

$$\| T\big(V(u)\big) - T\big(V(v)\big) \|_\infty \leq q_T \| V(u) - V(v) \|_\infty \quad (67)$$

Using (62),

$$\| T\big(V(u)\big) - T\big(V(v)\big) \|_\infty \leq q_T q_V \| u - v \|_\infty. \quad (68)$$

Therefore,

$$\| H(u) - H(v) \|_\infty \leq [(1-\theta) + \theta q_T q_V] \| u - v \|_\infty. \quad (70)$$

let

$$q_H = (1-\theta) + \theta q_T q_V. \quad (70)$$

Since $q_T q_V < 1$ and $0 < \theta \leq 1$,

$$0 \leq q_H < 1. \quad (71)$$

Hence H is a contraction.

Convergence of the Proposed Method

Theorem 2

Under the assumptions of Theorem 1, the sequence generated by

$$y_{n+1} = H(y_n) \quad (72)$$

converges uniformly to a unique fixed point $y^* \in X$.

Proof

Since $H: X \to X$is a contraction on the complete metric space X, Banach's fixed-point theorem guarantees the existence of a unique

$y^* \in X$
such that
$H(y^*) = y^*$ (73)
Furthermore,
$\lim_{n\to\infty} \| y_n - y^* \|_\infty = 0$ (74)
Thus, the VFPIM-RR sequence converges uniformly to the unique fixed point of the hybrid operator.

A Priori Error Estimate
From the contraction property,
$\| y_{n+1} - y^* \|_\infty \le q_H \| y_n - y^* \|_\infty.$ (75)
Consequently,
$\| y_n - y^* \|_\infty \le q_H^n \| y_0 - y^* \|_\infty.$ (76)
Since y^* is generally unknown, the following a posteriori estimate is more useful computationally:
$\| y_n - y^* \|_\infty \le \frac{q_H}{1-q_H} \| y_n - y_{n-1} \|_\infty.$ (77)
Thus, the successive-iteration difference provides a computable estimate that is theoretical and the exact error is used only for validation because the benchmark exact solution is known.

Residual-Based Verification
For each computed approximation y_n, let
$R_n(x) = -\left(p(x)y_n'(x)\right)' + q(x)y_n(x) - f(x).$ (78)
The residual norm will be
$\| R_n \|_\infty = \max_{a\le x\le b} | R_n(x) |.$ (79)
A converged approximation is accepted only when
$\| y_{n+1} - y_n \|_\infty < \varepsilon$ (80)
And
$\| R_{n+1} \|_\infty < \varepsilon.$ (81)
The boundary-condition error is additionally monitored through
$E_{\mathrm{BC}} = \max\{| y_n(a) - \alpha || y_n(b) - \beta |\}.$ (82)
Because the trial and correction spaces satisfy the essential boundary conditions,
$E_{\mathrm{BC}} = 0$ (83)
up to machine precision.

Complete VFPIM-RR Scheme
The complete procedure can therefore be summarized as
Step 1: Construct $y_0(x)$ using Rayleigh-Ritz;
Step 2: Compute $\tilde{y}_n = V(y_n)$;
Step 3: Compute $y_{n+1} = (1-\theta)y_n + \theta T(\tilde{y}_n)$;
Step 4: Evaluate $D_{n+1} = \| y_{n+1} - y_n \|_\infty$;
Step 5: Evaluate $R_{n+1} = \| L[y_{n+1}] - f \|_\infty$;
Step 6: Stop if $D_{n+1} < \varepsilon$ and $R_{n+1} < \varepsilon$;
Step 7: otherwise set $n \leftarrow n+1$ and repeat Step 2.

RESULTS AND DISCUSSION
Numerical Examples
Computational Setting
Both numerical examples are considered within the following model problem:
$-y''(x) + y(x) = f(x),\ 0 < x < 1,$
with $y(0) = y(1) = 0$.
The Rayleigh–Ritz basis is
$\phi_1(x) = x(1-x),$
For the one-term trial function $y_0(x) = c_1x(1-x)$, the Rayleigh–Ritz coefficient $c_1$is obtained from the stationarity condition
$c_1 = \frac{\int_0^1 f(x)\phi_1(x)\,dx}{\int_0^1[\phi_1'(x)^2+\phi_1(x)^2]\,dx}.$
The residual is
$R_n(x) = -y_n''(x) + y_n(x) - f(x).$
Since $p(x) = 1$, the VIM multiplier is
$\lambda(x, s) = x - s.$
The VIM correction is therefore
$$C_n(x) = \int_0^x (x-s)R_n(s)\,ds.$$
To preserve both boundary conditions,
$C_n^*(x) = C_n(x) - xC_n(1),$
and $\tilde{y}_n(x) = y_n(x) + C_n^*(x).$
For the fixed-point step, let
$$T(y)(x) = \int_0^1 G\,(x, s)[f(s) - y(s)]\,ds,$$
where
$G(x, s) = \begin{cases} x(1-s), & 0 \le x \le s \le 1, \\ s(1-x), & 0 \le s \le x \le 1. \end{cases}$
The hybrid iteration with $\theta = 1$ is
$$y_{n+1} = T(\tilde{y}_n).$$
For the model problem considered in the numerical experiments, the boundary-preserving VIM correction leads to an integral representation consistent with the Green's-function formulation used in the subsequent fixed-point step.
$y_0 \to \tilde{y}_0 \to y_1 \to \tilde{y}_1 \to y_2 \to \cdots.$
The stopping conditions are $\| y_{n+1} - y_n \|_\infty < 10^{-16}$, and $\| R_{n+1} \|_\infty < 10^{-16}$.

Example 1: Polynomial Exact Solution
Consider $-y''(x) + y(x) = 7x - x^3,\quad 0 < x < 1$ subject to $y(0) = 0,\ y(1) = 0.$
The exact solution is $y^*(x) = x - x^3.$
Indeed, $(y^*)'' = -6x$, and hence
$-(y^*)'' + y^* = 6x + x - x^3 = 7x - x^3.$

Rayleigh-Ritz Initialization
Using the one-term Rayleigh-Ritz trial function, let
$y_0(x) = c_1x(1-x)$. Substitution of the trial function into the Rayleigh–Ritz stationarity condition gives
$\left[\int_0^1\{\phi_1'(x)^2 + \phi_1(x)^2\}\,dx\right]c_1 = \int_0^1(7x - x^3)\phi_1(x)\,dx.$
Evaluation gives
$\int_0^1[\phi_1'^2 + \phi_1^2]\,dx = \frac{11}{15},$ and
$\int_0^1(7x - x^3)x(1-x)\,dx = \frac{11}{10}.$
Therefore, $c_1 = \frac{3}{2}.$
Thus, the Rayleigh–Ritz (RR) initial approximation is
$y_0(x) = \frac{3}{2}x(1-x)$. Its maximum absolute error is
$\| y_0 - y^* \|_\infty = 0.04811252$. The initial differential-equation residual is
$R_0(x) = -\frac{d^2}{dx^2}\left[\frac{3}{2}x(1-x)\right] + \frac{3}{2}x(1-x) - (7x - x^3),$
which simplifies to
$R_0(x) = x^3 - \frac{3}{2}x^2 - \frac{11}{2}x + 3.$
Hence $\| R_0 \|_\infty = 3$. Thus, although the Rayleigh–Ritz approximation satisfies both boundary conditions exactly, its relatively large residual indicates that further correction is required to satisfy the governing differential equation accurately.

First VIM Correction
For the numerical examples considered, the adopted sign convention in the VIM correction functional results in the Lagrange multiplier
$\lambda(x, s) = x - s,$
Using the adopted VIM multiplier, the first correction functional becomes
$C_0(x) = \int_0^x (x-s)R_0(s)\,ds.$
After imposing the right-end boundary condition,
$C_0^*(x) = C_0(x) - xC_0(1).$

The boundary-preserving VIM approximation is subsequently used as the input to the Green's-function fixed-point operator. With $\theta = 1$, $y_1 = T(\tilde{y}_0)$.
The first hybrid iterate is
$y_1(x) = \frac{x}{5040}(6x^6 - 21x^5 + 21x^4 - 5047x^2 + 5041)$.
This already reduces the maximum error from 4.81125×10^{-2} to 3.10299×10^{-5}

Second Hybrid Iteration
Applying the same procedure to $y_1(x)$ gives the second hybrid iterate
$y_2(x) = \frac{x}{39916800}(6x^{10} - 33x^9 + 55x^8 - 66x^6 + 66x^4 - 39916833x^2 + 39916805)$.
The error is now $\| y_2 - y^* \|_\infty = 1.99159 \times 10^{-8}$.
The corresponding maximum residual is 8.0613544×10^{-7}.
The complete iteration history is presented in Table 2.

Table 2: Iteration History of the VFPIM-RR Scheme for Example 1

n	$\| y_n - y^* \|_\infty$	$\| y_n - y_{n-1} \|_\infty$	$\| R_n \|_\infty$
0	4.8112521×10^{-2}	---------------------	3.0000000
1	3.1029898×10^{-5}	4.8082294×10^{-2}	1.2539085×10^{-3}
2	1.9915924×10^{-8}	3.1009984×10^{-5}	8.0613544×10^{-7}
3	$1.2778597 \times 10^{-11}$	1.9903145×10^{-8}	$5.1725714 \times 10^{-10}$
4	$8.1990540 \times 10^{-15}$	$1.2770398 \times 10^{-11}$	$3.3188473 \times 10^{-13}$

Hence, both the error and residual criteria are satisfied, as evidenced by their progressive reduction with each iteration. The rapid decay of these quantities indicates that the hybrid iteration scheme converges to the exact solution, while the small residual confirms that the computed approximation increasingly satisfies the governing differential equation and the associated boundary conditions. Therefore, the fourth iteration provides a sufficiently accurate approximation for Example 1, demonstrating the effectiveness and numerical stability of the proposed hybrid method.

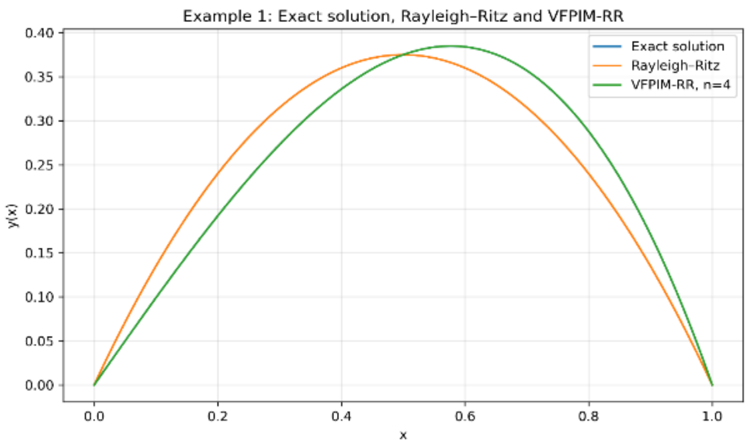

Figure 1: Exact Solution, Rayleigh–Ritz Initialization and final VFPIM-RR Approximation for Example 1

Interpretation of Figure 1
Figure 1 indicates that the initial Rayleigh–Ritz approximation captures the overall behaviour of the exact solution, although visible deviations remain within the interval. These deviations are progressively removed by the hybrid iterations, after which the computed solution is visually indistinguishable from the exact profile. Following the VFPIM-RR corrections, the numerical approximation becomes virtually indistinguishable from the exact solution. This graphical agreement is consistent with the reduction of the maximum error from 4.81×10^{-2} at initialization to 8.20×10^{-15} after four iterations, confirming the rapid refinement produced by the hybrid procedure.

Example 2: Trigonometric Exact Solution
Consider
$-y''(x) + y(x) = (1 + \pi^2)\sin(\pi x),\ 0 < x < 1,$
with $y(0) = y(1) = 0$.
The exact solution is $y^*(x) = \sin(\pi x)$.
Indeed, $(y^*)'' = -\pi^2 \sin(\pi x)$,
so $-(y^*)'' + y^* = (\pi^2 + 1)\sin(\pi x)$.

Rayleigh-Ritz Initialization
For the second example, the same boundary-compatible one-term Rayleigh–Ritz trial function is adopted:
$y_0(x) = c_1 x(1 - x)$.
The Rayleigh-Ritz coefficient is
$c_1 = \frac{\int_0^1 (1+\pi^2)\sin(\pi x) x(1-x)\, dx}{\int_0^1 [(1-2x)^2 + x^2(1-x)^2] dx}$.
This gives
$c_1 = \frac{120(1+\pi^2)}{11\pi^3} = 3.82430640673080$.
Thus
$y_0(x) = 3.82430640673080\, x(1 - x)$.
The initial maximum error is
$\| y_0 - y^* \|_\infty = 4.3923398 \times 10^{-2}$.
The relatively large initial residual indicates that, despite satisfying the boundary conditions, the one-term Rayleigh-Ritz approximation does not yet reproduce the differential equation with sufficient accuracy. The hybrid correction is therefore applied.
$R_0(x) = -y_0''(x) + y_0(x) - (1 + \pi^2)\sin(\pi x)$,
with $\| R_0 \|_\infty = 7.6486128$.

First Hybrid Iteration

Applying the boundary-preserving VIM correction followed by the Green's-function fixed-point operator yields the first hybrid iterate

$$y_1(x) \approx -0.010623073x^6 + 0.031869220x^5 \\ -0.053115367x^3 + 0.031869220x \\ +0.989734020\sin(\pi x).$$

The numerical results themselves are:

$\| y_1 - y^* \|_\infty = 1.4086547 \times 10^{-4}$,

and $\| R_1 \|_\infty = 1.8707364 \times 10^{-3}$.

Second Hybrid Iteration

A second application of the hybrid operator gives

$$y_2(x) \approx -2.1077526 \times 10^{-6}x^{10} + 1.0538763 \times 10^{-5}x^9 \\ -6.3232580 \times 10^{-5}x^7 + 2.6557683 \times 10^{-4}x^5 \\ -5.3747603 \times 10^{-4}x^3 + 3.2670166 \times 10^{-4}x \\ +0.99989461\sin(\pi x).$$

then, the corresponding maximum error decreases to 1.4003803×10^{-6}. and the maximum residual decreases to 1.5267448×10^{-5}.

Table 3: Iteration History of the VFPIM-RR Scheme for Example 2

n	$\| y_n - y^* \|_\infty$	$\| y_n - y_{n-1} \|_\infty$	$\| R_n \|_\infty$
0	4.3923398×10^{-2}	----------------------	7.6486128
1	1.4086547×10^{-4}	4.3782533×10^{-2}	1.8707364×10^{-3}
2	1.4003803×10^{-6}	1.3946509×10^{-4}	1.5267448×10^{-5}
3	1.4370349×10^{-8}	1.3860100×10^{-6}	1.5620587×10^{-7}
4	$1.4752499 \times 10^{-10}$	1.4222824×10^{-8}	1.6035391×10^{-9}
5	$1.5144889 \times 10^{-12}$	$1.4601050 \times 10^{-10}$	$1.6461895 \times 10^{-11}$
6	$1.5547716 \times 10^{-14}$	$1.4989412 \times 10^{-12}$	$1.6899752 \times 10^{-13}$

Table 4: Comparison of the Rayleigh–Ritz Initialization and VFPIM-RR Approximations

Example	Method	Maximum absolute error	Maximum residual
1	Rayleigh–Ritz	4.8113×10^{-2}	3.0000
1	VFPIM-RR, $n = 1$	3.1030×10^{-5}	1.2539×10^{-3}
1	VFPIM-RR, $n = 2$	1.9916×10^{-8}	8.0614×10^{-7}
1	VFPIM-RR, $n = 4$	8.1991×10^{-15}	3.3188×10^{-13}
2	Rayleigh–Ritz	4.3923×10^{-2}	7.6486
2	VFPIM-RR, $n = 1$	1.4087×10^{-4}	1.8707×10^{-3}
2	VFPIM-RR, $n = 2$	1.4004×10^{-6}	1.5267×10^{-5}
2	VFPIM-RR, $n = 4$	1.4752×10^{-10}	1.6035×10^{-9}
2	VFPIM-RR, $n = 6$	1.5548×10^{-14}	1.6900×10^{-13}

Comparison with VIM-only and VFPIM-RR Errors

To assess the contribution of the complete hybrid strategy, the VFPIM-RR results were compared with a VIM-only sequence generated from the same Rayleigh–Ritz initial approximation. The comparison is made using the maximum absolute error at each iteration.

Table 5: Comparison of VIM-only and VFPIM-RR Errors

Iteration	VIM-only error	VFPIM-RR error
0	4.3923×10^{-2}	4.3923×10^{-2}
1	1.7299×10^{-3}	1.4087×10^{-4}
2	1.4087×10^{-4}	1.4004×10^{-6}
3	1.3867×10^{-5}	1.4370×10^{-8}
4	1.4004×10^{-6}	1.4752×10^{-10}

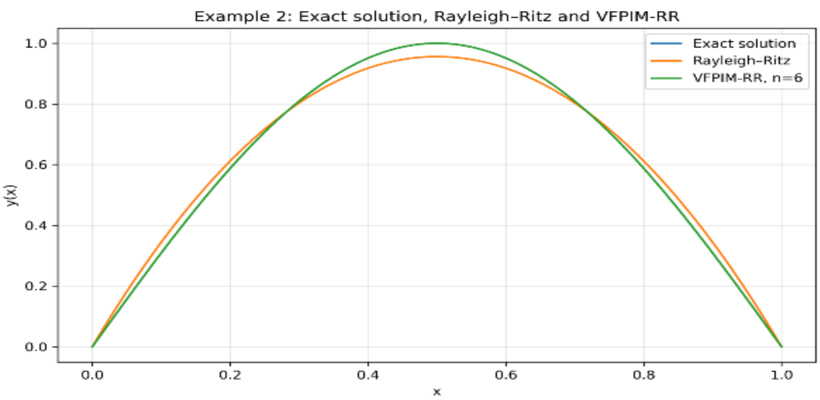

Figure 2: Exact Solution, Rayleigh-Ritz Initialization and Final VFPIM-RR Approximation for Example 2

Interpretation of Figure 2

Figure 2 demonstrates the ability of the VFPIM-RR scheme to correct the structural discrepancy between the polynomial Rayleigh–Ritz initialization and the non-polynomial exact solution $\sin(\pi x)$. Although the initial approximation follows the correct qualitative profile, noticeable deviations occur near the interior of the interval. After successive hybrid corrections, these deviations become negligible, and the final approximation closely overlaps the exact solution. The graphical result agrees with the corresponding error reduction reported in Table 3.

Values at Selected Points

For Example, 1,
$y^*(x) = x - x^3$.
For Example, 2,
$y^*(x) = \sin(\pi x)$.
Pointwise comparisons at selected locations in the computational interval are presented in Tables 6 and 7.

Table 6: Exact and Approximate Solution Values at Selected Points for Example 1

x	Exact	RR y_0	VFPIM-RR y_2	VFPIM-RR y_4
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.25	0.23437500	0.28125000	≈0.23437500	≈0.23437500
0.50	0.37500000	0.37500000	≈0.37500000	≈0.37500000
0.75	0.32812500	0.28125000	≈0.32812500	≈0.32812500
1.00	0.00000000	0.00000000	0.00000000	0.00000000

Table 7: Exact and Approximate Solution Values at Selected Points for Example 2

x	Exact	RR y_0	VFPIM-RR y_2	VFPIM-RR y_6
0.00	0.00000000	0.00000000	0.00000000	0.00000000
0.25	0.70710678	0.71668245	0.70710678	0.70710678
0.50	1.00000000	0.95607660	0.99999860	1.00000000
0.75	0.70710678	0.71668245	0.70710678	0.70710678
1.00	0.00000000	0.00000000	0.00000000	0.00000000

Error Reduction

The computed errors show a pronounced improvement after the Rayleigh–Ritz starting stage. In Example 1, the maximum error falls from $4.81 \times 10^{-2}$to 8.20×10^{-15}, while Example 2 improves from $4.39 \times 10^{-2}$to 1.55×10^{-14}.

Example 1

$\frac{\|y_0 - y^*\|_\infty}{\|y_4 - y^*\|_\infty} = 5.87 \times 10^{12}$.

Thus, the maximum error decreases from approximately 4.81×10^{-2} to 8.20×10^{-15}.

Example 2

The error decreases from
4.39×10^{-2} to 1.55×10^{-14}

Convergence Figures

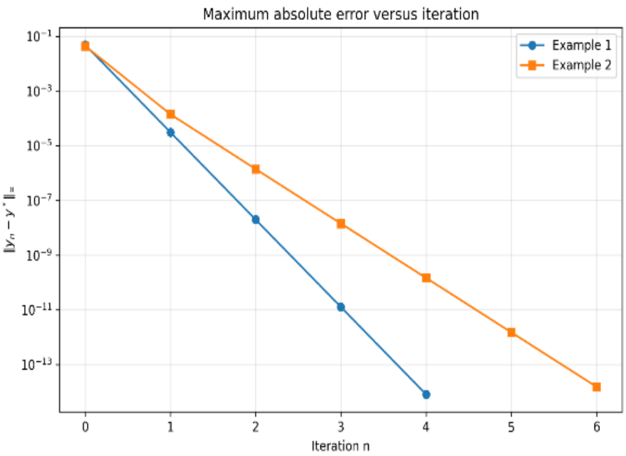

Figure 3: Maximum Absolute Error Versus Iteration

Interpretation of Figure 3

Figure 3 depicts the maximum absolute error as a function of the iteration number. The monotone decay of the error for both examples indicates stable numerical refinement. Example 1 reaches an error of 8.20×10^{-15}after four iterations, whereas Example 2 requires six iterations to attain 1.55×10^{-14}. The slightly larger number of iterations required by Example 2 is consistent with the greater difficulty associated with approximating its non-polynomial exact solution using the initial polynomial trial space.

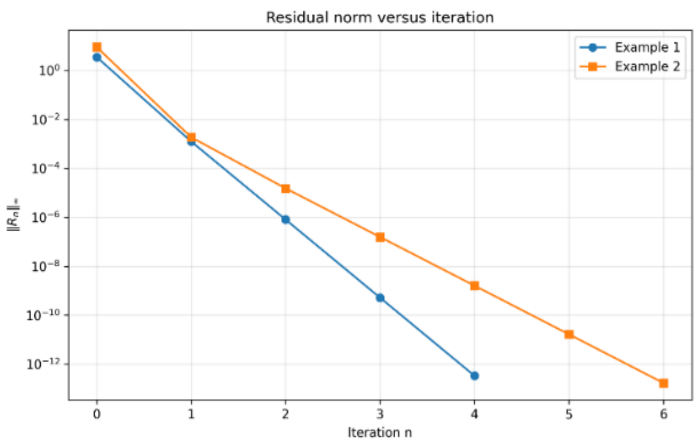

Figure 4: Maximum Differential-Equation Residual Versus Iteration

Interpretation of Figure 4

Figure 4 shows the maximum differential-equation residual as a function of iteration number. Figure 4 confirms that the reduction in solution error is accompanied by a corresponding reduction in the governing-equation residual. For Example, 1, the residual decreases from 3.00 at initialization to 3.32×10^{-13} after four iterations. For Example, 2, it decreases from 7.6486to 1.69×10^{-13}after six iterations. The simultaneous decay of the error and residual provides stronger evidence of convergence than either criterion considered independently.

Discussion

The numerical experiments show that the main benefit of VFPIM-RR lies in the interaction between initialization and subsequent correction. The Rayleigh–Ritz approximations satisfy the endpoint conditions exactly, but their initial residuals show that boundary compatibility alone is insufficient to produce a highly accurate solution. The VIM stage reduces this discrepancy, while the Green's-function fixed-point refinement provides further systematic improvement.

For Example, 1, the maximum error decreases from 4.81×10^{-2} to 8.20×10^{-15} within four hybrid iterations. For Example, 2, the corresponding reduction is from $4.39 \times 10^{-2}$to 1.55×10^{-14}after six iterations. The residuals decrease simultaneously, reaching $3.32 \times 10^{-13}$and 1.69×10^{-13}, respectively. The comparison with VIM alone is also informative: for Example, 2, VFPIM-RR produces smaller errors at the reported iterations, indicating that the fixed-point refinement contributes meaningfully beyond the variational correction alone.

The two examples also illustrate the effect of the initial trial space. The polynomial exact solution in Example 1 is approximated more rapidly, whereas the non-polynomial solution in Example 2 requires additional refinement. Nevertheless, both cases reach near-machine-precision accuracy, supporting the numerical effectiveness of the proposed hybrid strategy for the benchmark problems considered.

CONCLUSION

This study presented a Variational-Fixed Point Iteration Method with Rayleigh–Ritz Initialization (VFPIM-RR) for the numerical solution of second-order two-point boundary value problems. The method combines the boundary compatibility of the Rayleigh–Ritz approximation with the corrective capability of the Variational Iteration Method and the refinement provided by Green's-function-based fixed-point iteration. Under an appropriate contraction condition, convergence of the hybrid scheme was established using the Banach Fixed-Point Theorem.

The numerical examples demonstrate that the proposed approach consistently improves the initial Rayleigh-Ritz approximation. In particular, the maximum errors were reduced to approximately $8.20 \times 10^{-15}$and 1.55×10^{-14}, while the corresponding differential-equation residuals decreased to $3.32 \times 10^{-13}$and 1.69×10^{-13}, respectively. The comparison with VIM alone also indicates that the fixed-point refinement enhances the rate of numerical improvement.

Overall, the results show that VFPIM-RR provides an accurate, stable, and systematically convergent framework for second-order two-point boundary value problems. Its main strength is the deliberate use of a variationally informed initial approximation followed by successive correction and refinement. Future studies may extend the framework to nonlinear problems, variable-coefficient equations, higher-order boundary value problems, and broader classes of boundary conditions.

Compliance with Ethical Standards

Acknowledgments

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Declaration of Conflict of Interest

The authors declare that there are no financial, professional, or personal interests that could have influenced the research, analysis, or presentation of the results reported in this manuscript.

Statement of Ethical Approval

No ethical approval was required for this study. The work is entirely mathematical and computational and involves the analysis of second-order two-point boundary value problems without the use of human participants, animals, clinical materials, or personal data.

Statement of Informed Consent

Informed consent was not applicable to this study because it involved no human participants or human-subject data. The results presented are obtained solely through mathematical analysis and computational experimentation.

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