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Bivariate Conditional Weibull-Lognormal Distribution Model and Its Application

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ABSTRACT

This study proposes a new Bivariate Conditional Weibull-Lognormal Distribution (BCWLD), in which the marginal distribution of the explanatory variable follows a Weibull distribution while the conditional distribution of the response variable follows a Lognormal distribution. The joint probability density function was derived using the conditional distribution approach. The marginal density, posterior density, and theoretical joint moments of the proposed model were also established. Model parameters were estimated using the Maximum Likelihood Estimation method, with numerical optimization performed through the L-BFGS-B algorithm. Numerical results confirmed the validity of the marginal and posterior distributions, while goodness-of-fit assessment using the Cramer-von Mises statistic supported the suitability of the Weibull and Lognormal marginal assumptions. These findings demonstrate that the BCWLD provides a more flexible and accurate framework for modelling dependence between non-negative continuous variables and represents a useful addition to the family of conditional bivariate distributions.

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INTRODUCTION

Univariate probability distributions are extensively used in statistical modeling since they characterize the behavior of individual random variables seen in real-world situations. Widely utilized distributions, including the Weibull, lognormal, gamma, and normal distributions, are prevalent in domains like reliability analysis, finance, engineering, and health sciences, owing to their capacity to represent diverse data attributes such as skewness, nonnegativity, and varying tail behaviors (Algarni, 2022; Alonso, 2024; Obulezi et al., 2025).

In several real-world scenarios, the variables of interest are continuous and display unique traits like longevity patterns or positive skewness, necessitating the development of adaptable probabilistic models that accurately reflect their foundational distributions. (Li and Wang, 2013)

However, in many practical research, two variables are commonly concurrently observed and demonstrate dependency, but feature fundamentally distinct distributional properties. Existing bivariate models, including classical normal-based frameworks and several copula-based constructions, are often inadequate when one variable is nonnegative with lifetime-type behaviour and the other is positively skewed with heavy right-tailed characteristics (Kotz et al., 2000; Nelsen, 2006). As a consequence, these models may fail to completely reflect the genuine dependency structure between such variables in real data (Joe, 2014).This constraint presents a gap in the literature, since there is no sufficiently flexible bivariate framework expressly intended to jointly represent the dependency between a nonnegative continuous variable with lifespan features and a positively skewed continuous variable.

To overcome this gap, this paper suggests a bivariate conditional Weibull-Lognormal distribution model. The model posits that one variable follows a Weibull distribution to describe nonnegative lifespan behavior, while the conditional distribution of the second variable given the first follows a lognormal distribution to accommodate positive skewness and multiplicative variability. By enabling the parameters of the lognormal distribution to depend on the conditioning variable, the model offers a flexible framework for expressing nonlinear dependency between the two variables. This makes the suggested model useful for applications such as health, finance, and reliability studies where such mixed data sets regularly appear.

MATERIALS AND METHODS

Conditional Weibull-Lognormal Distribution

In many applied studies such as finance, health sciences, reliability analysis, and environmental modelling, it is well established that variables such as age, duration, or exposure time can be appropriately described by nonnegative lifetime distributions, with the Weibull distribution being one of the most widely used due to its flexibility in modelling different hazard behaviours (Weibull, 1951; Meeker and Escobar, 2021;).

On the other hand, variables such as body mass index (BMI), income levels, and some biological measures are generally positively skewed and may be appropriately characterized by the lognormal distribution owing to their multiplicative nature and right-tailed behavior (Aitchison and Brown, 1957).

Consequently, it is reasonable to explore the use of a bivariate conditional Weibull-Lognormal distribution in modelling the joint behaviour of such variables, as this framework can naturally accommodate the conditional dependence structure between a nonnegative continuous variable and a positively skewed continuous variable. The examination of a bivariate conditional Weibull-Lognormal distribution is well-supported by current research, demonstrating its promise in modeling the combined behavior of nonnegative and positively skewed variables.

The Weibull distribution is extensively utilized in reliability analysis due to its flexibility in representing various hazard behaviors, as demonstrated in studies that derive bivariate Weibull distributions and investigate their properties and applications in real-life datasets(John and Sankaran, 2017) .Conversely, the lognormal distribution successfully reflects the properties of positively skewed data, with applications identified in financial and dependability settings (Gupta and Gupta, 2012). The combination of these two distributions provides for a nuanced understanding of the conditional dependency between variables, as indicated by the creation of models that support varied dependence patterns, such as the bivariate conditional Weibull distribution (Gongsin and Saporu, 2020). This approach is especially beneficial in sectors like finance and health sciences, where such interactions are crucial for correct modeling and analysis.

Suppose a random variable X has a Weibull distribution with size and shape parameter λ and k respectively. The probability density function of X is given by

$$f_X(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{x}{\lambda}\right)^k\right] \qquad 0 < x < \infty \quad k > 0\,,\lambda > 0 \tag{1}$$

Consider another random variable Y, Assumed to depend on X, conditional on a realization X=x, we assume that Y follows a Lognormal distribution. That is, the conditional distribution of Y given X=x is given as

$$f_{Y|X}(y|x) = \frac{1}{y\sigma(x)\sqrt{2\pi}}\exp\left[-\frac{(ln y-\mu(x))^2}{2\sigma^2(x)}\right] 0 < y < \infty\ \sigma > 0, \tag{2}$$

The Joint Probability Density Function

A joint probability density function called Bivariate Conditional Weibull-Lognormal Distribution (BCWLD) for X and Y can be derived in multiplying equation (1) and (2) using the relation

$f(x,y) = f_x(x)f_{Y|X}(y|X = x)$

BCWLD is therefore given by

$$f(x,y) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{x}{\lambda}\right)^k\right].\frac{1}{y\sigma(x)\sqrt{2\pi}}\exp\left[-\frac{(\ln y-\mu(x))^2}{2\sigma^2(x)}\right]$$

$$f(x,y) = \frac{k}{\lambda y\sigma(x)\sqrt{2\pi}}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(\ln y- \mu(x))^2}{2\sigma^2(x)}\right] \tag{3}$$

Theorem 1: The relation

$$f(x,y) = \frac{k}{\lambda y\sigma(x)\sqrt{2\pi}}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(\ln y- \mu(x))^2}{2\sigma^2(x)}\right]$$

is a valid bivariate probability density function for $x\,\epsilon\,(0,\infty),\quad y\,\epsilon\,(0,\infty)$

$K > 0,\quad \lambda > 0,\quad \sigma(x) > 0,\quad \mu(x)\,\epsilon\,\mathbb{R}$

Proof

$$let\ Z = \int_0^\infty \int_0^\infty f(x,y)dy\,dx =$$

Substituting the Joint density

$$Z = \int_0^\infty \int_0^\infty \frac{k}{\lambda y\sigma(x)\sqrt{2\pi}}\left(\frac{x}{\lambda}\right)^{k-1} exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(\ln y- \mu(x))^2}{2\sigma^2(x)}\right] dy\,dx \tag{4}$$

Separate the terms involving $x\ and\ y$:

$$Z = \int_0^\infty \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} exp\left[-\left(\frac{x}{\lambda}\right)^k\right]\left[\int_0^\infty \frac{1}{y\sigma(x)\sqrt{2\pi}}\, exp\left[-\frac{(ln\ y- \mu(x))^2}{2\sigma^2(x)}\right] dy\right]dx \tag{5}$$

The integrand of the inner integral is the probability density function of a lognormal random variable with parameters$\mu(x) and\ \sigma(x)$. Hence its integral over (0,∞) equals 1

$$\int_0^\infty \frac{1}{y\sigma(x)\sqrt{2\pi}}\, exp\left[-\frac{(iny- \mu(x))^2}{2\sigma^2(x)}\right] dy = 1 \tag{6}$$

Hence,

$$Z = \int_0^\infty \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} exp\left[-\left(\frac{x}{\lambda}\right)^k\right] dx \tag{7}$$

Now let $p = \left(\frac{x}{\lambda}\right)^k$

Then $dp = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}dx$

So that $\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}dx = dp$

Also, $x = 0 \implies p = 0, x \longrightarrow \infty \implies p \longrightarrow \infty$

Therefore, $Z = \int_0^\infty e^{-p}\,dp$

Evaluating

$Z = [-e^{-p}]_0^\infty = 0 - (-1) = 1$

Thus

$$\int_0^\infty \int_0^\infty f(x,y)dy\,dx = 1$$

And therefore,

$$f(x,y) = \frac{k}{\lambda y\sigma(x)\sqrt{2\pi}}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(lny - \mu(x))^2}{2\sigma^2(x)}\right]$$

Is a valid bivariate probability density function [Z=1]

Hence the Bivariate Conditional Weibull-Lognormal Distribution belongs to the family of joint probability density functions. The three-dimensional joint density surfaces exhibit unimodal behavior with probability mass concentrated around moderate values of the Weibull variable. Variations in the conditional lognormal parameters primarily affect the height, spread, and smoothness of the density surface, while the weak slope parameters produce only mild dependence between the variables."

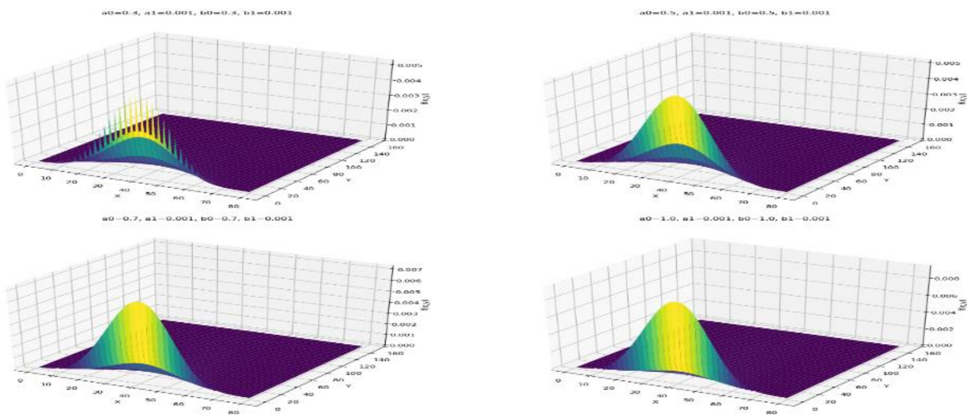

Figure 1: 3D Density Plots of BCWLD for Various Parameter Values

Marginal Density

$$f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx$$

Substitute the joint PDF

$$f_Y(y) = \int_0^\infty \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \cdot \frac{1}{y\sigma(x)\sqrt{2\pi}} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(In\, y - \mu(x))^2}{2\sigma^2(x)}\right]$$

Factor out terms independent of x

Only $\frac{1}{y}$ *does not depend on x,*

$$f_Y(y) = \frac{1}{y}\int_0^\infty \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \cdot \frac{1}{\sigma(x)\sqrt{2\pi}} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(In\, y - \mu(x))^2}{2\sigma^2(x)}\right] \quad (8)$$

Recognize the structure

$$f_Y(y) = \int_0^\infty f_X(x) f_{Y|X}(y|x)\, dx$$

$f_X(x)$ Is weibull

$f_{Y|X}(y|x)$ Is lognormal

$$f_Y(y) = \frac{1}{y}\int_0^\infty \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \cdot \frac{1}{\sigma(x)\sqrt{2\pi}} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(Iny - \mu(x))^2}{2\sigma^2(x)}\right] \quad (9)$$

Due to the dependence of the functions $\mu(x) and\ \sigma(x)\ on\ x,$ the integral obtained for the marginal density of Y cannot, in general, be expressed in a closed-form solution. Hence, the marginal density $f_Y(y)$ is computed using numerical integration methods Table 1, the shape of the marginal density confirms that the distribution of Y is unimodal and right-skewed. Most observations are concentrated around the peak region, while the probability of extreme large values diminishes steadily. This behavior is consistent with the characteristics typically observed in lognormal-type distributions, indicating that the proposed Bivariate Conditional Weibull-Lognormal model yields a realistic and well-behaved marginal distribution for Marginal Distribution of Y Plot.

Table 1: Area under Marginal Density F_y (Y)-Weibull-Lognormal Model

a_0	a_1	b_0	b_1	Numerical Integral
0.3	0.001	0.3	0.001	0.987192
0.5	0.001	0.5	0.001	0.961540
0.7	0.001	0.7	0.001	0.902098

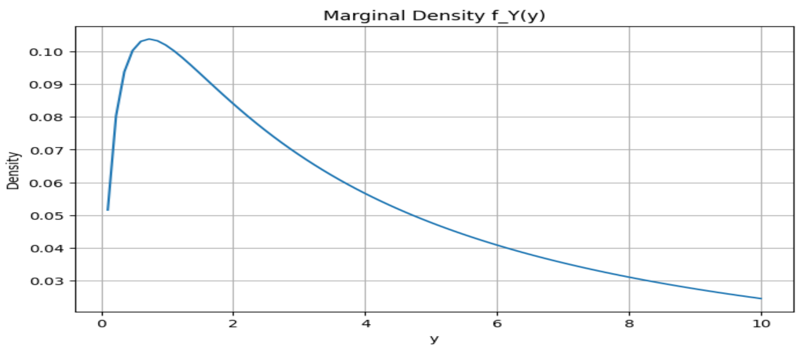

Figure 2: Marginal Distribution of Y Plot

Posterior Density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \cdot \frac{1}{y\sigma(x)\sqrt{2\pi}} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(Iny-\mu(x))^2}{2\sigma^2(x)}\right]}{f_Y(y)} \quad (10)$$

$$f_{X|Y}(x/y) = \frac{\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \cdot \frac{1}{y\sigma(x)\sqrt{2\pi}} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(Iny - \mu(x))^2}{2\sigma^2(x)}\right]}{f_Y(y)}$$

Simplify $(cancel\ 1/y)$ since $f_Y(y)\ also\ contains\ 1/y,\ it\ cancels\ out$:

$$f_{X|Y}(x/y) = \frac{\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \frac{1}{\sigma(x)\sqrt{2\pi}} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(In\, y-\mu(x))^2}{2\sigma^2(x)}\right]}{\int_0^\infty \frac{k}{\lambda}\left(\frac{u}{\lambda}\right)^{k-1} \frac{1}{\sigma(u)\sqrt{2\pi}} \exp\left[-\left(\frac{u}{\lambda}\right)^k - \frac{(In\, y-\mu(u))^2}{2\sigma^2(u)}\right] du} \quad (11)$$

$$f_{X|Y}(x/y) = \frac{f_X(x) f_{Y|X}(y|x)}{\int_0^\infty f_X(u) f_{Y|X}(y|u) du}$$

Since the functions μ(u)and σ(u)are dependent on u, the integral in the denominator of the posterior density cannot often be computed analytically. Therefore, the posterior density f_(X|Y) (x/y)) does not contain a closed-form equation and is determined numerically. the posterior integral equals 1.0 for all parameter settings, confirming that the posterior density is properly normalized and mathematically valid Table 2, The posterior distribution concentrates around a specific range of (X) values given (Y=50), demonstrating the model's ability to capture the dependence between (X) and (Y) Fig 3.

Table 2: Posterior Integral Varying Parameter Settings

a_0	a_1	b_0	b_1	Posterior Integral
0.3	0.001	0.3	0.001	1.0
0.5	0.001	0.5	0.001	1.0
0.7	0.001	0.7	0.001	1.0
1.0	0.001	1.0	0.001	1.0
1.2	0.001	1.2	0.001	1.0

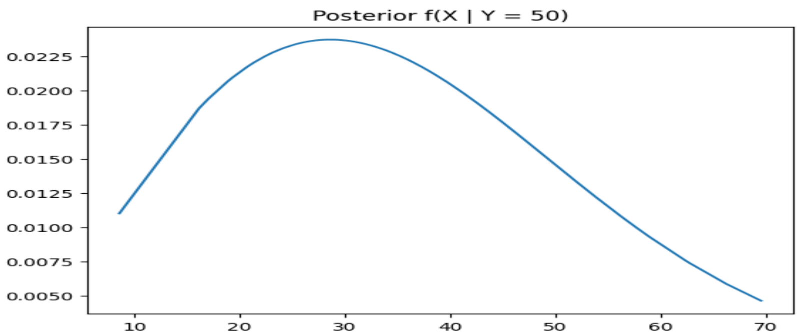

Figure 3: Posterior Density of Y plot

Joint Moments

The joint moment of order (r,s) for the Conditional Weibull-Lognormal distribution is defined as

$$E(X^rY^s) = \int_0^\infty \int_0^\infty x^r\, y^s\, f(x,y)dy\, dx. \qquad (12)$$

Substituting the joint density

$$f(x,y) = \frac{k}{\lambda y\sigma(x)\sqrt{2\pi}}\left(\frac{x}{\lambda}\right)^{k-1} exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(iny-\ \mu(x))^2}{2\sigma^2(x)}\right],$$

Gives

$$E(X^rY^s) = \int_0^\infty x^r\,\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}\left[\int_0^\infty \frac{y^{s-1}}{\sigma(x)\sqrt{2\pi}}\exp\left(-\frac{(Iny-\mu(x))^2}{2\sigma^2(x)}\right)dy\right]dx. \qquad (13)$$

Since $Y|X=x$ follows a lognormal distribution with parameters $\mu(x)$ and $\sigma^2(x)$, its s-th moment is

$$E(Y^s|X=x) = exp\left[s\mu(x)+\frac{s^2}{2}\sigma^2(x)\right]. \qquad (14)$$

Therefore,

$$E(X^rY^s) = \int_0^\infty x^r\,\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} exp\left[-\left(\frac{x}{\lambda}\right)^k + s\mu(x)+\frac{s^2}{2}\sigma^2(x)\right]dx. \qquad (15)$$

For the exponential link functions

$\mu(x) = \exp(a_0+a_1x)$, $\quad \sigma(x) = \exp(b_0+b_1x)$,

The joint moment becomes

$$E(X^rY^s) = \int_0^\infty x^r\,\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} exp\left[-\left(\frac{x}{\lambda}\right)^k + se^{a_0+a_1x} + \frac{s^2}{2}e^{2(b_0+b_1x)}\right]dx. \qquad (16)$$

This expression represents the theoretical joint raw moment of order (r,s) for the proposed conditional weibull-lognormal distribution.

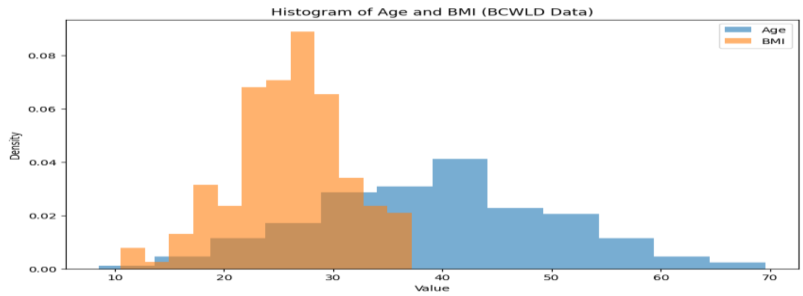

Figure 4: Data Histogram

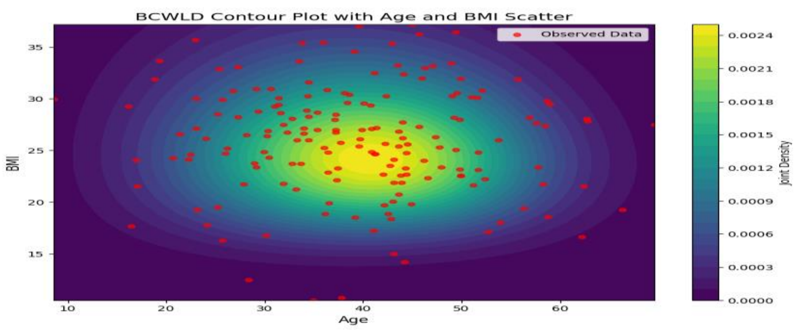

Figure 5: Contour Plot of BCWLD with Scatter of Age and BMI

Method of Parameter Estimation

The method of parameters of the BCWLD can be estimated using the maximum likelihood method of estimation from the joint density

$$f(x,y) = \frac{k}{\lambda y\sigma(x)\sqrt{2\pi}}\left(\frac{x}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{x}{\lambda}\right)^k - \frac{(lny-\ \mu(x))^2}{2\sigma^2(x)}\right]$$

Let the parameter vector be:

$\theta = (k,\lambda,a_0,a_1,b_{0,}b_1)$

Where:

k,λ Are the Weibull shape and scale parameters

a_0,a_1 Are the parameters associated with? $\mu(x)$

$,b_{0,}b_1$ Are the parameters associated with? $\sigma(x)$

Likelihood function

Given a random sample$(x_{i,}y_i)$, $\qquad i = 1,2,\dots,n,$

$$L(\theta) = \prod_{i=1}^{n} f_{X,Y}(x_i,y_i) \quad (17)$$

Substituting joint PDF:

$$L(\theta) = \prod_{i=1}^{n} \frac{k}{\lambda}\left(\frac{x_i}{\lambda}\right)^{k-1} \cdot \frac{1}{y_i\sigma(x_i)\sqrt{2\pi}} \exp\left[-\left(\frac{x_i}{\lambda}\right)^{k} - \frac{(In\ yi-\mu(x_i))^2}{2\sigma^2(x_i)}\right] \quad (18)$$

Taking the Log-likelihood function

Taking logarithms:

$$l(\theta) = \sum_{i=1}^{n} \ln\ f_{X,Y}\ (x_i,y_i)$$

Expanded log-likelihood

$$l(\theta) = \sum_{i=1}^{n} \left[In\ k - \mathrm{k}In\ \lambda + (k-1)\ In\ x_i - Iny_i - In\ \sigma(x_i) - \left(\frac{x_i}{\lambda}\right)^{k} - \frac{(In\ y_i-\mu(x_i))^2}{2\sigma^2(x_i)}\right] \quad (19)$$

Substitution of functional forms

Assume

$\mu(x_i) = \exp(a_0 + a_1x_i), \sigma(x_i) = \exp(b_0 + b_1x_i)$

Then

$In\ \sigma(x_i) = b_0 + b_1x_i$

$\sigma^2(x_i) = \exp(2(b_0 + b_1x_i))$

The maximum likelihood estimators are obtained by solving

$$\frac{\partial l(\theta)}{\partial \theta_j} = 0$$

For each parameter:

$k, \lambda, , a_0, a_1, b_{0,} b_1$

Since the resulting likelihood equations do not admit closed-form solutions, the estimators must be obtained numerically. Therefore: $\hat{O} = arg \max_{\theta} e(\theta)$ Must be obtained numerically. Since the likelihood equations produced from the first-order partial derivatives of the log-likelihood function are very nonlinear, explicit analytical solutions for the parameter estimations are not achievable. Consequently, the maximum likelihood estimates of the parameter vector, θ=(k,λ,a_0,a_1 〖,b〗_(0,) b_1) were calculated numerically using the L-BFGS-B optimization method written in Python. The inverse of the observed Hessian matrix was used to estimate the covariance matrix of the estimators, from which the standard errors and accompanying 95% confidence intervals were obtained.

Table 3: Maximum Likelihood Estimates, Standard Error and confidence interval

Parameters	MLE Estimates	Standard Errors	95% confidence interval
K	3.8668	0.2268	3.4223, 4.3113
λ	43.4825	0.9039	41.7107, 45.2542
a_0	1.1847	0.0192	1.1470, 1.2224
a_1	-0.0002	0.0004	-0.0012,0.0006
b_0	-1.3904	0.2102	-1.8023,-0.9784
b_1	-0.0031	0.0052	-0.0132,0.0071

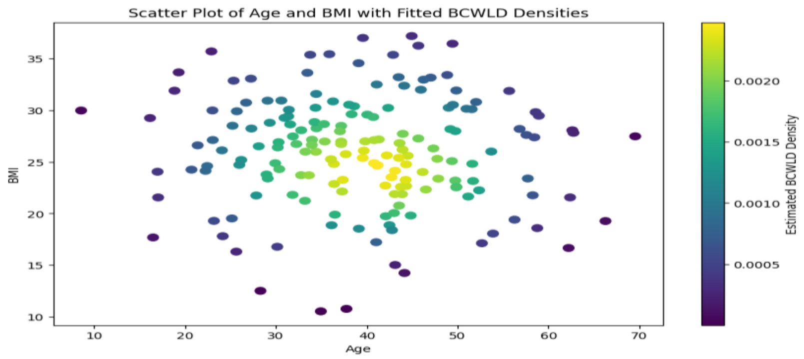

Figure 6: Age and BMI with Estimated Parameters

Model Fit Comparison

To examine the suitability of the Bivariate Conditional Weibull-Lognormal Distribution (BCWLD) model, we compared its performance with a rival model, the Bivariate Conditional Weibull Distribution. The examination applied multiple statistical criteria, notably Log-Likelihood (LLH), Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Each of these metrics gives a complementary perspective: LLH examines overall model likelihood, AIC and BIC punish over-parameterization across all criteria, the BCWLD consistently obtained the most favorable findings, defined by the highest LLH, the lowest AIC and BIC. These findings suggest that the BCWLD finds the optimal balance in goodness-of-fit consequently, the BCWLD appears as the best acceptable model for capturing the dependency structure between the Age and BMI

Table 5. Additionally, the marginal distributions of X and Y were checked using the Cramer-von Mises statistic with bootstrap p-values to establish agreement with the theoretical Weibull and Lognormal distributions Table 7

Table 5: Model Fit and Comparison

Model	Log-likelihood(LLH)	Akaike information criterion (AIC)	Bayesian information criterion (BIC)
Bivariate Conditional Weibull-Lognormal distribution (BCWLD)	-1200.5982	2413.1965	2432.0814
Bivariate Conditional Weibull distribution	-1344.8278	2695.6556	2705.0981

Table 6: Marginal Weibull and Lognormal Distribution Parameter Estimate

Parameter Estimate			
Weibull distribution		**Lognormal Distribution**	
K	3.8667	M	3.2338
λ	43.4823	**σ**	0.2211

Table 7: Cramer Von Mises Test for Marginal Weibull and Lognormal Distribution Result

	Weibull Distribution	**Lognormal Distribution**
CvM Statistic	0.0367	0.3357
Bootstrap P-Value	0.725	0.1080

Hypothesis

H_0: Data fits the distribution (at 5% level)

H_1: Data doesn't fits the distribution (at 5% level)

The test for goodness-of-fit based on CvM statistic reveal a value of 0.0367 for Weibull distribution and 0.3357 for Lognormal distribution with p-value = 0.725 > 0.05, 0.1080 > 0.05 we fail to reject the null hypothesis and conclude that the data fits the weibull and Lognormal distribution at 5% level,

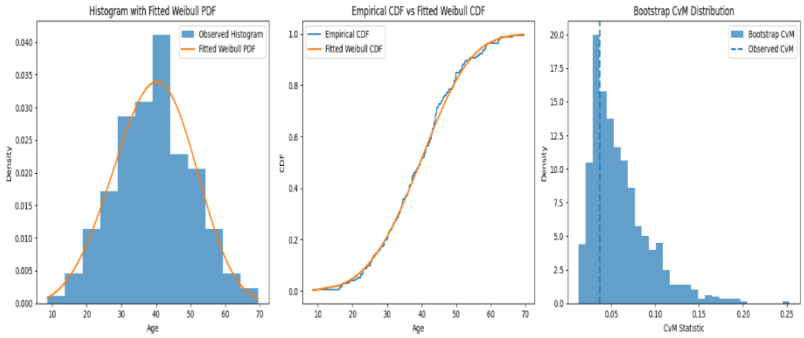

Figure7: A histogram of Age (X) with fitted weibull PDF, Empirical CDF vs fitted weibull CDF, and bootstrap CvM distribution

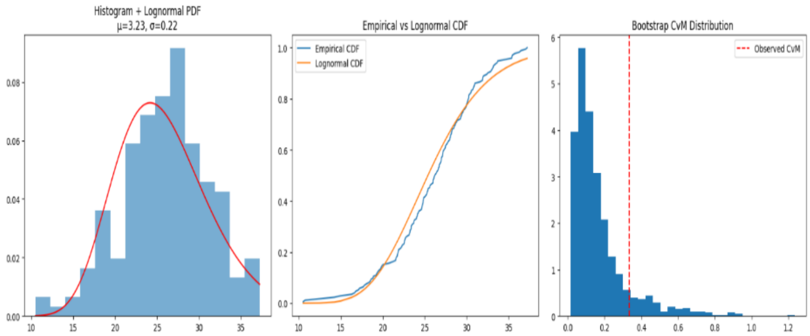

Figure 8: A histogram of BMI (Y) with fitted Lognormal PDF, Empirical CDF vs fitted Lognormal CDF, and bootstrap CvM distribution

RESULTS AND DISCUSSION

To highlight the practical use of the proposed Bivariate Conditional Weibull-Lognormal Distribution (BCWLD), the BCWLD was applied to a real-world dataset consisting of Age and BMI data. The choice was based on the fact that these variables possesses some distributional characteristics so it is plausible to use them for the model: Age is a non-negative continuous variable, usually exhibits similar features as lifespan data, hence is appropriate to be described by Weibull distribution while BMI is a positive continuous variable, typically right-skewed and have multiplicative variability, so it would be appropriate to be described by the Lognormal distribution. The dataset was from Kaggle synthetic health data, and the data includes a column for age and a column for body mass index (bmi).

Discussion

The findings of this study show that the proposed Bivariate Conditional Weibull–Lognormal Distribution (BCWLD) can be used to model two positive continuous variables with different distributional characteristics. The fitted model combines a Weibull marginal distribution for Age with a conditional lognormal distribution for BMI. This supports the conditional modelling approach discussed by Gongsin and Saporu (2020), who demonstrated that useful bivariate distributions can be constructed by combining a marginal distribution with a conditional distribution. However, unlike their bivariate conditional Weibull model, where both components belong to the Weibull family, the present BCWLD allows the two variables to belong to different distributional families. This directly addresses the limitation identified in the literature review concerning bivariate models that retain the same or closely related distributions for both variables.

The estimated Weibull shape and scale parameters were 3.8668 and43.4825, respectively, and these values were very close to the estimates obtained when the Weibull distribution was fitted separately to Age. This provides support for the Weibull component of the BCWLD. The finding is consistent with the usefulness of Weibull-based models reported in previous studies. Almetwally et al. (2020) showed that bivariate Weibull models can adequately represent dependent positive lifetime data, while Pathak et al. (2023) demonstrated the flexibility of generalized Weibull models for dependent lifetime observations. Similarly, Poonia et al. (2024) reported that Weibull-related bivariate models can provide good fits to correlated reliability data. However, these earlier models-maintained Weibull-type distributions for both variables, while the present study combines the Weibull form with a conditional lognormal component.

The estimates of the conditional parameters also provide information about the dependence between Age and BMI. The slope estimates $a_1 = -0.0002$ and $b_1 = -0.0031$ were small, and their 95% confidence intervals included zero. This indicates that changes in Age produced only small changes in the conditional location and dispersion of BMI. The fitted model therefore suggests a weak dependence between the two variables in the dataset rather than a strong relationship. This result is comparable to the finding of Almetwally et al. (2020), who's FGM bivariate Weibull model was found useful for dependent lifetime data with weak correlation. It also supports the argument of Shahbaz et al. (2022) that a useful bivariate distribution should provide a mechanism for examining marginal, conditional and dependence properties rather than considering the two variables separately.

The graphical results provide further support for the estimated dependence structure. The contour and fitted density plots show a smooth concentration of the Age–BMI observations, without a very strong shift in BMI across the range of Age. This agrees with the small values of the estimated conditional slope parameters. The result illustrates one of the major advantages of conditional specification discussed in the literature review: the parameters of the conditional distribution can be allowed to vary with the conditioning variable so that changes in location and dispersion can represent dependence directly. This is consistent with the conditional modelling framework employed by Gongsin and Saporu (2020) and other conditionally specified models reviewed in the study.

The goodness-of-fit result also provides support for the chosen distributional components. For Age, the Cramér–von Mises statistic was 0.0367 with a bootstrap p-value of 0.7250. Since the p-value is greater than0.05, there is no evidence against the Weibull specification. For the separately fitted Lognormal distribution of BMI, the Cramér–von Mises statistic was 0.3357 with a p-value of 0.1080, which is also greater than 0.05. The ability of the lognormal distribution to represent positive and skewed observations agrees with Nadarajah and Lyu (2022), who demonstrated the usefulness of multivariate lognormal distributions for positive and strongly skewed financial and insurance data. It is also consistent with Alamri et al. (2026), who successfully incorporated Lognormal margins into a bivariate dependence model for positive skewed data. Nevertheless, the BMI goodness-of-fit result should be interpreted cautiously because the BCWLD specifies $Y \mid X = x$ as Lognormal rather than requiring the unconditional marginal distribution of Y to be exactly Lognormal.

The most important empirical finding is the better performance of the BCWLD compared with the Bivariate Conditional Weibull Distribution. The proposed model produced a maximised log-likelihood of-1200.5982, compared with -1344.8278 for the competing model. It also recorded lower AIC and BIC values of 2413.1965 and2432.0814, respectively. These results provide strong evidence that allowing BMI to follow a conditional Lognormal distribution gives a better representation of the Age–BMI data than forcing the response variable into another Weibull specification. This finding supports the general results of Poonia et al. (2024) and Pathak et al. (2023), who showed through model comparisons that newly developed bivariate distributions can improve fit when their distributional structures better reflect the characteristics of the data.

The result also extends the direction of previous Lognormal-based bivariate studies. Nadarajah and Lyu (2022) and Alamri et al. (2026) showed that lognormal distributions are useful for modelling dependent positive and skewed variables, but their models remained largely within the Lognormal family. The present BCWLD differs by allowing the conditioning variable to follow a Weibull distribution while the response follows a conditional Lognormal distribution. Therefore, its better AIC and BIC values provide empirical support for the main argument of this study that combining different distributional families can be useful when two related positive variables have different characteristics.

CONCLUSION

A flexible model,BCWLD, for modeling dependence of positive continuous variables has been presented herein. The theoretical validation of the model was provided by its valid joint probability density function and well-behavior of its marginal and posterior distributions, while the model application provided effective depiction of the dependence of variables. Comparisons with BCW have clearly shown that the BCWLD offer a better explanation of dependencies among data. The inclusion of a Lognormal in the conditional aspect enables a more adequate representation of a positive skewed response, in this case BMI. BCWLD offer an interesting contribution to the family of Conditional Bivariate distributions and can be useful in fields such as age, health, reliability, survival and so on.

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