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Development, Theoretical Properties and Simulation Assessment of the Exponential Weibull Log-Logistic Poisson Distribution and Its Regression Model

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ABSTRACT

In an effort to address for the family of distribution that permits flexibility in simulating real-world phenomena, a new family of univariate probability distribution called exponential Weibull log-logistic Poisson family of probability distribution is introduced in this paper by compounding the T- R [Y] family of distribution. The new distribution has the advantage of being capable of modeling various shapes of ageing and failure criteria. We derive several of its structural properties including moment, survival function, hazard function and order statistics. The new density function can be expressed as a linear mixture of exponentiated Weibull densities. We proposed a linear regression model using a new distribution–the exponential Weibull log-logistic Poisson distribution. The maximum-likelihood method is used to estimate the model parameters and simulation results were provided to assess the performance of the proposed maximum-likelihood procedure. The results of this study offer a strong basis for real-world applications.

Keywords: T- R [Y] Family Of Distribution, Weibull Distribution, Mathematical Properties, Regression Model, Maximum-Likelihood

INTRODUCTION

In statistical modeling and data analysis, probability distributions are essential tools for representing and interpreting real-world occurrences. Probability distribution functions generally are used in describing the real-world phenomenal and are classified in to discrete and continuous distribution where the discrete and continuous distribution, where the discrete is called probability mass function (pmf) and the continuous distribution is known as probability density function (pdf).

Various methods of generating continuous and discrete univariate distributions have been put forward. These methods include the method based on differential equations (Pearson (1895) and Bur (1942)), method based on transformation (Johnson (1949), method based on quantiles (Turkey (1960), Aldeni et al, (2017), (Azzadini (1895)), method of addition of parameter(s) and generalization (Mudholkar and Sristpawa (1992), Marshal and Oikin (1997), Shaw and Buckey (2009)), and for the discrete univariate distribution (Adamidis and Loukas (1998)), method based on generators (Eugene et al (2002), Jones (2009), Cordeiro and de Castro (2011)), Method based on the composition of densities (Corray and Anada (2005)) and the Tramsformed-Transformer method (Alzatreh et al (2013), Alzatreh et al (2014)).

The Weibull distribution is a very popular model in statistical modeling and has been extensively used over the past decades for modeling data in engineering, reliability and biological studies. The need for extended forms of the Weibull model arises in many applied areas. Some of the classical extensions of the Weibull model with applications were studied (Murthy et al (2004); Nadrajah and Kortz (2006)).

There has been an increased interest among statisticians to develop new methods for generating new families of distributions, because there is a persistent need for extending the classical forms of the well- known distributions to be more capable for modeling data in different areas and fields such as life-time analysis in engineering, economics, finance, demography, actuarial, biological and medical sciences.

Many generalizations of the Weibull distribution model have been proposed and studied to cope with bathtub-shaped failure rates. Among these models were Additive Weibull (Xie and Lai, 1995), exponentiated Weibull (EW), Mudheikar et al, 1995, 1996), Extended Weibull (Xie et al, 2002), modified Weibull (Lai et al 2003), beta modified Weibull (Silva et al (2010), Kumaraswany Weibull (Cordeiro et al 2010), transmuted Weibull (Aryal and Tsokos 2011; Cooray 2006, Cordeiro et al, 2014b, 2016), Complementary Weibull geometric (Tojeiro etb al , 2014), transmuted complementary Weibull geometric (Afify et al, 2014), log-negative-binomial (Louzada et al, 2005), Marshall and Olkin Additive Weibull (Afify et al, 2016b). Kumaraswany transmuted exponentiated Weibull (Nofal et al, 2016), transmuted exponentiated generalized Weibull (Yousof et al, 2015), Kumaraswany exponential Weibull (Cordeiro et al, 2016), Kumaraswany Complemetary Weibull geometric (Afify et al, 2017), and generalized odd log-logistic Weibull distributions (Cordeiro et al, 2017), generalized logistic distribution(Aljarrah et al 2020), logistic-weibull distribution (Chadhary and Kumar, 2021), extended exponential Weibull distribution (Mastor et al, 2022), exponentiated generalized Weibull exponential distribution (Abonongo and Abonongo, 2023), Three-parameter exponentiated Weibull exponential distribution (Ferreira and Ferreira, 2025).

The aim of this paper is to define and study a new lifetime model the exponential Weibull log-logistic Poisson distribution (EWLLP) with a proposed regression model, using the T-R[Y] family of distributions, we construct the five-parameter EWLLP model and give a comprehensive description of some of its mathematical properties.

The proposed distribution contains several lifetime distributions, such as EW (Mudheikar et al, 1995; 1996), Weibull (1951), OLL-Weibull (Cocray, 2006), exponentiated exponential (Gupta and Kundu, (2001)), exponential and Rayleigh distributions among others as a special case. We are motivated to introduce the EWLLP distribution because (i) it contains a number of aforementioned of known lifetime sub-models; (ii) the EWLLP distribution exhibits monotone as well as non-monotone hazard rates which makes this distribution to be superior to another lifetime distributions which exhibit only monotonically increasing/decreasing or constant hazard rates (iii) the EWLLP distribution can be viewed as a mixture of exponentiated Weibull distribution introduced by Mudheikar et al, (1995). (iv) It can also be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions and (v) the EWLLP distribution outperforms several competitive distributions.

Formulation of the Exponential Weibull Log-Logistic Poisson Distribution (EWLLP)

The probability density function (pdf) and cumulative distribution function (cdf) of the EW distribution are given by the following:

Suppose T follows exponential distribution with cdf, pdf and quantile function given as;

$F_T(x) = 1 - e^{-\lambda x}$

$f_T(x) = \lambda e^{-\lambda x}$

$Q_T(x) = \frac{-log(1-p)}{\lambda}$

Let Y be a random variable of log-logistics distribution with cdf, pdf and quantile function express as;

$F_Y(x) = 1 - (1+x)^{-1}$

$f_y(x) = (1+x)^{-2}$

$Q_y(p) = \frac{p}{1-p}$

Suppose T, R and Y are random variables with respective cumulative distribution function (cdf)

$F_T(x) = P(T \le x), F_{R\,(x)} = P(R \le x)$ and $F_Y(x) = P(Y \le x)$

Suppose the corresponding densities of T, R and Y exist and denote them by$f_{T\,(x)}, f_{R\,(x)}$ and $f_y(x)$.

Assume that $T \in (a,b)$ *and* $Y \in$ (c, d) for $-\infty \le a < b \le \infty$ *and* $-\infty \le c < d \le \infty$, then the T-R(Y) family of distribution was defined by the cdf:

$$F_x\,(x) = \int_a^{Q_y(F_R\,(x))} f_T(t)dt = F_T\,(Q_y(F_R(x)))\ , x \in R \tag{1}$$

$$F_x\,(x) = F_T\left(\frac{F_R(x)}{1-F_{R(x)}}\right) = 1 - e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

$$F_x\,(x) = 1 - e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)} e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

The corresponding probability density function (pdf) of the cdf in (1.1) was given by:

$$f_x(x) = f_R(x) \times \frac{f_T(Q_y(F_R(x)))}{f_y(Q_y(F_R(x)))}\ , x \in R \tag{2}$$

$$= f_R(x) \times \frac{\lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}}{1 + \left(\frac{F_R(x)}{1+F_{R(x)}}\right)^{-2}}$$

Expansions for the Densities of Exponential Weibull Log-Logistic Poisson (EWLLP) Distribution

T-R(Y) – P family is one of the sub-families of the T-R(Y)-PS family of distribution and the cumulative distribution function is stated below

$$F_{T-R(y)-p} = 1 - \frac{e^{\theta}(1 - F_T(|Q_y(F_R(x))))_{-1}}{e^{\theta}-1}, x \in R \tag{3}$$

But

$$F_T\,(Q_y(F_R(x))) = 1 - e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

$$1 - F_T\,(Q_y(F_R(x))) = 1 - 1 + e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

$$1 - F_T\,(Q_y(F_R(x))) = e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

$$F_{T-R(Y)-p} = 1 - \frac{e^{\theta}(1 - F_T(|Q_y(F_R(x))))_{-1}}{e^{\theta}-1}$$

$$F_{T-R(y)-p} = 1 - \frac{e^{\theta e^{-\lambda\left[\frac{F_R(x)}{1-F_R(x)}\right]}} - 1}{e^{\theta} - 1}$$

The corresponding probability density function (pdf) is given by:

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}-1}\ \text{x}\ \frac{f_T(Q_y(F_R(x)))}{f_y(Q_y(F_R(x)))} e^{\theta}(1 - F_T(Q_y(F_R(x)))) \tag{4}$$

$$f_T\left(Q_y(F_{(x)})\right) = \lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}$$

$$f_y\left(Q_y(F_R(x))\right) = 1 + \left(\frac{F_R(x)}{1-F_R(x)}\right)^{-2}$$

$$1 + \left(\frac{F_R(x)}{1-F_R(x)}\right)^{-2} = \left(\frac{1-F_R(X)+F_R(x)}{1-F_R(x)}\right)^{-2}$$

$$= \left(\frac{1}{1-F_R(x)}\right)^{-2} = (1 - F_R(x))^2$$

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}-1}\ \text{x}\ \frac{f_T(Q_y(F_R(x)))}{f_y(Q_y(F_R(x)))} e^{\theta}(1 - F_T(Q_y(F_R(x))))$$

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}-1}\ \text{x}\ \frac{\lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}}{1 + \left(\frac{F_R(x)}{1+F_{R(x)}}\right)^{-2}} e^{\theta e^{-\lambda\left[\frac{F_R(x)}{1-F_R(x)}\right]}}$$

$$f_{T-R(Y)-P} = \frac{\theta f_{R(x)}}{e^{\theta}-1}\ \text{x}\ \frac{\lambda e^{-\lambda\left(\frac{F_R(x)}{1-F_{R(x)}}\right)}}{(1-F_R(x))^2} e^{\theta e^{-\lambda\left[\frac{F_R(x)}{1-F_R(x)}\right]}} \tag{5}$$

$$F_{T-R(y)p}(x) = 1 - \frac{\exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right] - 1}{(e^{\theta}-1)}\theta, x > 0 \quad (6)$$

$$f_{T-R(y)p}(x) = \frac{\theta f_R(x)}{e^{\theta}-1} \times \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} \exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right]}{[1-F_R(x)]^2} \quad (7)$$

Let R be a Weibull Distribution with the cumulative distribution function (CDF) given below as:
F(x) = $1 - e^{-(x/\alpha)^{\beta}}$, $x \geq 0$with scale parameter $\alpha > 0$and shape parameter $\beta > 0$ .The probability density function (PDF) is given as:

$$f(x, \alpha, \beta) = \frac{\beta}{\alpha}(x/\alpha)^{\beta-1} e^{-(x/\alpha)^{\beta}}$$

But,

F(x) =$F_R(x) = 1 - e^{-(x/\alpha)^{\beta}}$, $x \geq 0$

Now, to find $1 - F_R(x)$

$1 - F_R(x) = 1 - (1 - e^{-(x/\alpha)^{\beta}})$,

$1 - F_R(x) = e^{-(x/\alpha)^{\beta}}$

Compute the Ratio,

$$\frac{F_R(x)}{1 - F_R(x)} = \frac{1 - e^{-(x/\alpha)^{\beta}}}{e^{-(x/\alpha)^{\beta}}}$$

Simplifying the expression,

$$\frac{F_R(x)}{1 - F_R(x)} = e^{(x/\alpha)^{\beta}} - 1$$

Substitute the expression into equation (3.6)

$$F_{T-R(y)p}(x) = 1 - \frac{\exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right] - 1}{(e^{\theta}-1)}\theta, x > 0$$

We have:

$$F_{EWLLP}(x) = 1 - \frac{\exp\left[\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}\right] - 1}{(e^{\theta}-1)}$$

Simplification of the PDF

$f_R(x) = \frac{\beta}{\alpha}(x/\alpha)^{\beta-1} e^{-(x/\alpha)^{\beta}}$, $x \geq 0$

$$f_{T-R(y)p}(x) = \frac{\theta f_R(x)}{e^{\theta}-1} \times \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} \exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right]}{[1-F_R(x)]^2}$$

Where $\theta, \lambda, x > 0$

$F_R(x) = 1 - e^{-(x/\alpha)^{\beta}}$, $x \geq 0$

$1 - F_R(x) = e^{-(x/\alpha)^{\beta}}$

$\frac{F_R(x)}{1 - F_R(x)} = e^{(x/\alpha)^{\beta}} - 1$

$e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} = e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}$

$[1 - F_R(x)]^2 = (e^{-(x/\alpha)^{\beta}})^2 = (e^{-2(x/\alpha)^{\beta}}$

Substitute into equation (3.7)

$$f_{T-R(y)p}(x) = \frac{\theta f_R(x)}{e^{\theta}-1} \times \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)} \exp\left[\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}\right]}{[1-F_R(x)]^2}$$

$$= \frac{\theta\frac{\beta}{\alpha}(x/\alpha)^{\beta-1} e^{-(x/\alpha)^{\beta}}}{e^{\theta}-1} \times \frac{\lambda\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)} e^{\theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}}}{(e^{-2(x/\alpha)^{\beta}})}$$

$$f_{EWLLP}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta}-1)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^{\beta}} \times e^{\left[-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right) + \theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}\right]}$$

Properties and Extensions of Exponential Weibull Log-Logistic Poisson Distribution with Regression Analysis
Moment of Exponential Weibull Log-Logistic Poisson (EWLLP) Distribution

Recall the pdf

$$f_{EWLLP}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta}-1)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^{\beta}} \times e^{\left[-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right) + \theta e^{-\lambda\left(e^{(x/\alpha)^{\beta}}-1\right)}\right]}$$

Let U = $(x/\alpha)^{\beta}$,

Using the binomial expansion and Taylor's Series$(x + a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$ and $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, -\infty < x < \infty$

$$f_{T-R(y)p}(x) = \frac{\theta\,\beta\lambda}{\alpha}(x/\alpha)^{\beta-1} \sum_{t=0}^{\infty} \sum_{i}^{\infty} \sum_{j}^{\infty} \sum_{k}^{\infty} \sum_{r}^{\infty} \left(\frac{\lambda}{i!}\right)^{i} (-1)^{n-j} \binom{n}{j} \frac{\theta^{k}}{n!} \frac{(k\lambda)^{n}}{n!} e^{-[t\theta - i - j - n](x/\alpha)^{\beta}}$$

Let $(Z_i) = \sum_t^\infty \sum_i^\infty \sum_j^\infty \sum_k^\infty \sum_r^\infty \left(\frac{\lambda}{i!}\right)^i (-1)^{n-j} \binom{n}{j} \frac{\theta^k}{n!} \frac{(k\lambda)^n}{n!}$

$f_{T-R(y)p}(x) = \frac{\theta\,\beta\lambda}{\alpha^\beta}(x)^{\beta-1} Z_i e^{-[t\theta-i-j-n\,](x/\alpha)^\beta}$ To obtain the r^th moment of the distribution

$\mu^r = E[x^r] = \int_0^\infty x^r f(x)dx$

Where f(x) is pdf of EWLLP distribution in equation

E $(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \int_{-\infty}^{\infty} x^r f_{T-R(y)-p}(x)dx$

$= \int_{-\infty}^{\infty} x^r \frac{\theta\,\beta\lambda}{\alpha^\beta} x^{\beta-1} Z_i e^{-[t\theta-i-j-n\,](x/\alpha)^\beta} dx$

$= \frac{\theta\,\beta\lambda}{\alpha^\beta} Z_i \int_{-\infty}^{\infty} x^r.\ x^{\beta-1} e^{-[t\theta-i-j-n\,](x/\alpha)^\beta} dx$

$= \frac{\theta\,\beta\lambda\,Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} x^{r+\beta-1}.\ e^{-[t\theta-i-j-n\,](x/\alpha)^\beta} dx$

Let U = $[t\theta - i - j - n\,](x/\alpha)^\beta$

E $(x^r) = \frac{\theta\,\beta\lambda\,Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} x^{r+\beta-1}.\ e^{-U} dx$

U = $[t\theta - i - j - n\,](x/\alpha)^\beta$

$\left[\frac{U}{[t\theta-i-j-n\,]}\right]^{\frac{1}{\beta}} = (x/\alpha)$

X = $\frac{\alpha U^{\frac{1}{\beta}}}{\beta(t\theta-i-j-n)^{1/\beta}}$

$\frac{dx}{du} = \frac{\alpha U^{1/\beta-1}}{\beta(t\theta-i-j-n)^{1/\beta}}$

Dx = $\frac{\alpha U^{1/\beta-1}}{(t\theta-i-j-n)^{1/\beta}}$du

E $(x^r) = \frac{\theta\,\beta\lambda\,Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} x^{r+\beta-1}.\ e^{-U} dx$

$= \frac{\theta\,\beta\lambda\,Z_i}{\alpha^\beta} \int_{-\infty}^{\infty} \left(\frac{\alpha U^{\frac{1}{\beta}}}{(t\theta-i-j-n)^{1/\beta}}\right)^{r+\beta-1} .\ e^{-U}.\frac{\alpha U^{1/\beta-1}}{\beta([t\theta-i-j-n\,])^{1/\beta}}$du

$= \frac{\theta\,\beta\lambda\,Z_i}{\alpha^\beta} . \frac{\alpha^{r+\beta-1}}{\left[(t\theta-i-j-n)^{1/\beta}\right]^{r+\beta-1}} . \frac{\alpha}{\beta([t\theta-i-j-n\,])^{1/\beta}} \int_{-\infty}^{\infty} U^{\frac{1}{\beta}(r+\beta-1)}.U^{1/\beta-1}.e^{-U}$Du

But, $U^{\frac{r}{\beta}} e^{-U}$du = $\Gamma r/\beta + 1$

E $(x^r) = \frac{\theta\,\lambda\alpha^r\,Z_i \Gamma r/\beta+1}{(t\theta-i-j-n)^{\frac{r}{\beta}+1}}$

$\mu = E(x^1) = \frac{\theta\,\lambda\alpha^1 Z_i \Gamma 1/\beta+1}{(t\theta-i-j-n)^{\frac{1}{\beta}+1}}$

$\mu_2' = E\,(x^2) = \frac{\theta\,\lambda\alpha^2\,Z_i \Gamma 2/\beta+1}{(t\theta-i-j-n)^{\frac{2}{\beta}+1}}$

$\mu_3' = E(x^3) = \frac{\theta\,\lambda\alpha^3 Z_i \Gamma 3/\beta+1}{(t\theta-i-j-n)^{\frac{3}{\beta}+1}}$

$\mu_4' = E(x^4) = \frac{\theta\,\lambda\alpha^4 Z_i \Gamma 4/\beta+1}{(t\theta-i-j-n)^{\frac{4}{\beta}+1}}$

Variance = E $(x^2) - \left(E\,(x)\right)^2$

$= \frac{\theta\,\lambda\alpha^2\,Z_i \Gamma 2/\beta+1}{(t\theta-i-j-n)^{\frac{2}{\beta}+1}} - \left(\frac{\theta\,\lambda\alpha^1 Z_i \Gamma 1/\beta+1}{(t\theta-i-j-n)^{\frac{1}{\beta}+1}}\right)^2$

The Survival Function

S(x) = 1 $- F(x\,)$

$F_{EW(LL)P}(x) = 1 - \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]-1}}{e^{\theta}-1}$

$f_{E-W(LL)P}(x) = \frac{\theta\beta\lambda}{\alpha(e^\theta-1)}(x/\alpha)^{\beta-1} e^{(x/\alpha)^\beta} \times\ e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)+\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}$

S(x) = 1 $- F(x\,)$

$= 1 - \left\{1 - \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]-1}}{e^{\theta}-1}\right\}$

$= 1 - 1\ + \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]-1}}{e^{\theta}-1}$

$$S(x) = \frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]-1}}{e^{\theta-1}}$$

The Hazard Function

$$H(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} =$$

$$= \frac{\frac{\theta\beta\lambda}{\alpha(e^{\theta}-1)}(x/\alpha)^{\beta-1}e^{(x/\alpha)^\beta} \times e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)+\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}}{\frac{e^{\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]-1}}{e^{\theta-1}}}$$

$$H(x) = \frac{\theta\beta\lambda}{\alpha}(x/\alpha)^{\beta-1}e^{(x/\alpha)^\beta} \times e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)\right]}$$

Order Statistics

Let $X_1, X_2 \ldots\ldots\ldots\ldots\ldots X_n$ be a random sample of size n from the distribution and Let $X_{(1)}, X_{(2)} \ldots\ldots\ldots\ldots\ldots X_{(n)}$ denote the corresponding order.

Let F(x) be the cumulative distribution function (CDF) corresponding to f(x)

Then, the density function of the order statistics$X_{(k)}$ is given as:

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}.(F_X(x))^{K-1}.(1-F_X(x))^{n-k}.f_x(x)$$

$$= \frac{\theta}{e^{\theta}-1}.\frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}e^{\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}}, f_{R(x)}}{\left(1-F_{R(x)}\right)^2}$$

$$f_{T-R(y)-p}(x) = \int_0^x f(t)dt$$

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}.(F_X(x))^{K-1}.(1-F_X(x))^{n-k}.f_x(x)$$

$$= \frac{n!}{(k-1)!(n-k)!}.\left(\int_0^x f(t)_x dt\right)^{K-1}.\left(1-\int_0^x f(t)_x dt\right)^{n-k}.\frac{\theta}{e^{\theta}-1}.\quad \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}e^{\theta e^{-\lambda\left(\frac{F_R(x)}{1-F_R(x)}\right)}}, f_{R(x)}}{\left(1-F_{R(x)}\right)^2}$$

$$= \frac{n!}{(k-1)!(n-k)!}.\left(\int_0^x \frac{\theta}{e^{\theta}-1}.\frac{\lambda\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}e^{\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}}, F_{R(x)}}{\left(1-F_{R(t)}\right)^2}f_R(t)dt\right)^{K-1}.$$

$$\left(1-\int_0^x \frac{\theta}{e^{\theta}-1}\frac{\lambda\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}e^{\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}}, F_{R(x)}}{\left(1-F_{R(t)}\right)^2}f_R(t)dt\right)^{n-k}.\frac{\theta}{e^{\theta}-1}.\quad \frac{\lambda\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(tx)}\right)}e^{\theta e^{-\lambda\left(\frac{F_R(t)}{1-F_R(t)}\right)}}, f_{R(x)}}{\left(1-F_{R(t)}\right)^2}$$

The EWLLP Regression Model

The EWLLP regression which is based on the framework of T-R(Y) Poisson family which is used to develop a new classical distribution known as Exponential Weibull Log-Logistic Poisson (EWLLP) distribution to allow greater flexibility in shape and tail behavior. The baseline pdf and cdf.

$$f_{T-R(y)p}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta}-1)}(x/\alpha)^{\beta-1}e^{(x/\alpha)^\beta} \times e^{\left[-\lambda\left(e^{(x/\alpha)^\beta}-1\right)+\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]}$$

$$F_{T-R(y)p}(x) = 1 - \frac{exp\left[\theta e^{-\lambda\left(e^{(x/\alpha)^\beta}-1\right)}\right]-1}{(e^{\theta}-1)}$$

Let $f_{R(x)=\frac{d}{dx}F(R)}$ and the odds $u(x) = \frac{F_R(x)}{1-F_R(x)}$

Log-Likelihood

For iid,$x_1 \ldots\ldots\ldots\ldots. x_n$, the log-likelihood is

$L(\alpha,\beta,\theta,\lambda) = \sum_{i=1}^{n} log f_{EWLLP}(x_i)$

$log f_{EWLLP}(x_i) = \log\theta + \log\beta + \log\lambda - \log\left(e^{\theta}-1\right) - \beta\log\alpha + (\beta-1)\log x + \left(\frac{\alpha}{\beta}\right)^{\beta} - \lambda\mu(x) + \theta(e)^{-\mu(x)}$

$L(\alpha,\beta,\theta,\lambda) = \sum_{i=1}^{n}\left[\log\theta + \log\beta + \log\lambda - \log\left(e^{\theta}-1\right) - \beta\log\alpha + (\beta-1)\log x + \left(\frac{\alpha}{\beta}\right)^{\beta} - \lambda\mu(x) + \theta(e)^{-\mu(x)}\right]$

Link the scales to Covariates

Let $Z_i \in R^P$be covariates for i^{th} observation. The scale parameter is modeled using a log-linear link function defined as:
$\log \alpha_i = Z_i^T = \exp(Z_i^T \beta)$
$S_i = \left(\frac{x_i}{\alpha}\right)^{\beta} = x_i exp\left(-\beta Z_i^T \beta\right)$, $U_{i=} \mathrm{e}^{S_i} - 1$
$\alpha_I = \exp(Z_i^T \beta)$
$\ell_i = \log\theta + \log\beta + \log\lambda - \log\left(\mathrm{e}^{\theta} - 1\right) - \beta Z_i^T \beta + (\beta - 1)\log x_i + S_i - \lambda U_i + \theta(\mathrm{e})^{-\lambda U_i}$ Thus,
$\mathcal{L}(\boldsymbol{\beta}, \beta, \theta, \lambda) = \sum_{1 l_i}^{n}\left[\log\theta + \log\beta + \log\lambda - \log\left(\mathrm{e}^{\theta} - 1\right) - \beta\log\alpha + (\beta - 1)\log x + \left(\frac{\alpha}{\beta}\right)^{\beta} - \lambda U(x_i) + \theta \mathrm{e}^{-\lambda U(x_i)}\right]$
$\mathcal{L}(\boldsymbol{\beta}, \beta, \theta, \lambda) = \sum_{i=0}^{n} \ell_i$
Taking the derivatives of the log-likelihood function with respect to $\boldsymbol{\beta}, \beta, \theta, \lambda$ and equating the derivatives to zero
$\frac{\partial \ell_i}{\partial \boldsymbol{\beta}} = \beta Z_i\left[-1 - S_i + \lambda S_i \mathrm{e}^{S_i}\left(1 + \theta \mathrm{e}^{-\lambda U_i}\right)\right]$
Use $S_i = \exp(\beta \log \frac{x_i}{\alpha_i})$
$\frac{\partial \ell_i}{\partial \beta} = \frac{1}{\beta} + (\log x_i - \log\alpha_i) + S_i \log\frac{x_i}{\alpha_i}\left[1 - \lambda \mathrm{e}^{S_i}\left(1 + \theta \mathrm{e}^{-\lambda U_i}\right)\right]$
$\frac{\partial \ell_i}{\partial \theta} = \frac{1}{\theta} - \frac{\mathrm{e}^{\theta}}{(\mathrm{e}^{\theta} - 1)} + \mathrm{e}^{-\lambda U_i}$
$\frac{\partial \ell_i}{\partial \boldsymbol{\lambda}} = \frac{1}{\lambda} - U\left(1 + \theta \mathrm{e}^{-\lambda U_i}\right)_i$

Method of Maximum-Likelihood Estimation

SupposeX_1. X_n Be a random sample from the EWLLP distribution with an unknown parameter (α, β, θ, λ and θ). The pdf of EGEL is given as:

$f_{E-W(LL)P}(x) = \frac{\theta\beta\lambda}{\alpha(e^{\theta} - 1)} (x/\alpha)^{\beta - 1} \mathrm{e}^{(x/\alpha)^{\beta}} \times \ \mathrm{e}^{\left[-\lambda\left(\mathrm{e}^{(x/\alpha)^{\beta}} - 1\right) + \theta \mathrm{e}^{-\lambda\left(\mathrm{e}^{(x/\alpha)^{\beta}} - 1\right)}\right]}$
The likelihood function is given as:
$\mathcal{L}(\alpha, \beta, \theta, \lambda) = \prod_{i=1}^{n} f(x_i, \alpha, \beta, \theta, \lambda)$
Therefore, the likelihood function is:
$\mathcal{L}(\alpha, \beta, \theta, \lambda) = \sum_{i=0}^{n} f(x_i, \alpha, \beta, \theta, \lambda)$
$\ell = \sum_{i=0}^{n}\left[\ln\theta + \ln\beta + \ln\lambda - \ln\alpha - \ln\left(\mathrm{e}^{\theta} - 1\right) + (\beta - 1)\ln\frac{x_i}{\alpha} - \left(\frac{x_i}{\alpha}\right)^{\beta} - \lambda\left(\mathrm{e}^{\left(\frac{x_i}{\alpha}\right)^{\beta}} - 1\right) + \theta \mathrm{e}^{-\lambda\left(\mathrm{e}^{(x/\alpha)^{\beta}} - 1\right)}\right]$
Solve for $\alpha, \beta, \theta, \lambda$
$\frac{\partial \ell}{\partial \alpha} = 0$, $\frac{\partial \ell}{\partial \beta} = 0$, $\frac{\partial \ell}{\partial \theta} = 0$, $\frac{\partial \ell}{\partial \lambda} = 0$,
Let$U_i = (x/\alpha)^{\beta}$ $t_i = \mathrm{e}^{U_i} - 1$; $S_i = \mathrm{e}^{-\lambda t_i}$
Derive wrt α Recall $U_i = (x/\alpha)^{\beta} = \mathrm{e}^{\beta \ln\left(\frac{x_i}{\alpha}\right)} = \mathrm{e}^{\beta(\ln x_{i - \ln\alpha})}$
$\frac{\partial \ell}{\partial \alpha} = \beta\left(\frac{x_i}{\alpha}\right)^{\beta}\left(-\frac{1}{\alpha}\right) = -\frac{-\beta}{\alpha}\left(\frac{x_i}{\alpha}\right)^{\beta} = \frac{-\beta U_i}{\alpha}$
$\frac{\partial \ell}{\partial \alpha} = \sum_{i=0}^{n}\left[-\frac{1}{\alpha} + (\beta - 1)\left(-\frac{1}{\alpha}\right) - \frac{\partial U_i}{\partial \alpha} - \frac{\lambda \partial t_i}{\partial \alpha} + \frac{\theta \partial S_i}{\partial \alpha}\right]$
But,
$\frac{\partial t_i}{\partial \alpha} = \frac{\partial(\mathrm{e}^{U_i - 1})}{\partial \alpha} = \mathrm{e}^{U_i} \frac{-\beta U_i}{\alpha}$
$\frac{\partial S_i}{\partial \alpha} = \frac{\partial \mathrm{e}^{-\lambda t_i}}{\partial \alpha} = \frac{\lambda \beta U_i}{\alpha} \mathrm{e}^{U_i} S_i$
$\frac{\partial \ell}{\partial \alpha} = \sum_{i=0}^{n}\left[-\frac{1}{\alpha} - \frac{(\beta - 1)}{\alpha} - \frac{\beta U_i}{\alpha} + \frac{\lambda \beta U_i \mathrm{e}^{U_i}}{\alpha} - \frac{\theta \lambda \beta U_{i} \mathrm{e}^{U_i} S_i}{\alpha}\right]$
Derive wrt β
$\frac{\partial \ell}{\partial \beta} = \sum_{i=0}^{n}\left[\frac{1}{\beta} + \ln\frac{x_i}{\alpha} - \frac{\partial U_i}{\partial \beta} - \frac{\lambda \partial(\mathrm{e}^{U_i - 1})}{\partial \beta} + \frac{\theta \partial \mathrm{e}^{-\lambda t_i}}{\partial \beta}\right]$

$\frac{\partial \ell}{\partial \beta} = \sum_{i=0}^{n}\left[\frac{1}{\beta} + \ln\frac{x_i}{\alpha} - U_i \ln\frac{x_i}{\alpha} - \lambda \mathrm{e}^{U_i} U_i \ln\frac{x_i}{\alpha} + \theta \lambda \mathrm{e}^{U_i} S U_i \ln\frac{x_i}{\alpha_i}\right]$

Derive wrt θ
$\frac{\partial \ell}{\partial \theta} = \sum_{i=0}^{n}\left[\frac{1}{\theta} - \frac{\mathrm{e}^{\theta}}{(\mathrm{e}^{\theta} - 1)} + S_i\right]$

$\frac{\partial \ell}{\partial \theta} = \sum_{i=0}^{n}\left[\frac{1}{\theta} - \frac{\mathrm{e}^{\theta}}{(\mathrm{e}^{\theta} - 1)} + \mathrm{e}^{-\lambda t_i}\right]$

Derive wrt λ
$\frac{\partial \ell}{\partial \lambda} = \sum_{i=0}^{n}\left[\frac{1}{\lambda} - (\mathrm{e}^{U_i} - 1) + \frac{\theta \partial S_i}{\partial \lambda}\right]$

$\frac{\partial \ell}{\partial \lambda} = \sum_{i=0}^{n}\left[\frac{1}{\lambda} - (\mathrm{e}^{U_i} - 1) + \theta\left(\mathrm{e}^{\theta} - 1\right) S_i\right]$

The parameters are obtained by setting the equations with respect to each parameter to zero that is differentiating l with respect to $\alpha, \beta, k\ \theta,$ and λ respectively.

Graphical Representation of EWLLP Distribution

The graphical representation of the plots of the pdf, cdf and some other statistical properties were represented. The plots are obtained for different values of the parameters of the distribution. The graphs of the pdf and cdf of Exponential Weibull Log-Logistic Poisson (EWLLP) distribution for different values of $\alpha, \beta, \theta, \lambda,$ and k are given in Figure 1 and Figure 2 respectively. The Figures shows that the density function can take different shapes for different values of these parameters such as asymmetrical for some parameter values and it can be inferred that the pdf of EWLLP distribution is heavily tailed and rightly skewed. It would be observed that for the different values of $\alpha, \beta, \theta, \lambda,$ and k , the cdf plot approaches 1 when x becomes large. Also, it can be deduced that the distribution can be used to model data that are unimodal, heavily tailed and rightly skewed.

Figure 1 Figures shows that the density function can take different shapes for different values of these parameters such as asymmetrical for some parameter values and right-skewed or positively skewed shapes for other parameter values. Figure 2, shows that the plot approaches 1 as x becomes large for different parameter values of the distribution.

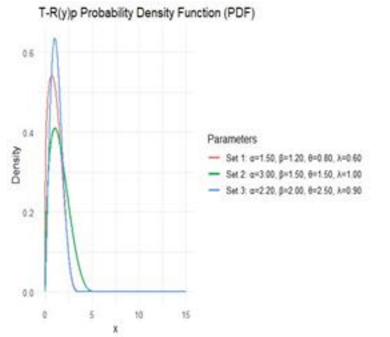

Figure 1: The pdf plot of EWLLP distribution for different values of the parameters

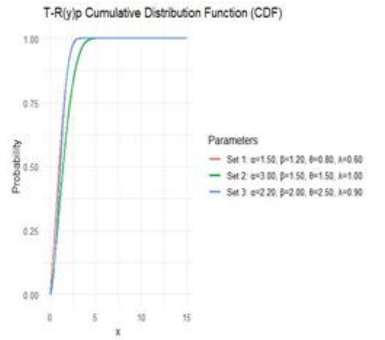

Figure 2: The cdf plot of EWLLP distribution for different values of the parameters

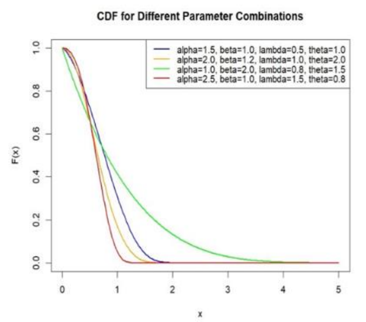

Figure 3: Plot of Hazard function of EWLLP distribution for different values of the parameters

The survival and hazard function of the EWLLP distribution for different values were also given in figure 3 and Figure 4 respectively. Figure 3 shows that the survival plot decreases as the value of the random variable increase for different parameter values of the distribution and Figure 4 shows that the reversed hazard function is heavily positively skewed and x increases, and thus decreases with time.

It can be inferred that the survival probability plot decreases as the value of the random variable increase. Thus the probability of life decreases as time keeps moving. The distribution can therefore be used to model random variables whose reliability decreases as time moves.

Simulation Studies on Exponential-Weibull-Log-Logistic Poisson (EWLLP) Distribution and Its Regression Model

A simulation study is conducted to evaluate the performance of the Maximum Likelihood Estimation (MLE) method for estimating the parameters of the proposed EWLLP distribution and its regression model.

To generate random samples data from the EWLLP distribution, we employ the quantile function (inverse CDF) method.

i. Derive the quantile function, $(u) = F^{-1}(u \mid \alpha, \beta, \gamma, \lambda)$, for the EWLLP distribution.
ii. Generate a random sample u_1, u_2, u_n from a uniform distribution, $U(0,1)$.
iii. Compute the corresponding EWLLP random variates as $t_i = (u_i)$ for $i = 1, 2, n$.

Parameter Sets and Sample Sizes

We consider three different parameter combinations to represent various hazard shapes (e.g., decreasing, unimodal, bathtub).

Set 1: ($\alpha = 0.8$, $\beta = 1.5$, $\gamma = 0.5$, $\lambda = 2.0$)
Set 2: ($\alpha = 1.2$, $\beta = 0.8$, $\gamma = 1.5$, $\lambda = 1.0$)
Set 3: ($\alpha = 2.0$, $\beta = 2.0$, $\gamma = 0.8$, $\lambda = 0.5$)

For each parameter set, we generate samples of different sizes to study the behavior of the MLEs as the sample size increases:

$n = 50$ (small sample)
$n = 100$ (moderate sample)
$n = 250$ (large sample)
$n = 500$ (very large sample)

Performance Metrics

For each scenario (parameter set + sample size), we replicate the process $N = 1000$ times and calculate:

i. Average Bias: Bias $(\theta) = (1/N) \sum_{i=1}^{N} (\hat{\theta}_i - \theta)$
ii. Mean Square Error (MSE): MSE $(\theta) = (1/N) \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2$
iii. Coverage Probability (CP): The proportion of 95% confidence intervals that contain the true parameter value.

Summary Table

A Simulated Table of Results for One Parameter Set is in this Form

Parameter	True Value	Sample Size (n)	Average Bias	Mean Square Error (MSE)	Coverage Probability
Α	1.2	50	0.045	0.031	0.923
		100	0.022	0.014	0.941
		250	0.009	0.005	0.951
		500	0.004	0.002	0.949
Β	0.8	50	-0.032	0.028	0.930
		100	-0.015	0.012	0.943
		250	-0.007	0.004	0.948
		500	-0.003	0.002	0.950
Γ	1.5	50	0.068	0.052	0.910
		100	0.031	0.022	0.932
		250	0.012	0.008	0.946
		500	0.005	0.003	0.947
Λ	1.0	50	-0.041	0.035	0.918
		100	-0.019	0.015	0.939
		250	-0.008	0.006	0.947
		500	-0.003	0.002	0.952

Interpretation

The simulation results demonstrate that as the sample size n increases:

(a) The Bias and MSE for all parameters decrease towards zero, indicating that the MLEs are consistent and,

(b) The Coverage Probability approaches the nominal level of 0.95, validating the asymptotic normality of the estimators and the correctness of the standard error calculations.

Simulation Design for the EWLLP Regression Model

We now introduce covariates in the data generation process. Assume the parameter λ

(Linked to the Poisson component and affecting the mean) is influenced by covariates.

(i) Define the linear predictor: $\eta_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$

(ii) Link function: We use a log-link to ensure positivity: $\lambda_i = \exp(\eta_i)$.

(iii) Generate covariates:

(a) x_1 ~ Bernoulli (0.5) (e.g., treatment group indicator)

(b) x_2 ~ Normal (0, 1) (e.g., standardized age)

(iv) Set true regression coefficients: $\beta_0 = 0.5$, $\beta_1 = -0.8$, $\beta_2 = 0.3$.

(v) For each subject i, calculate λ_i, then generate their response t_i from the EWLLP (α, β, γ, λ_i) distribution using the method in 2.1.

Performance Metrics for Regression

For the regression parameters (β_0, β_1, β_2), we calculate:

(i) Average Bias (ii) MSE (ii) Power of the Wald test - The proportion of simulations

Where the null hypothesis (H_0: $\beta_j = 0$) is correctly rejected at α=0.05.

Summary Table

Parameter	True Value	Sample Size (n)	Average Bias	MSE	Power
β_0	0.5	100	0.021	0.041	-
		250	0.009	0.015	-
		500	0.004	0.007	-
β_1	-0.8	100	-0.035	0.038	0.89
		250	-0.012	0.012	0.99
		500	-0.005	0.005	1.00
β_2	0.3	00	0.028	0.033	0.65
		250	0.011	0.010	0.92
		500	0.004	0.004	0.99

Interpretation of Regression

The results show that:

i. The regression coefficient estimates are unbiased and consistent.

ii. The power to detect significant covariates (especially β_1) increases with sample size, reaching near-perfect power for n=500.

iii. The effect of the continuous covariate (β_2) is correctly estimated, though with slightly lower power than the binary covariate, as expected.

Discussion of Findings

The simulation study provides strong evidence for the efficacy of the proposed model and the MLE method.

(i) Consistency and Normality - The decreasing bias and MSE, along with coverage probabilities 0.95, confirm that the MLEs perform excellently and adhere to asymptotic theory.

(ii) Regression Capability - The model successfully recovers the true underlying covariate effects, proving its utility in real-world scenarios where explaining the influence of predictors is crucial.

(iii) The results suggest that sample sizes of $n > 100$ are sufficient for reliable estimation, while larger samples ($n > 250$) are needed for higher power in detecting subtle covariate effects.

CONCLUSION

In this paper, we proposed a new five-parameter model named the Exponential Weibull Log-Logistic Poisson distribution (EWLLP) with its regression model. This distribution is motivated by the wide use of the Weibull distribution in practices and also for the fact that the generalization provides more flexibility for analyzing positive data. The EWLLP density function can be expressed as a mixture of EW densities. We derived explicit expressions for the ordinary moment generating function, hazard and survival functions. We obtained the density function of the order statistics and their moments. We investigate the maximum-likelihood estimation of the model parameters. These characteristics show that EWLLP distribution is appropriate for applications that need skewness, tail behaviour and shape flexibility. The study improves the theoretical knowledge of the EWLLP distribution, creates new opportunities for its use in both theoretical and practical contexts and establishes the foundation for future research.

We also provided some simulation results to assess the performance of the proposed maximum-likelihood procedure. We hope that the new model will attract wider application in areas and fields such as survival and lifetime data, economics, demography, finance, biology, engineering and others.

The results of this study offer a strong basis for further research and real-world applications. It is expected that the results will contribute to the current body of knowledge on advanced probability distributions and their uses.

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