FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 16, June, 2026, pp 578 – 588
DOI: https://doi.org/10.33003/fjs-2026-1016-5844

Higher-Order Implicit Hybrid Runge-Kutta Schemes for Direct Numerical Solution of Third-Order Initial Value Problems

***[1]Subair Ahmed Olajuwon, [1]Ademola Mudashiru Badmus, [1]Aminu Abdullahi and [2]Muhammed Sani Muhammed**

[1]Nigerian Defence Academy, Kaduna State, Nigeria.
[2]Federal University Dutse, Jigawa State, Nigeria.

*Corresponding authors' email: ahmed.subair@nda.edu.ng
ORCID: https://orcid.org/0000-0001-8851-4599

Received: 31 Aug. 2026
Accepted: 01 Sep. 2026
Published: 08 Sep. 2026

ABSTRACT
Direct numerical solution of higher-order initial value problems remains an important problem in the numerical solution of ordinary differential equations particularly when conventional approaches require the reduction of higher-order equations to equivalent systems of first-order equations. This study develops two high-order implicit hybrid Runge-Kutta type schemes for the direct numerical solution of third-order initial value problems through the reformulation of linear multistep methods. Power series interpolation is employed to construct the two linear multistep method formulations, which are subsequently transformed into implicit hybrid Runge-Kutta type schemes with step lengths $k = 3$ and$k = 4$. The resulting methods are analyzed in terms of consistency, order, zero stability and convergence. The analysis establishes that the proposed schemes possess orders eight and nine for $k = 3$ and $k = 4$, respectively. The methods are applied to selected linear and nonlinear third-order initial value problems with known exact solutions, and their accuracy is assessed using absolute errors and comparative numerical experiments with existing methods. The numerical results demonstrate that both schemes provide accurate solutions, with the $k = 4$ generally yielding smaller absolute errors than the $k = 3$ formulation.

Keywords: LMM, Implicit Hybrid Runge-Kutta, Third-order IVPs, Error Analysis, Direct numerical methods, Power series interpolation

INTRODUCTION

Initial value problems (IVPs) involving higher-order ordinary differential equations (ODEs) arise in areas of mathematical modeling and scientific computing, particularly in problems whose governing equations naturally involve higher-order derivatives (Ascher & Petzold, 1998). A general third-order IVP can be expressed as

$$\frac{d^3y(x)}{dx^3} = f(x, y, y', y'') \quad , y(x_0) = y_0, y'(x_0) = y_0', y''(x_0) = y_0'' \tag{1}$$

Traditionally, the numerical solution of (1) is obtained by reducing it to an equivalent system of first-order ODEs and subsequently applying a suitable integrator. Adeyefa & Kuboye (2020), Duromola et al. (2024), and Lawal et al. (2026) explained that this conversion makes the problem incompatible with standard solvers because it introduces several drawbacks such as an increase in the computational cost, larger memory demand and enhanced propagation of local truncation errors across coupled state variables. These limitations have motivated the development of numerical methods capable of solving higher order IVPs directly.

Runge–Kutta (R-K) methods are among the most widely used methods for solving IVPs because of their flexibility and accuracy (Butcher, 2008). However, the construction of higher-order implicit R-K methods can become algebraically demanding as the number of stages increases. The associated order conditions grow rapidly with the number of stages, making the derivation of high-order schemes increasingly complicated and, in some cases problem specific (Badmus & Subair, 2024). These challenges have prompted researchers to explore alternative formulation pathways.

Linear multistep methods (LMMs) provide an alternative framework for the numerical solution of ODEs. Their relatively simple algebraic structure makes it possible to construct high-order methods through polynomial interpolation and related techniques (Lambert, 1991). However, conventional LMMs have limitations of dependency on previously computed solution values and their stability restrictions, particularly when high-order methods are considered. These limitations have encouraged the development of hybrid formulations that combine features of LMMs with those of R-K methods.

Hybrid methods incorporate additional off-step points into the numerical formulation and can provide greater flexibility in the construction of high-order schemes. Adesanya, (2014) demonstrated the use of hybrid formulations in the direct numerical treatment of high-order ODEs. Such approaches provide a basis for combining the systematic formulation of LMMs with the structural features of R-K methods.

Recent literature shows promising progress in solving third-order ODEs directly, yet many existing schemes lack consistency, exhibit limited stability regions, or are not generalized for third-order IVPs (Kuboye, 2020). Despite these developments, there is a need for high-order numerical methods that provide a systematic formulation for the direct solution of third-order IVPs while retaining desirable stability and convergence properties.

Definition of Terms

i. Power Series about the origin provides an infinite series representation of the exact solution of an IVP, provided the solution is analytic in a neighborhood of the origin.

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n, \quad /x/< R \tag{2}$$

ii. A LMM of order p is expressed as

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} - h^p \sum_{j=0}^{k} \beta_j f_{n+j} \quad , \quad j = 0,1,2, \dots k \tag{3}$$

Where y_{n+j} and f_{n+j} are interpolation and collocation points respectively, α_j and β_j are constants and p determines the order. It is implicit if$\beta_k \neq 0$, otherwise; explicit.

iii. An S-stage R-K type methods for direct integration of third-order ODE is given by

$$y_{n+1} = y_n + \alpha_i h y_n' + \frac{(\alpha_i h)^2}{2} y_n'' + h^3 \sum_{j=1}^{s} a_{ij} k_i \tag{4}$$

$$y_{n+1}' = y_n' + \alpha_i h y_n'' + h^2 \sum_{j=1}^{s} \overline{a}_{ij} k_i$$

$$y_{n+1}'' = y_n'' + h \sum_{j=1}^{s} \overline{\overline{a_{ij}}} k_i \quad , \quad for \quad i = 1, \dots, s$$

$$k_i = f\left(x_n + \alpha_i h, y_n + \alpha_i h y_n' + \frac{(\alpha_i h)^2}{2} y_n'' + h^3 \sum_{j=1}^{s} a_{ij} k_i, y_n' + \alpha_i^2 h y'' + h^2 \sum_{j=1}^{s} \overline{a}_{ij} k_i, y_n'' + h \sum_{j=1}^{s} \overline{\overline{a_{ij}}} k_i\right)$$

The real parameters $\alpha_i, k_i, a_{ij}, \overline{a}_{ij}, \overline{\overline{a_{ij}}}$ define the method above and its Butcher array is given as

$$\begin{array}{c|c|c|c} \alpha & \overline{\overline{A}} & \overline{A} & A \\ \hline & \overline{\overline{b^T}} & \overline{b}' & b \end{array} \quad \text{(Fawzi \& Jumaa, 2022)} \tag{5}$$

For the consistency of this method

$$\sum_{i=1}^{s} b_i = \frac{1}{6} \tag{6}$$

iv. Zero Stability: The LMM (3) is said to satisfy the root conditions if all the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple (Fatunla, 1988).

MATERIALS AND METHODS

Given a power series solution of the form

$$y(x) = \sum_{j=0}^{q+r-1} \alpha_j x^j = y_{n+j} \tag{7}$$

$$y'(x) = \sum_{j=3}^{q+r-1} j \alpha_j x^{j-1} = f_{n+j} \tag{8}$$

$(q + r - 1)$ is the degree of polynomials and α_j's are the unknown parameters to be determined.

Construction of first-order LMM at $k = 3$

Firstly, interpolate (7) at $x = x_{n+j}, j = 0, \frac{3}{2}, \frac{5}{2}$ and collocate (8) at $x = x_{n+j}, j = \frac{1}{2}, 1, 2, \frac{5}{2}, \frac{11}{4}, 3)$ to form the d-matrix as follows

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 & x_{n+\frac{3}{2}}^8 \\ 1 & x_{n+\frac{5}{2}} & x_{n+\frac{5}{2}}^2 & x_{n+\frac{5}{2}}^3 & x_{n+\frac{5}{2}}^4 & x_{n+\frac{5}{2}}^5 & x_{n+\frac{5}{2}}^6 & x_{n+\frac{5}{2}}^7 & x_{n+\frac{5}{2}}^8 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 & 6x_{n+\frac{1}{2}}^5 & 7x_{n+\frac{1}{2}}^6 & 8x_{n+\frac{1}{2}}^7 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 & 6x_{n+\frac{5}{2}}^5 & 7x_{n+\frac{5}{2}}^6 & 8x_{n+\frac{5}{2}}^7 \\ 0 & 1 & 2x_{n+\frac{11}{4}} & 3x_{n+\frac{11}{4}}^2 & 4x_{n+\frac{11}{4}}^3 & 5x_{n+\frac{11}{4}}^4 & 6x_{n+\frac{11}{4}}^5 & 7x_{n+\frac{11}{4}}^6 & 8x_{n+\frac{11}{4}}^7 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_{\frac{3}{2}} \\ \alpha_{\frac{5}{2}} \\ \beta_{\frac{1}{2}} \\ \beta_1 \\ \beta_2 \\ \beta_{\frac{5}{2}} \\ \beta_{\frac{11}{4}} \\ \beta_3 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{3}{2}} \\ y_{n+\frac{5}{2}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{11}{4}} \\ f_{n+3} \end{pmatrix} \tag{9}$$

In addition, rigorous algebraic restructuring of (9) yield the continuous formulation and continuous scheme respectively given as

$$y(x) = \alpha_0 y_n + \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + \alpha_{\frac{5}{2}} y_{n+\frac{5}{2}} = h[\beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_{\frac{11}{4}} f_{n+\frac{11}{4}} + \beta_3 f_{n+3}] \tag{10}$$

and

$$\begin{aligned} y(x) = {} & \left(1 - \frac{188672x}{2776h} + \frac{7413056x^2}{416475h^2} - \frac{3361408x^3}{138825h^3} + \frac{2654368x^4}{138825h^4} - \frac{421504x^5}{46275h^5} + \frac{119552x^6}{46275h^6} - \frac{55808x^7}{138825h^7} + \frac{11008x^8}{416475h^8}\right) y_n + \\ & \left(\frac{173800x}{5553h} - \frac{2514470x^2}{16659h^2} + \frac{1792768x^3}{5553h^3} - \frac{1970767x^4}{5553h^4} + \frac{398680x^5}{1851h^5} - \frac{45160x^6}{617h^6} + \frac{72608x^7}{5553h^7} - \frac{15952x^8}{16659h^8}\right) y_{n+\frac{3}{2}} + \left(\frac{-75592x}{3085h} + \frac{6160966x^2}{46275h^2} - \right. \\ & \left. \frac{13819264x^3}{46275h^3} + \frac{5179423x^4}{15425h^4} - \frac{3181832x^5}{15425h^5} + \frac{3267448x^6}{46275h^6} - \frac{195488x^7}{15425h^7} + \frac{43088x^8}{46275h^8}\right) y_{n+\frac{5}{2}} + \left(\frac{-258830x}{49977} + \frac{6221249x^2}{299862h} - \frac{25445972x^3}{749655h^2} + \right. \\ & \left. \frac{29651873x^4}{999540h^3} - \frac{3760426x^5}{249885h^4} + \frac{667366x^6}{149931h^5} - \frac{178264x^7}{249885h^6} + \frac{35932x^8}{749655h^7}\right) f_{n+\frac{1}{2}} + \left(\frac{-14465x}{12957} - \frac{1211891x^2}{155484h} + \frac{10204253x^3}{388710h^2} - \frac{675294x^4}{21595h^3} + \right. \\ & \left. \frac{409286x^5}{21595h^4} - \frac{244640x^6}{38871h^5} + \frac{23616x^7}{21595h^6} - \frac{15152x^8}{194355h^7}\right) f_{n+1} + \left(\frac{301895x}{16659} - \frac{5116738x^2}{49977h} + \frac{58570733x^3}{249885h^2} - \frac{22338314x^4}{83295h^3} + \frac{13962196x^5}{83295h^4} - \right. \\ & \left. \frac{2915224x^6}{49977h^5} + \frac{885376x^7}{83295h^6} - \frac{197824x^8}{249885h^7}\right) f_{n+2} + \left(\frac{4994x}{1851} - \frac{580243x^2}{55530h} + \frac{489548x^3}{27765h^2} - \frac{164029x^4}{12340h^3} + \frac{11074x^5}{3085h^4} + \frac{13918x^6}{27765h^5} - \frac{13918x^7}{27765h^6} - \right. \\ & \left. \frac{1288x^8}{3085h^7}\right) f_{n+\frac{5}{2}} + \left(\frac{389120x}{349839} - \frac{9392128x^2}{1049517h} + \frac{127259648x^3}{5247585h^2} - \frac{56274944x^4}{1749195h^3} + \frac{40815616x^5}{1749195h^4} - \frac{9865216x^6}{1049517h^5} + \frac{3444736x^7}{1749195h^6} - \frac{876544x^8}{5247585h^7}\right) f_{n+\frac{11}{4}} + \\ & \left(\frac{-605x}{1851} + \frac{53053x^2}{22212h} - \frac{116023x^3}{18510h^2} + \frac{75448x^4}{9255h^3} - \frac{18006x^5}{3085h^4} + \frac{4328x^6}{1851h^5} - \frac{4544x^7}{9255h^6} + \frac{1168x^8}{27765h^7}\right) \end{aligned} \tag{11}$$

Equation (11) above is evaluated at $x = y_{n+j}, j = \frac{1}{2}, 1, 2, \frac{11}{4}, 3$, while the first derivative evaluated at $x = y_{n+j}, j = 0, \frac{3}{2}$ which gives the following seven discrete schemes

$$\frac{3712}{138825}y_n + y_{n+\frac{1}{2}} - \frac{9856}{5553}y_{n+\frac{3}{2}} + \frac{34621}{46275}y_{n+\frac{5}{2}} = -\frac{148199}{749655}hf_{n+\frac{1}{2}} - \frac{26338}{38871}hf_{n+1} + \frac{20476}{49977}hf_{n+2} + \frac{6299}{27765}hf_{n+\frac{5}{2}} + \frac{66560}{1049517}hf_{n+\frac{11}{4}} + \frac{94}{9255}hf_{n+3}$$

$$\frac{-161}{15425}y_n + y_{n+1} - \frac{1615}{617}y_{n+\frac{3}{2}} + \frac{25111}{15425}y_{n+\frac{5}{2}} = \frac{3007}{111060}hf_{n+\frac{1}{2}} - \frac{22041}{86380}hf_{n+1} + \frac{9974}{9255}hf_{n+2} + \frac{3909}{12340}hf_{n+\frac{5}{2}} - \frac{4096}{194355}hf_{n+\frac{11}{4}} - \frac{21}{12340}hf_{n+3}$$

$$\frac{-329}{138825}y_n + y_{n+2} + \frac{1448}{5553}y_{n+\frac{3}{2}} - \frac{58232}{46275}y_{n+\frac{5}{2}} = \frac{746}{149931}hf_{n+\frac{1}{2}} - \frac{2441}{194355}hf_{n+1} - \frac{110542}{249885}hf_{n+2} - \frac{10846}{27765}hf_{n+\frac{5}{2}} + \frac{524288}{5247585}hf_{n+\frac{11}{4}} - \frac{131}{9255}hf_{n+3}$$

$$\frac{-665}{1421568}y_n + y_{n+\frac{11}{4}} + \frac{876887}{22745088}y_{n+\frac{3}{2}} - \frac{7870445}{7581696}y_{n+\frac{5}{2}} = \frac{2356475}{245469504}hf_{n+\frac{1}{2}} - \frac{355135}{159215616}hf_{n+1} - \frac{2069705}{51176448}hf_{n+2} + \frac{12602755}{90980352}hf_{n+\frac{5}{2}} + \frac{1023935}{8396136}hf_{n+\frac{11}{4}} - \frac{46585}{7581696}hf_{n+3}$$

$$\frac{29}{46275}y_n + y_{n+3} - \frac{77}{1851}y_{n+\frac{3}{2}} - \frac{14793}{15425}y_{n+\frac{5}{2}} = -\frac{419}{333180}hf_{n+\frac{1}{2}} + \frac{47}{17276}hf_{n+1} + \frac{214}{5553}hf_{n+2} + \frac{869}{1340}hf_{n+\frac{5}{2}} + \frac{40960}{116613}hf_{n+\frac{11}{4}} + \frac{967}{12340}hf_{n+3}$$

$$11886336y_n - 54747000y_{n+\frac{3}{2}} + 42860664y_{n+\frac{5}{2}} = -9059050hf_{n+\frac{1}{2}} - 1952775hf_{n+1} + 31698975hf_{n+2} + 4719330hf_{n+\frac{5}{2}} + 1945600hf_{n+\frac{11}{4}} - 571725hf_{n+3} - 1749195hf_n$$

$$-4032y_n - 3270960y_{n+\frac{3}{2}} + 3274992y_{n+\frac{5}{2}} = 10325hf_{n+\frac{1}{2}} - 50247hf_{n+1} + 1952958hf_{n+2} + 789453hf_{n+\frac{5}{2}} + 135680hf_{n+\frac{11}{4}} - 14553f_{n+3} - 699678hf_{n+\frac{3}{2}} \quad (12)$$

Construction of first-order LMM at $k = 4$

Here, equation (7) is interpolated at $x = x_{n+j}, j = 0, \frac{1}{4}, \frac{1}{2}$ and (8) is collocated at $x = x_{n+j}, j = \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, 2, 3, 4$. The degree of the polynomial of this method is 6 and its continuous formulation is expressed as

$$y(x) = \alpha_0 y_n + \alpha_{\frac{1}{4}} y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} = h\left[\beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} + \beta_{\frac{5}{4}} f_{n+\frac{5}{4}} + \beta_2 f_{n+2} + \beta_3 f_{n+3} + \beta_4 f_{n+4}\right] \quad (13)$$

The continuous scheme of this method is evaluated at $x = y_{n+j}, j = \frac{3}{4}, 1, \frac{5}{4}, 2, 3, 4$, while the first derivative of the continuous scheme is also evaluated at $x = y_{n+j}, j = 0, \frac{1}{4}$ which gives the following eight discrete schemes

$$\frac{9028912747}{1333338000224}y_n - \frac{5196840795}{41666812507}y_{n+\frac{1}{4}} - \frac{1176068007531}{1333338000224}y_{n+\frac{1}{2}} + y_{n+\frac{3}{4}} = \frac{3484290595291}{23333415003920}hf_{n+\frac{1}{2}} + \frac{3216730994909}{21666742503640}hf_{n+\frac{3}{4}} - \frac{4190615866503}{170667264028672}hf_{n+1} + \frac{119145143555}{25666756504312}hf_{n+\frac{5}{4}} - \frac{73051288151}{853336320143360}hf_{n+2} + \frac{17477180849}{5973354241003520}hf_{n+3} - \frac{20047752657}{170837931292700672}hf_{n+4}$$

$$\frac{76323573}{41666812507}y_n - \frac{2139553792}{41666812507}y_{n+\frac{1}{4}} - \frac{39603582288}{41666812507}y_{n+\frac{1}{2}} + y_{n+1} = \frac{4536629912}{41666812507}hf_{n+\frac{1}{2}} + \frac{833047572288}{2708342812955}hf_{n+\frac{3}{4}} + \frac{16590759057}{166667250028}hf_{n+1} - \frac{13206804032}{3208344563039}hf_{n+\frac{5}{4}} + \frac{12404423}{416668125070}hf_{n+2} - \frac{774555}{1166670750196}hf_{n+3} + \frac{256791}{11916708377002}hf_{n+4}$$

$$\frac{3185689419}{333334500056}y_n - \frac{6509101275}{41666812507}y_{n+\frac{1}{4}} - \frac{284447379275}{333334500056}y_{n+\frac{1}{2}} + y_{n+\frac{5}{4}} = \frac{186065315795}{1166670750196}hf_{n+\frac{1}{2}} + \frac{19047872025}{83333625014}hf_{n+\frac{3}{4}} + \frac{13063613679225}{42666816007168}hf_{n+1} + \frac{579537691385}{64166891126078}hf_{n+\frac{5}{4}} - \frac{12113087275}{42666816007168}hf_{n+2} + \frac{2348937405}{298667712050176}hf_{n+3} - \frac{966583125}{3285344832551936}hf_{n+4}$$

$$\frac{-26722844379}{41666812507}y_n + \frac{331802542080}{41666812507}y_{n+\frac{1}{4}} - \frac{346746510208}{41666812507}y_{n+\frac{1}{2}} + y_{n+2} = \frac{-706404164032}{208334062535}hf_{n+\frac{1}{2}} + \frac{11886158043648}{2708342812955}hf_{n+\frac{3}{4}} - \frac{152958831543}{41666812507}hf_{n+1} + \frac{95599060480}{41666812507}hf_{n+\frac{5}{4}} + \frac{43767413144}{208334062535}hf_{n+2} - \frac{339966333}{208334062535}hf_{n+3} + \frac{27950454}{541668562591}hf_{n+4}$$

$$\frac{504687627125}{41666812507}y_n - \frac{5959139328000}{41666812507}y_{n+\frac{1}{4}} + \frac{5412784888368}{41666812507}y_{n+\frac{1}{2}} + y_{n+3} = \frac{17271317055080}{291667687549}hf_{n+\frac{1}{2}} - \frac{2527845073600}{41666812507}hf_{n+\frac{3}{4}} + \frac{8077971812625}{166667250028}hf_{n+1} - \frac{5076953464000}{291667687549}hf_{n+\frac{5}{4}} + \frac{192832354825}{83333625014}hf_{n+2} + \frac{324820024045}{116667075196}hf_{n+3} - \frac{1521295875}{583335375098}hf_{n+4}$$

$$\frac{-5274220765275}{41666812507}y_n + \frac{60708984455168}{41666812507} - \frac{55476430502400}{41666812507}y_{n+\frac{1}{2}} + y_{n+4} = -\frac{2451710318240}{41666812507}hf_{n+\frac{1}{2}} + \frac{314432232683520}{541668562591}hf_{n+\frac{3}{4}} - \frac{18216118533900}{41666812507}hf_{n+1} + \frac{70168042649600}{458334937577}hf_{n+\frac{5}{4}} - \frac{73051288151}{853336320143360}hf_{n+2} + \frac{17477180849}{5973354241003520}hf_{n+3} + \frac{1495711174860}{5958354188501}hf_{n+4}$$

$$-1463496949474560y_n + 5547900251013120y_{n+\frac{1}{4}} - 4084403301538560y_{n+\frac{1}{2}} = -1150091005844352hf_{n+\frac{1}{2}} + 594866987545600hf_{n+\frac{3}{4}} - 297541197323370hf_{n+1} + 74178007747584hf_{n+\frac{5}{4}} - 1832891740680hf_{n+2} + 71091027722hf_{n+3} - 3017387028hf_{n+4} + 125125437958521hf_n$$

$$99259732091520y_n + 2371391219957760y_{n+\frac{1}{4}} - 2470650952049280y_{n+\frac{1}{2}} = -475545638833408hf_{n+\frac{1}{2}} + 168213386761728hf_{n+\frac{3}{4}} - 75046733266505hf_{n+1} + 17623003473920hf_{n+\frac{5}{4}} - 401718759013hf_{n+2} + 14956121973hf_{n+3} - 622487303hf_{n+4} + 277334304046592hf_{n+\frac{1}{4}} \quad (14)$$

Following the order and error constant analysis established by Soomro et al. (2022), the two discrete LMM schemes (12) and (14) are confirmed to achieve uniform orders of 8 and 9, respectively. Since the subsequent implicit hybrid R-K type formulations are algebraic reformulations of their corresponding LMM schemes, without altering the underlying discrete approximation, the reformulated methods preserve orders of accuracy, in accordance with the equivalence between such formulations (Butcher, 2008).

Reformulation of first-order LMMs into Implicit R-K type schemes for third-order ODEs.
The reformulation begins with the zero-stability analysis of (12) as follows:

$$\begin{pmatrix} 1 & 0 & -\frac{9856}{5553} & 0 & \frac{34621}{46275} & 0 & 0 \\ 0 & 1 & -\frac{1615}{617} & 0 & \frac{25111}{15425} & 0 & 0 \\ 0 & 0 & \frac{1448}{5553} & 0 & -\frac{58232}{46275} & 0 & 0 \\ 0 & 0 & \frac{876887}{22745088} & 0 & -\frac{7870445}{7581696} & 1 & 0 \\ 0 & 0 & -\frac{77}{1851} & 0 & -\frac{14793}{15425} & 0 & 1 \\ 0 & 0 & -54747000 & 0 & 42860664 & 0 & 0 \\ 0 & 0 & -3270960 & 0 & 3274992 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+\frac{11}{4}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3712}{138825} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{161}{15425} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{329}{138825} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{665}{1421568} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-29}{46275} \\ 0 & 0 & 0 & 0 & 0 & 0 & -11886336 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4032 \end{pmatrix} \begin{pmatrix} y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{pmatrix} +$$

$$\begin{pmatrix} \frac{-148199}{749655} & \frac{-26338}{38871} & 0 & \frac{20476}{49977} & \frac{6299}{27765} & \frac{-66560}{1049517} & \frac{94}{9255} \\ \frac{3007}{111060} & \frac{-22041}{86380} & 0 & \frac{9974}{9255} & \frac{3909}{12340} & \frac{-4096}{194355} & \frac{-21}{12340} \\ \frac{746}{149931} & \frac{-2441}{194355} & 0 & \frac{-110542}{249885} & \frac{-10846}{27765} & \frac{524288}{5247585} & \frac{-131}{9255} \\ \frac{2356475}{2456469504} & \frac{-355135}{159215616} & 0 & \frac{-2069705}{51176448} & \frac{12602755}{90980352} & \frac{1023935}{8396136} & \frac{-46585}{7581696} \\ \frac{-419}{333180} & \frac{47}{17276} & 0 & \frac{214}{5553} & \frac{869}{12340} & \frac{40960}{116613} & \frac{967}{12340} \\ -9059050 & -1952775 & 0 & 31698975 & 4719330 & 1945600 & -571725 \\ 10325 & -50247 & 699678 & 1952958 & 789453 & -135680 & 14553 \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{11}{4}} \\ f_{n+3} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1749195 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{4}} \\ f_n \end{pmatrix} \tag{15}$$

Normalizing (15) by multiplying with $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-4553}{52895656800} & \frac{19001}{21158262720} \\ 0 & 0 & 0 & 0 & 0 & \frac{-4027}{48487685400} & \frac{13}{22039857} \\ 0 & 0 & 0 & 0 & 0 & \frac{-361}{4310016480} & \frac{859}{783639360} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1517}{18182882025} & \frac{488}{330597855} \\ 0 & 0 & 0 & 0 & 0 & \frac{-59}{705275424} & \frac{1975}{1410550848} \\ 0 & 0 & 0 & 0 & 0 & \frac{-905773}{10833030512640} & \frac{7643933}{5416515256320} \\ 0 & 0 & 0 & 0 & 0 & \frac{-41}{489774600} & \frac{17}{12244365} \end{pmatrix}$ to obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+\frac{11}{4}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{pmatrix} +$$

$$\begin{pmatrix} \frac{107293}{181440} & \frac{-3727}{6720} & \frac{19001}{30240} & \frac{-4271}{7560} & \frac{3559}{6720} & \frac{-200}{567} & \frac{4381}{60480} \\ \frac{2227}{2835} & \frac{-103}{840} & \frac{26}{63} & \frac{-3047}{7560} & \frac{41}{105} & \frac{-8192}{31185} & \frac{137}{2520} \\ \frac{345}{448} & \frac{243}{2240} & \frac{859}{1120} & \frac{-18}{35} & \frac{1053}{2240} & \frac{-24}{77} & \frac{143}{2240} \\ \frac{440}{567} & \frac{8}{105} & \frac{976}{945} & \frac{-193}{945} & \frac{8}{21} & \frac{-8192}{31185} & \frac{52}{945} \\ \frac{28025}{36288} & \frac{125}{1344} & \frac{1975}{2016} & \frac{125}{1512} & \frac{955}{1344} & \frac{-200}{567} & \frac{275}{4032} \\ \frac{7180745}{9289728} & \frac{310123}{3440640} & \frac{7643933}{7741440} & \frac{2019127}{30965760} & \frac{1476079}{1720320} & \frac{-4213}{18144} & \frac{1925957}{30965760} \\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & 0 & \frac{41}{280} \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+\frac{11}{4}} \\ f_{n+3} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{4553}{30240} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4027}{27720} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{361}{2464} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1517}{10395} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{295}{2016} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{905773}{6193152} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{41}{280} \end{pmatrix} \begin{pmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{4}} \\ f_n \end{pmatrix} \qquad (16)$$

$$\rho(R) = det\left[R\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] = 0$$

This show that $\rho(R) = \left[\begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & R-1 \end{pmatrix}\right] = R^7 - R^6 = 0$

This implies $R_1 = R_2 = R_3 = R_6 = 0$ and $R_7 = 1$. Hence; the method is zero stable.

Having arranged (15) into the form of (5), the value of $\bar{\bar{A}}$ is multiplied by $\frac{1}{3}$ and its resulting parameters is subsequently used to spell out the values of $\bar{A}$ and A which are expressed as;

$$\begin{array}{c|c} \alpha & \bar{\bar{A}} \\ \hline & \bar{\bar{b}}^T \end{array} \quad \text{as}$$

C	$\bar{\bar{A}}$							
0	0	0	0	0	0	0	0	0
$\frac{1}{6}$	$\frac{4553}{90720}$	$\frac{107293}{544320}$	$\frac{-3727}{20160}$	$\frac{19001}{90720}$	$\frac{-4271}{22680}$	$\frac{3559}{20160}$	$\frac{-200}{1701}$	$\frac{4381}{181440}$
$\frac{1}{3}$	$\frac{4027}{83160}$	$\frac{2227}{8505}$	$\frac{-103}{2520}$	$\frac{26}{189}$	$\frac{-3047}{22680}$	$\frac{41}{315}$	$\frac{-8192}{93555}$	$\frac{137}{7560}$
$\frac{1}{2}$	$\frac{361}{7392}$	$\frac{115}{448}$	$\frac{81}{2240}$	$\frac{859}{3360}$	$\frac{-6}{35}$	$\frac{351}{2240}$	$\frac{-8}{77}$	$\frac{143}{6720}$
$\frac{2}{3}$	$\frac{1517}{31185}$	$\frac{440}{1701}$	$\frac{8}{315}$	$\frac{967}{2835}$	$\frac{-193}{2835}$	$\frac{8}{63}$	$\frac{-8192}{93555}$	$\frac{52}{2835}$
$\frac{5}{6}$	$\frac{295}{6048}$	$\frac{28025}{108854}$	$\frac{125}{4032}$	$\frac{1975}{6048}$	$\frac{125}{4536}$	$\frac{955}{4032}$	$\frac{-200}{1701}$	$\frac{275}{12096}$
$\frac{11}{12}$	$\frac{905773}{18579456}$	$\frac{7180745}{27869184}$	$\frac{310123}{10321920}$	$\frac{7643933}{23224320}$	$\frac{2019127}{92897280}$	$\frac{1476079}{5160960}$	$\frac{-4213}{54432}$	$\frac{1925957}{92897280}$
1	$\frac{41}{840}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{34}{105}$	$\frac{9}{280}$	$\frac{9}{35}$	0	$\frac{41}{840}$
1	$\frac{41}{840}$	$\frac{9}{35}$	$\frac{9}{280}$	$\frac{34}{105}$	$\frac{9}{280}$	$\frac{9}{35}$	0	$\frac{41}{840}$

$$\begin{array}{c|c} \alpha & \bar{A} \\ \hline & \bar{b}^T \end{array} \quad \text{as}$$

C	$\bar{A}$							
	0	0	0	0	0	0	0	0
	$\frac{11621831}{1916006400}$	$\frac{209749}{12441600}$	$\frac{-18097}{774144}$	$\frac{1364569}{43545600}$	$\frac{-374867}{11612160}$	$\frac{329363}{9676800}$	$\frac{-44749}{1871100}$	$\frac{899929}{174182400}$
	$\frac{35699}{2494800}$	$\frac{58253}{1020600}$	$\frac{-653}{15120}$	$\frac{3271}{56700}$	$\frac{-7619}{136080}$	$\frac{2111}{37800}$	$\frac{-53696}{1403325}$	$\frac{1831}{226800}$
	$\frac{529783}{23654400}$	$\frac{107939}{1075200}$	$\frac{-5967}{143360}$	$\frac{2277}{25600}$	$\frac{-11843}{143360}$	$\frac{29097}{358400}$	$\frac{-1277}{23100}$	$\frac{24937}{2150400}$
	$\frac{57131}{1871100}$	$\frac{4061}{28350}$	$\frac{-137}{3780}$	$\frac{5989}{42525}$	$\frac{-169}{1620}$	$\frac{983}{9450}$	$\frac{-11008}{155925}$	$\frac{2509}{170100}$
	$\frac{987125}{25546752}$	$\frac{1947425}{10450944}$	$\frac{-24625}{774144}$	$\frac{114475}{580608}$	$\frac{-737125}{6967296}$	$\frac{7325}{55296}$	$\frac{-2825}{32076}$	$\frac{42475}{2322432}$
	$\frac{1904306833}{44590694400}$	$\frac{4632557479}{22295347200}$	$\frac{-28934609}{990904320}$	$\frac{2501118037}{11147673600}$	$\frac{-307957463}{2972712960}$	$\frac{383153639}{2477260800}$	$\frac{-4209227}{43545600}$	$\frac{859293157}{44590694400}$
	$\frac{4321}{92400}$	$\frac{321}{1400}$	$\frac{-3}{112}$	$\frac{529}{2100}$	$\frac{-57}{560}$	$\frac{249}{1400}$	$\frac{-192}{1925}$	$\frac{9}{400}$
$\frac{1}{2}$	$\frac{4321}{92400}$	$\frac{321}{1400}$	$\frac{-3}{112}$	$\frac{529}{2100}$	$\frac{-57}{560}$	$\frac{249}{1400}$	$\frac{-192}{1925}$	$\frac{9}{400}$

$$\begin{array}{c|c} \alpha & A \\ \hline & b \end{array}$$ as

C	*A*							
	0	0	0	0	0	0	0	0
	$\frac{15468791}{36787322880}$	$\frac{6888223}{8360755200}$	$\frac{-99973}{74317824}$	$\frac{8899517}{4180377600}$	$\frac{-417913}{159252480}$	$\frac{17489}{5308416}$	$\frac{-451777}{179625600}$	$\frac{9837389}{16721510400}$
	$\frac{507649}{239500800}$	$\frac{96043}{13996800}$	$\frac{-3523}{483840}$	$\frac{7577}{777600}$	$\frac{-131389}{13063680}$	$\frac{12997}{1209600}$	$\frac{-4586}{601425}$	$\frac{36287}{21772800}$
	$\frac{1306699}{252313600}$	$\frac{2063407}{103219200}$	$\frac{-66569}{4587520}$	$\frac{74563}{3440640}$	$\frac{-296287}{13762560}$	$\frac{254063}{11468800}$	$\frac{-1369}{88704}$	$\frac{75989}{22937600}$
	$\frac{1722463}{179625600}$	$\frac{328987}{8164800}$	$\frac{-7613}{362880}$	$\frac{23687}{583200}$	$\frac{-40651}{1088640}$	$\frac{4877}{129600}$	$\frac{-36392}{1403325}$	$\frac{89921}{16329600}$
	$\frac{3423275}{222953472}$	$\frac{9711475}{143327232}$	$\frac{-73375}{2752512}$	$\frac{3936375}{55738368}$	$\frac{-36847625}{668860416}$	$\frac{235775}{4128768}$	$\frac{-76625}{1959552}$	$\frac{1840775}{222953472}$
	$\frac{80235165461}{4280706662400}$	$\frac{25739291941}{305764761600}$	$\frac{-2777795669}{95126814720}$	$\frac{528307175}{6115295232}$	$\frac{-18209699827}{285380444160}$	$\frac{16417236803}{237817036800}$	$\frac{-5591773}{119439360}$	$\frac{42183930041}{4280706662400}$
	$\frac{13289}{591360}$	$\frac{4587}{44800}$	$\frac{-113}{3584}$	$\frac{7139}{6720}$	$\frac{-1297}{17920}$	$\frac{743}{8960}$	$\frac{-106}{1925}$	$\frac{1041}{89600}$
$\frac{1}{6}$	$\frac{13289}{591360}$	$\frac{4587}{44800}$	$\frac{-113}{3584}$	$\frac{7139}{67200}$	$\frac{-1297}{17920}$	$\frac{743}{8960}$	$\frac{-106}{1925}$	$\frac{1041}{89600}$

This gives an eight-stage implicit hybrid R-K method as follows

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + h^3\left[\frac{13289}{591360}k_1 + \frac{4587}{44800}k_2 - \frac{113}{3584}k_3 + \frac{7139}{67200}k_4 - \frac{1297}{17920}k_5 + \frac{743}{8960}k_6 - \frac{106}{1925}k_7 + \frac{1041}{89600}k_8\right] \quad (17)$$

$$y'_{n+1} = y'_n + hy''_n + h^2\left[\frac{4321}{92400}k_1 + \frac{321}{1400}k_2 - \frac{3}{112}k_3 + \frac{529}{2100}k_4 - \frac{57}{560}k_5 + \frac{249}{1400}k_6 - \frac{192}{1925}k_7 + \frac{9}{400}k_8\right]$$

$$y''_{n+1} = y''_n + h\left[\frac{41}{840}k_1 + \frac{9}{35}k_2 + \frac{9}{280}k_3 + \frac{34}{105}k_4 + \frac{9}{280}k_5 + \frac{9}{35}k_6 + 0k_7 + \frac{41}{840}k_8\right]$$

Where

$$k_1 = f(x_n, y_n, y'_n, y''_n)$$

$$k_2 = f\Big[x_n + \frac{1}{6}h,\ y_n + \frac{1}{6}hy'_n + \frac{\left(\frac{1}{6}h\right)^2}{2}y''_n + h^3\left(\frac{15468791}{36787322880}k_1 + \frac{6888223}{8360755200}k_2 - \frac{99973}{74317824}k_3 + \frac{8899517}{4180377600}k_4 - \frac{417913}{159252480}k_5 + \frac{17489}{5308416}k_6 - \frac{451777}{179625600}k_7 + \frac{9837389}{16721510400}k_8\right), y'_n + \frac{1}{36}hy''_n + h^2\left(\frac{11621831}{1916006400}k_1 + \frac{209749}{12441600}k_2 - \frac{18097}{774144}k_3 + \frac{1364569}{43545600}k_4 - \frac{374867}{11612160}k_5 + \frac{329363}{9676800}k_6 - \frac{44749}{1871100}k_7 + \frac{899929}{174182400}k_8\right), y''_n + h\left(\frac{4553}{90720}k_1 + \frac{107293}{544320}k_2 - \frac{3727}{20160}k_3 + \frac{19001}{90720}k_4 - \frac{4271}{22680}k_5 + \frac{3559}{20160}k_6 - \frac{200}{1701}k_7 + \frac{4381}{181440}k_8\right)\Big]$$

$$k_3 = f\Big[x_n + \frac{1}{3}h,\ y_n + \frac{1}{3}hy'_n + \frac{\left(\frac{1}{3}h\right)^2}{2}y''_n + h^3\left(\frac{507649}{239500800}k_1 + \frac{96043}{13996800}k_2 - \frac{3523}{483840}k_3 + \frac{7577}{777600}k_4 - \frac{131389}{13063680}k_5 + \frac{12997}{1209600}k_6 - \frac{4586}{601425}k_7 + \frac{36287}{21772800}k_8\right), y'_n + \frac{1}{9}hy''_n + h^2\left(\frac{35699}{2494800}k_1 + \frac{58253}{1020600}k_2 - \frac{653}{15120}k_3 + \frac{3271}{56700}k_4 - \frac{7619}{136080}k_5 + \frac{2111}{37800}k_6 - \frac{53696}{1403325}k_7 + \frac{1831}{226800}k_8\right), y''_n + h\left(\frac{4027}{83160}k_1 + \frac{2227}{8505}k_2 - \frac{103}{2520}k_3 + \frac{26}{189}k_4 - \frac{3047}{22680}k_5 + \frac{41}{315}k_6 - \frac{8192}{93555}k_7 + \frac{137}{7560}k_8\right)\Big]$$

$$k_4 = f\Big[x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hy'_n + \frac{\left(\frac{1}{2}h\right)^2}{2}y''_n + h^3\left(\frac{1306699}{252313600}k_1 + \frac{2063407}{103219200}k_2 - \frac{66569}{4587520}k_3 + \frac{74563}{3440640}k_4 - \frac{296287}{13762560}k_5 + \frac{254063}{11468800}k_6 - \frac{1369}{88704}k_7 + \frac{75989}{22937600}k_8\right), y'_n + \frac{1}{4}hy''_n + h^2\left(\frac{529783}{23654400}k_1 + \frac{107939}{1075200}k_2 - \frac{5967}{143360}k_3 + \frac{2277}{25600}k_4 - \frac{11843}{143360}k_5 + \frac{29097}{258400}k_6 - \frac{1277}{23100}k_7 + \frac{24937}{2150400}k_8\right), y''_n + h\left(\frac{361}{7392}k_1 + \frac{115}{448}k_2 + \frac{81}{2240}k_3 + \frac{859}{3360}k_4 - \frac{6}{35}k_5 + \frac{351}{2240}k_6 - \frac{8}{77}k_7 + \frac{143}{6720}k_8\right)\Big]$$

$$k_5 = f\Big[x_n + \frac{2}{3}h,\ y_n + \frac{2}{3}hy'_n + \frac{\left(\frac{2}{3}h\right)^2}{2}y''_n + h^3\left(\frac{1722463}{179625600}k_1 + \frac{328987}{8164800}k_2 - \frac{7613}{362880}k_3 + \frac{23687}{583200}k_4 - \frac{40651}{1088640}k_5 + \frac{4877}{129600}k_6 - \frac{36392}{1403325}k_7 + \frac{89921}{16329600}k_8\right), y'_n + \frac{4}{9}hy''_n + h^2\left(\frac{57131}{1871100}k_1 + \frac{4061}{28350}k_2 - \frac{137}{3780}k_3 + \frac{5989}{42525}k_4 - \frac{169}{1620}k_5 + \frac{983}{9450}k_6 - \frac{11008}{155925}k_7 + \frac{2509}{170100}k_8\right), y''_n + h\left(\frac{1517}{31185}k_1 + \frac{440}{1701}k_2 + \frac{8}{35}k_3 + \frac{976}{2835}k_4 - \frac{193}{2835}k_5 + \frac{8}{63}k_6 - \frac{8192}{93555}k_7 + \frac{52}{2835}k_8\right)\Big]$$

$$k_6 = f\Big[x_n + \frac{5}{6}h,\ y_n + \frac{5}{6}hy'_n + \frac{\left(\frac{5}{6}h\right)^2}{2}y''_n + h^3\left(\frac{3423275}{222953472}k_1 + \frac{9711475}{143327232}k_2 - \frac{73375}{2752512}k_3 + \frac{3836375}{55738368}k_4 - \frac{36847625}{668860416}k_5 + \frac{235775}{4128768}k_6 - \frac{76625}{1959552}k_7 + \frac{1840775}{222953472}k_8\right), y'_n + \frac{25}{36}hy''_n + h^2\left(\frac{987125}{25546752}k_1 + \frac{1947425}{10450944}k_2 - \frac{24625}{774144}k_3 + \frac{114475}{580608}k_4 - \frac{737125}{6967296}k_5 + \frac{7325}{55296}k_6 - \frac{2825}{32076}k_7 + \frac{42475}{2322432}k_8\right), y''_n + h\left(\frac{295}{6048}k_1 + \frac{28025}{108864}k_2 + \frac{125}{4032}k_3 + \frac{1975}{6048}k_4 + \frac{125}{4536}k_5 + \frac{955}{4032}k_6 - \frac{200}{1701}k_7 + \frac{275}{12096}k_8\right)\Big]$$

$$k_7 = f\Big[x_n + \frac{11}{12}h,\ y_n + \frac{11}{12}hy'_n + \frac{\left(\frac{11}{12}h\right)^2}{2}y''_n + h^3\left(\frac{80235165461}{4280706662400}k_1 + \frac{25739291941}{305764761600}k_2 - \frac{2777795669}{95126814720}k_3 + \frac{528307175}{6115295232}k_4 - \frac{18209699827}{285380444160}k_5 + \frac{16417236803}{237817036800}k_6 - \frac{5591773}{119439360}k_7 + \frac{42183930041}{4280706662400}k_8\right), y'_n + \frac{121}{144}hy''_n + h^2\left(\frac{1904306833}{44590694400}k_1 + \frac{462557479}{2225934720}k_2 - \frac{28934609}{990904320}k_3 + \frac{2501118037}{11147673600}k_4 - \frac{307957463}{2972712960}k_5 + \frac{383153639}{2477260800}k_6 - \frac{4209227}{43545600}k_7 + \frac{895293157}{44590694400}k_8\right), y''_n + h\left(\frac{905773}{18579456}k_1 + \frac{7180745}{27869184}k_2 + \frac{310123}{10321920}k_3 + \frac{7643933}{23224320}k_4 + \frac{2019127}{92897280}k_5 + \frac{1476079}{5160960}k_6 - \frac{4213}{54432}k_7 + \frac{1925957}{92897280}k_8\right)\Big]$$

$$k_8 = f\Big[x_n + h,\ y_n + hy'_n + \frac{h^2}{2}y''_n + h^3\left(\frac{13289}{591360}k_1 + \frac{4587}{44800}k_2 - \frac{113}{3584}k_3 + \frac{7139}{67200}k_4 - \frac{1297}{17920}k_5 + \frac{743}{8960}k_6 - \frac{106}{1925}k_7 + \frac{1041}{89600}k_8\right), y'_n + hy''_n + h^2\left(\frac{4321}{92400}k_1 + \frac{321}{1400}k_2 - \frac{3}{112}k_3 + \frac{529}{2100}k_4 - \frac{57}{560}k_5 + \frac{249}{1400}k_6 - \frac{192}{1925}k_7 + \frac{9}{400}k_8\right), y''_n + h\left(\frac{41}{840}k_1 + \frac{9}{35}k_2 + \frac{9}{280}k_3 + \frac{34}{105}k_4 + \frac{9}{280}k_5 + \frac{9}{35}k_6 + 0k_7 + \frac{41}{840}k_8\right)\Big]$$

Also, following the same analysis in getting the R-K method conversion from the equation (14) of $k = 4$, a nine-stage implicit hybrid R-K method is obtained and it is expressed as

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + h^3\left[\frac{136361}{8164800}k_1 - \frac{53552}{1964655}k_2 + \frac{32026}{127575}k_3 - \frac{1963904}{4975425}k_4 + \frac{161333}{408240}k_5 - \frac{1180048}{9823275}k_6 + \frac{1106969}{28576800}k_7 + \frac{3267619}{471517200}k_8 + \frac{41347}{233513280}k_9\right] \quad (18)$$

$$y'_{n+1} = y_n' + hy_n'' + h^2\left[\frac{10607}{85050}k_1 - \frac{66560}{130977}k_2 + \frac{392192}{297675}k_3 - \frac{2025472}{1658475}k_4 + \frac{628}{1215}k_5 + \frac{375808}{3274425}k_6 + \frac{23741}{297675}k_7 + \frac{709964}{9823275}k_8 + \frac{30332}{8513505}k_9\right]$$

$$y''_{n+1} = y_n'' + h\left[\frac{-97571}{28350}k_1 + \frac{27017216}{1091475}k_2 - \frac{7245824}{99225}k_3 + \frac{62881792}{552825}k_4 - \frac{267664}{2835}k_5 + \frac{37830656}{1091475}k_6 - \frac{186386}{99225}k_7 + \frac{1755472}{3274425}k_8 + \frac{1784869}{28378350}k_9\right]$$

Where

$$k_1 = f(x_n, y_n, y_n', y_n'')$$

$$k_2 = f[x_n + \tfrac{1}{16}h,\ y_n + \tfrac{1}{16}hy_n' + \frac{(\frac{1}{16}h)^2}{2}\ y_n'' + h^3 \Big(\frac{3149264621}{136982613196800}k_1 + \frac{75334213}{2060090081280}k_2 - \frac{331802689}{7491236659200}k_3 + \frac{32212667}{652138905600}k_4 - \frac{243084367}{6849130659840} + \frac{124891967}{10300450406400}k_6 - \frac{314624011}{479439146188800}k_7 + \frac{434083399}{7910745912115200}k_8 - \frac{112968551}{27423919161999360}k_9\Big), y_n' + \frac{1}{256}hy_n'' + h^2\Big(\frac{310461983}{356725555200}k_1 + \frac{11804197}{5364817920}k_2 - \frac{7126219}{2786918400}k_3 + \frac{17703473}{6793113600}k_4 - \frac{28929073}{17836277760}k_5 + \frac{12623021}{26824089600}k_6 - \frac{19978081}{1248539443200}k_7 + \frac{17173873}{20600900812800}k_8 - \frac{451499}{10202350878720}k_9\Big), y_n'' + h\Big(\frac{142834651}{7431782400}k_1 + \frac{39354967}{558835200}k_2 - \frac{22719479}{406425600}k_3 + \frac{1721563}{35380800}k_4 - \frac{9901433}{371589120}k_5 + \frac{3866497}{558835200}k_6 - \frac{4669709}{26011238400}k_7 + \frac{3056609}{429185433600}k_8 - \frac{2271989}{7439214182400}k_9\Big)]$$

$$k_3 = f[x_n + \tfrac{1}{8}h,\ y_n + \tfrac{1}{8}hy_n' + \frac{(\frac{1}{8}h)^2}{2}\ y_n'' + h^3\Big(\frac{61143343}{535088332800}k_1 + \frac{13639819}{40236134400}k_2 - \frac{243337}{836075520}k_3 + \frac{746251}{2547417600}k_4 - \frac{4818869}{26754416640}k_5 + \frac{2101741}{40236134400}k_6 - \frac{3485273}{1872809164800}k_7 + \frac{671449}{6180270243840}k_8 - \frac{501719}{76517631590400}k_9\Big), y_n' + \frac{1}{64}hy_n'' + h^2\Big(\frac{1429061}{696729600}k_1 + \frac{803977}{104781600}k_2 - \frac{35507}{7620480}k_3 + \frac{32077}{6633900}k_4 - \frac{443}{155520}k_5 + \frac{82099}{104781600}k_6 - \frac{57133}{2438553600}k_7 + \frac{4369}{4023613440}k_8 - \frac{37087}{697426329600}k_9\Big), y_n'' + h\Big(\frac{133883}{7257600}k_1 + \frac{104407}{1091475}k_2 - \frac{2197}{396900}k_3 + \frac{16409}{552825}k_4 - \frac{26101}{1451520}k_5 + \frac{482}{99225}k_6 - \frac{6689}{50803200}k_7 + \frac{8923}{1676505600}k_8 - \frac{19}{82555200}k_9\Big)]$$

$$k_4 = f[x_n + \tfrac{3}{16}h,\ y_n + \tfrac{3}{16}hy_n' + \frac{\left(\frac{3}{16}h\right)^2}{2}\ y_n'' + h^3\Big(\frac{471290557}{1691143372800}k_1 + \frac{127619209}{127166054400}k_2 - \frac{52002737}{92484403200}k_3 + \frac{215359}{322043904}k_4 - \frac{33720959}{84557168640}k_5 + \frac{14269999}{1271660054400}k_6 - \frac{865139}{236760072192}k_7 + \frac{18836087}{97663529779200}k_8 - \frac{18252083}{1692834516172800}k_9\Big), y_n' + \frac{9}{256}hy_n'' + h^2\Big(\frac{14126737}{4404019200}k_1 + \frac{4500691}{331161600}k_2 - \frac{905351}{240844800}k_3 + \frac{25777}{3354624}k_4 - \frac{940343}{220200960}k_5 + \frac{54757}{47308800}k_6 - \frac{103699}{3082813440}k_7 + \frac{55073}{36333158400}k_8 - \frac{320207}{4408423219200}k_9\Big), y_n'' + h\Big(\frac{1712843}{91750400}k_1 + \frac{642711}{6899200}k_2 + \frac{127833}{5017600}k_3 + \frac{30299}{436800}k_4 - \frac{116169}{4587520}k_5 + \frac{43521}{6899200}k_6 - \frac{51357}{321126400}k_7 + \frac{33457}{5298585600}k_8 - \frac{24837}{91842150400}k_9\Big)]$$

$$k_5 = f[x_n + \tfrac{1}{4}h,\ y_n + \tfrac{1}{4}hy_n' + \frac{\left(\frac{1}{4}h\right)^2}{2}\ y_n'' + h^3\Big(\frac{4309871}{8360755200}k_1 + \frac{182849}{89812800}k_2 - \frac{343531}{457228800}k_3 + \frac{7499}{5686200}k_4 - \frac{58865}{83607552}k_5 + \frac{122657}{628689600}k_6 - \frac{25087}{4180377600}k_7 + \frac{142141}{482833612800}k_8 - \frac{129121}{8369115955200}k_9\Big), y_n' + \frac{1}{16}hy_n'' + h^2\Big(\frac{47611}{10886400}k_1 + \frac{63631}{3274425}k_2 - \frac{2749}{1190700}k_3 + \frac{1718}{127575}k_4 - \frac{559}{108864}k_5 + \frac{4783}{3274425}k_6 - \frac{793}{19051200}k_7 + \frac{571}{314344800}k_8 - \frac{71}{838252800}k_9\Big), y_n'' + h\Big(\frac{8401}{453600}k_1 + \frac{103184}{1091475}k_2 + \frac{1894}{99225}k_3 + \frac{4576}{42525}k_4 + \frac{583}{90720}k_5 + \frac{4303}{1091475}k_6 - \frac{403}{3175200}k_7 + \frac{551}{104781600}k_8 - \frac{1}{4365900}k_9\Big)]$$

$$k_6 = f[x_n + \tfrac{5}{16}h,\ y_n + \tfrac{5}{16}hy_n' + \frac{\left(\frac{5}{16}h\right)^2}{2}\ y_n'' + h^3\Big(\frac{4521680725}{5479304527872}k_1 + \frac{1414749025}{412018016256}k_2 - \frac{256034825}{299649466368}k_3 + \frac{61624075}{26085556224}k_4 - \frac{1332971875}{1369826131968}k_5 + \frac{124043575}{412018016256}k_6 - \frac{172261475}{19177565847552}k_7 + \frac{135672575}{316429836484608}k_8 - \frac{120539675}{5484783832399872}k_9\Big), y_n' + \frac{25}{256}hy_n'' + h^2\Big(\frac{78912475}{14269022208}k_1 + \frac{554725}{21897216}k_2 - \frac{804425}{780337152}k_3 + \frac{59275}{2985984}k_4 - \frac{10629625}{3567255552}k_5 + \frac{2334625}{1072963584}k_6 - \frac{384275}{7134511104}k_7 + \frac{1921925}{824036032512}k_8 - \frac{119225}{1098714710016}k_9\Big), y_n'' + h\Big(\frac{5572115}{297271296}k_1 + \frac{2070575}{22353408}k_2 + \frac{429425}{16257024}k_3 + \frac{127475}{1415232}k_4 + \frac{4238875}{74317824}k_5 + \frac{621545}{22353408}k_6 - \frac{219925}{1040449536}k_7 + \frac{130825}{17167417344}k_8 - \frac{94525}{297568567296}k_9\Big)]$$

$$k_7 = f[x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hy_n' + \frac{\left(\frac{1}{2}h\right)^2}{2}\ y_n'' + h^3\Big(\frac{1124353}{522547200}k_1 + \frac{397321}{39293100}k_2 - \frac{45893}{285776800}k_3 + \frac{1934}{199017}k_4 - \frac{81131}{26127360}k_5 + \frac{12461}{3572100}k_6 + \frac{5617}{73156608}k_7 - \frac{60157}{30177100800}k_8 - \frac{5843}{47551795200}k_9\Big),\ y_n' + \frac{1}{4}hy_n'' + h^2\Big(\frac{2917}{340200}k_1 + \frac{14128}{297675}k_2 - \frac{668}{42525}k_3 + \frac{26528}{331695}k_4 - \frac{3197}{68040}k_5 + \frac{161552}{3274425}k_6 + \frac{227}{95256}k_7 - \frac{281}{7144200}k_8 + \frac{79}{48648600}k_9\Big), y_n'' + h\Big(\frac{53}{113400}k_1 + \frac{256768}{1091475}k_2 - \frac{44512}{99225}k_3 + \frac{528896}{552825}k_4 - \frac{4849}{5670}k_5 + \frac{610048}{1091475}k_6 + \frac{10489}{198450}k_7 - \frac{2753}{6548850}k_8 + \frac{1523}{113513400}k_9\Big)]$$

$$k_8 = f[x_n + \tfrac{3}{4}h,\ y_n + \tfrac{3}{4}hy_n' + \frac{\left(\frac{3}{4}h\right)^2}{2}\ y_n'' + h^3\Big(\frac{537247}{103219200}k_1 + \frac{1788711}{7761600}k_2 - \frac{31}{1128960}k_3 + \frac{12709}{491400}k_4 - \frac{43781}{5160960}k_5 + \frac{140689}{7761600}k_6 + \frac{2306743}{361267200}k_7 + \frac{269161}{1192181760}k_8 - \frac{713777}{103322419200}k_9\Big),\ \ y_n' + \frac{9}{16}hy_n'' + h^2\Big(\frac{2399}{134400}k_1 + \frac{113}{5775}k_2 + \frac{578}{2940}k_3 - \frac{926}{2925}k_4 +$$

$$\frac{1321}{3360}k_5-\frac{4393}{40425}k_6+\frac{173}{2400}k_7+\frac{6623}{1552320}k_8-\frac{10723}{134534400}k_9\Big), y_n''+h\Big(\frac{1973}{5600}k_1-\frac{31728}{13475}k_2+\frac{9222}{1225}k_3-\frac{82336}{6825}k_4+\frac{11919}{1120}k_5-\frac{53808}{13475}k_6+\frac{22341}{39200}k_7+\frac{90463}{1293600}k_8-\frac{933}{113513400}k_9\Big)\Big]$$

$$k_9=f\Big[x_n+h,\ y_n+hy_n'+\frac{h^2}{2}y_n''+h^3\Big(\frac{136361}{8164800}k_1-\frac{53552}{1964655}k_2+\frac{32026}{127575}k_3-\frac{1963904}{4975425}k_4+\frac{161333}{408240}k_5-\frac{1180048}{9823275}k_6+\frac{1106969}{28576800}k_7+\frac{3267619}{471517200}k_8+\frac{41347}{233513280}k_9\Big), y_n'+hy_n''+h^2\Big(\frac{10607}{85050}k_1-\frac{66560}{130977}k_2+\frac{392192}{297675}k_3-\frac{2025472}{1658475}k_4+\frac{628}{1215}k_5+\frac{375808}{3274425}k_6+\frac{23741}{297675}k_7+\frac{709964}{9823275}k_8+\frac{30332}{8513505}k_9\Big), y_n''+h\Big(-\frac{97571}{28350}k_1+\frac{27017216}{1091475}k_2-\frac{7245824}{99225}k_3+\frac{62881792}{552825}k_4-\frac{267664}{2835}k_5+\frac{37830656}{1091475}k_6-\frac{186386}{99225}k_7+\frac{1755472}{3274425}k_8+\frac{1784869}{28378350}k_9\Big)\Big]$$

RESULTS AND DISCUSSION

The newly reformulated implicit hybrid R-K methods (17) and (18) were used to solve the following linear and nonlinear IVPs of third-order IVPs.

i. $y'''(x)=y^2+\cos^2(x)-\cos(x)-1$
$y(0)=y''(0)=0\,, y'(0)=1\,,\ \ h=0.1\ ,\ y(x)=\sin(x)$

ii. $y'''(x)=y-y'+y''$
$y(0)=1\,,\ y'(0)=0,\ y''(0)=-1\,, h=0.01\,,\ y(x)=\cos x$

iii. $y'''(x)=-y$
$y(0)=1,\ \ y'(0)=-1,\ y''(0)=1\,,\ h=0.1\,,\ y(x)=e^{-x}$

iv. $y'''(x)=\frac{3y'(x)}{4[y(x)]^4}$
$y(0)=1,\ y'(0)=\frac{1}{2}\,,\ y''(0)=\frac{1}{4},\ h=0.1\,,\ y(x)=\sqrt{x+1}$

Table of Result

Table 1: Approximate Solutions of Problem 1 with New Reformulated R-K Type Methods

X	Theoretical Solution	Reformulated R-K method at $k=3$	Reformulated R-K method at $k=4$
0.1	0.09983341665	0.09983349984	0.09983349984
0.2	0.19866933080	0.19867115126	0.19867115125
0.3	0.29552020666	0.29553109966	0.29553109964
0.4	0.38941834231	0.38945578261	0.38945578258
0.5	0.47942553860	0.47952088513	0.47952088509

Table 2: Absolute Errors of Problem 1 with New Reformulated R-K Type Methods

X	Errors of Reformulated R-K method at $k=3$	Errors of Reformulated R-K method at $k=4$
0.1	8.3193 x 10^{-8}	8.3193 x 10^{-8}
0.2	1.8205 x 10^{-6}	1.8205 x 10^{-6}
0.3	1.0893 x 10^{-5}	1.0893 x 10^{-5}
0.4	3.7440 x 10^{-5}	3.7440 x 10^{-5}
0.5	9.5347 x 10^{-5}	9.5346 x 10^{-5}

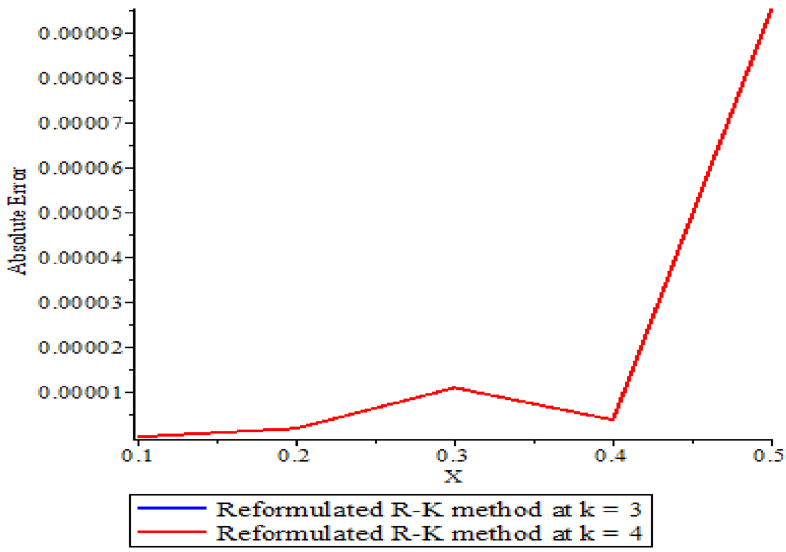

Figure 1: Error Graph of Problem 1

Table 3: Approximate Solutions of Problem 2 with New Reformulated R-K Type Methods and Muhammed and Adeniyi, (2014)

X	Theoretical Solution	Approximate solution of Muhammed and Adeniyi, (2014)	Reformulated R-K method at $k = 3$	Reformulated R-K method at $k = 4$
0.01	0.9999500004	0.9999506724	0.999950000166110	0.999950000352135
0.02	0.9998000067	0.9998013508	0.999800004489364	0.999800006962578
0.03	0.9995500337	0.9995520507	0.999550026277426	0.999550033548625
0.04	0.9992001067	0.9992027951	0.999200088819352	0.999200110151158
0.05	0.9987502604	0.9987536198	0.998750225383255	0.998750265869475

Table 4: Absolute Errors of Problem 2 with New R-K Type Methods Compared With Muhammed and Adeniyi, (2014)

X	Error of Muhammed and Adeniyi, (2014) at $k = 3$	Error of Reformulated R-K method at $k = 3$	Error of Reformulated R-K method at $k = 4$
0.01	6.72000 x 10^{-7}	2.33890 x 10^{-10}	4.78650 x 10^{-11}
0.02	1.34410 x 10^{-6}	2.21063 x 10^{-9}	2.62578 x 10^{-10}
0.03	2.01700 x 10^{-6}	7.42257 x 10^{-9}	1.51375 x 10^{-10}
0.04	2.68840 x 10^{-6}	1.78806 x 10^{-8}	3.45115 x 10^{-9}
0.05	3.35940 x 10^{-6}	3.50167 x 10^{-8}	5.46947 x 10^{-9}

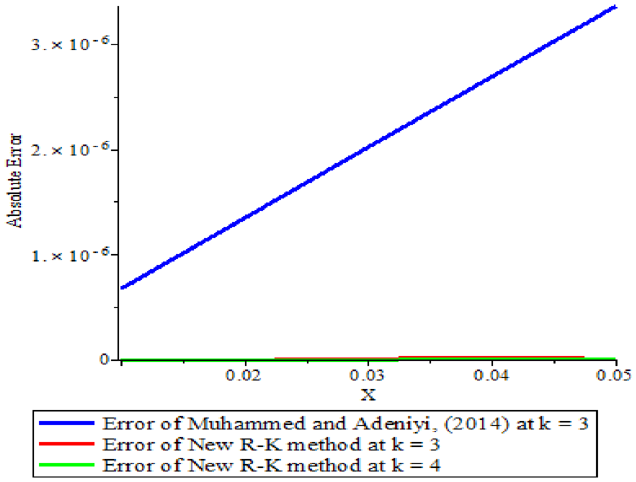

Figure 2: Error Graph of Problem 2

Table 5: Approximate Solutions of Problem 3 with New Reformulated R-K Type Methods and Ogunware Et Al., 2015 (HYBRID)

X	Theoretical Solution	Approximate solution of Ogunware et al., (2015)	Reformulated R-K method at $k = 3$	Reformulated R-K method at $k = 4$
0.1	0.904837418035959520	0.904837418026245840	0.904837418035960	0.904837418035960
0.2	0.818730753077981820	0.818730753015859850	0.818730753078282	0.818730753078282
0.3	0.740818220681717770	0.740818220522834860	0.740818220681718	0.740818220689698
0.4	0.670320046035639330	0.670320045749979720	0.670320046035640	0.670320046035640
0.5	0.606530659712633420	0.606530659251041330	0.606530659712634	0.606530659711145
0.6	0.548811636094026390	0.548811635406465050	0.548811636094027	0.548811636098742
0.7	0.496585303791409470	0.496585302838955580	0.496585303791410	0.496585303790258
0.8	0.449328964117221560	0.449328962847427850	0.449328964117222	0.449328964104142
0.9	0.406569659740599050	0.406569658101199830	0.406569659740600	0.406569659795365
1.0	0.367879441171442220	0.367879439119434160	0.367879441171443	0.367879441124963
1.1	0.332871083698079500	0.332871081181678460	0.332871083698080	0.332871083667984
1.2	0.301194211912202030	0.301194208881219510	0.301194211912202	0.301194211907496

Table 6: Absolute Errors of Problem 3 with New R-K Type Methods Compared With Ogunware Et Al (2015) (HYBRID)

X	Error of Ogunware et al., (2015) at $k = 3$	Error of Reformulated R-K method at $k = 3$	Error of Reformulated R-K method at $k = 4$
0.1	9.7136 x 10^{-12}	0.00	0.00
0.2	6.2121 x 10^{-11}	1.2600 x 10^{-13}	1.2600 x 10^{-13}
0.3	1.5888 x 10^{-10}	7.4060 x 10^{-12}	5.7400 x 10^{-13}
0.4	2.8565 x 10^{-10}	4.1900 x 10^{-13}	4.1900 x 10^{-13}

X	Error of Ogunware et al., (2015) at $k = 3$	Error of Reformulated R-K method at $k = 3$	Error of Reformulated R-K method at $k = 4$
0.5	4.6159 x 10^{-10}	1.1780 x 10^{-12}	3.1100 x 10^{-13}
0.6	6.8756 x 10^{-10}	4.9650 x 10^{-12}	2.5000 x 10^{-13}
0.7	9.5245 x 10^{-10}	1.0360 x 10^{-12}	1.1600 x 10^{-13}
0.8	1.2697 x 10^{-9}	1.3025 x 10^{-11}	5.5000 x 10^{-13}
0.9	1.6393 x 10^{-9}	5.8254 x 10^{-11}	3.4890 x 10^{-13}
1.0	2.0520 x 10^{-9}	4.8965 x 10^{-11}	2.4850 x 10^{-13}
1.1	2.5164 x 10^{-9}	3.3513 x 10^{-11}	3.4170 x 10^{-13}
1.2	3.0309 x 10^{-9}	1.0944 x 10^{-11}	6.2380 x 10^{-13}

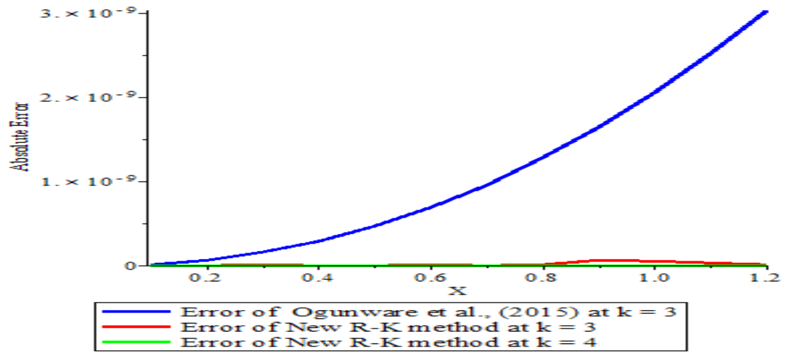

Figure 3: Error Graph of Problem 3.

Table 7: Approximate Solutions of Problem 4 with New Reformulated R-K Type Methods

X	Theoretical Solution	Reformulated R-K scheme at $k = 3$	Reformulated R-K scheme at $k = 4$
0.1	1.04880884820	1.04880657780	1.04880920900
0.2	1.09544511500	1.09541871790	1.09544866140
0.3	1.14017542510	1.14008062350	1.14018699910
0.4	1.18321595660	1.18299008540	1.18324197190
0.5	1.22474487140	1.22430992900	1.22479295580

Table 8: Absolute Errors of Problem 4 with New R-K type Methods

X	Error of Reformulated R-K method at $k = 3$	Error of Reformulated R-K method at $k = 4$
0.1	2.2704 x 10^{-6}	3.6080 x 10^{-7}
0.2	2.6397 x 10^{-5}	3.5464 x 10^{-6}
0.3	9.4802 x 10^{-5}	1.1574 x 10^{-5}
0.4	2.2587 x 10^{-4}	2.6015 x 10^{-5}
0.5	3.3494 x 10^{-4}	4.8084 x 10^{-5}

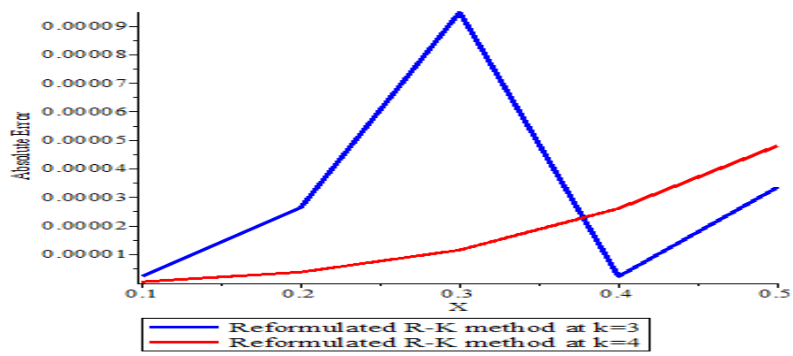

Figure 4: Error Graph of Problem 4

Numerical Results and Discussions

The proposed reformulated implicit R-K schemes with step lengths $k = 3$ and $k = 4$ were applied to selected linear and nonlinear third-order IVPs. The numerical results and absolute errors of each problem are presented in Tables $1 - 8$, while the corresponding graphs are illustrated in Figures $1 - 4$.

For the test problems 1 and 4, both proposed schemes produced numerical solutions that closely agree with the theoretical solutions, with small absolute errors at the selected mesh points. These results support the higher-order behavior established by the theoretical analysis. In particular, the $k = 4$ scheme generally produced smaller absolute errors than the $k = 3$ scheme, reflecting the advantage of its higher algebraic order.

For Problem 2, Muhammed and Adeniyi (2014) obtained an absolute error of 10^{-6} at $x = 0.05$, whereas the proposed R-K type scheme of $k = 3$ reduces their reported error by approximately two orders of magnitude given its absolute error of 10^{-8} while that of $k = 4$ scheme produced errors of 10^{-9}. For Problem 3, the proposed schemes were compared with the hybrid block method of Ogunware et al. (2015) where they reported an absolute error of 10^{-9} at $x = 1.2$. However, R-K type scheme of $k = 3$ and $k = 4$ produced errors of 10^{-11} and10^{-13}, respectively. The comparison at the intermediate mesh points also shows consistently smaller errors.

The graphical representations further indicate that the proposed schemes maintain low error levels over the computational intervals. This behavior is consistent with the theoretical analysis of the methods and provides numerical evidence of their effectiveness for the direct solution of third-order IVPs. Overall, the numerical experiments demonstrate that reformulating the LMM into implicit hybrid R-K schemes provides and effective high-order approach for the direct solution of third-order IVPs.

CONCLUSION

This study developed two hybrid R-K type schemes through the reformulation of LMMs for the direct numerical solution of third-order IVPs. The resulting schemes, with $k = 3$ and$k = 4$, were established to have orders eight and nine, respectively. The methods were shown to satisfy the relevant consistency, stability and convergence requirements.

Numerical experiments on selected linear and nonlinear third-order IVPs demonstrated that both schemes provide accurate numerical solutions, with the $k = 4$ scheme generally producing smaller absolute errors. The results therefore demonstrate the effectiveness of the proposed reformulation strategy and its potential as a systematic approach for constructing high-order R-K type schemes for third-order IVPs. Future work may investigate the extension of the reformulation framework to higher-order and stiff differential equations.

REFERENCES

Adeyefa, E. O., & Kuboye, J. O. (2020). Derivation of new numerical model capable of solving second and third order ordinary differential equations directly. *IAENG International Journal of Applied Mathematics, 50(2), 177-185.*

Adesanya, A. O., Odekunle, M. R., & Udoh, M. O. (2014). A class of hybrid block methods for direct solution of fourth order ordinary differential equations. *International Journal of Mathematics and Statistics Invention*, 2*(9), 23-30.*

Ascher, U. M., & Petzold, L. R. (1998). Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations. *Society for Industrial and Applied Mathematics.* http://dx.doi.org/10.1137/1.9781611971392

Badmus, A. M., & Subair, A. O. (2024). Implicit block and Runge-Kutta type methods for solution of second-order ordinary differential equations. *Applied Mathematics and Computational Intelligence (AMCI), 13(1), 109–127.* https://doi.org/10.58915/amci.v13iNo.1.229

Butcher, J. C. (2008). *Numerical methods for ordinary differential equations (2nd ed.).* John Wiley & Sons, Ltd. https://doi.org/10.1002/9780470753767

Duromola, M. K., Lawal, R. S., & Akinmoladun, O. M. (2024). Numerical integration of linear hybrid multistep block method for third-order ordinary differential equations (IVPs). *Scientific African, 24, Article e02129.* https://doi.org/10.1016/j.sciaf.2024.e02129

Fatunla, S. O. (1988). *Numerical Methods for Initial Value Problems in Ordinary Differential Equations.* Academic Press. https://doi.org/10.1016/C2013-0-10643-5

Fawzi, F. A., & Jumaa, M. H. (2022). The implementations special third-order ordinary differential equations (ODE) for 5th-order 3rd-stage diagonally implicit type Runge-Kutta method (DITRKM). *Ibn AL-Haitham Journal for Pure and Applied Sciences, 35(1), 92–101.* https://doi.org/10.30526/35.1.2803

Kuboye, J. O., Quadri, O. F., & Elusakin, O. R. (2020). Solving Third Order Ordinary Differential Equations Directly Using Hybrid Numerical Models. *Journal of the Nigerian Society of Physical Sciences, 2(2), 69-76.* https://doi.org/10.46481/jnsps.2020.43

Lawal, R. S., Adeusi, E. K., & Okafor, N. P. (2026). Effective method of solving one-third third-order IVPs with two off-grid points ordinary differential equations. *FUDMA Journal of Sciences, 10(5), 39–45.* https://doi.org/10.33003/fjs-2026-1005-4948

Muhammed, U., & Adeniyi, R. B. (2014). A Three Step Implicit Hybrid Linear Multistep Method for The Solution of Third Order ODEs. *Gen. Math. Notes, 25(1), 62-74.*

Ogunware, B. G., Omole, E. O., & Omole, O. O. (2015). Hybrid and Non-hybrid Implicit Schemes for Solving Third Order ODEs Using Block Method as Predictors. *Mathematical Theory and Modelling, 5(3), 10-25.*

Soomro, H., Zainuddin, N., Daud, H., & Sunday, J. (2022). Optimized hybrid block Adams method for solving first order ordinary differential equations. *Computers, Materials & Continua, 72(2), 2947–2961.* https://doi.org/10.32604/cmc.2022.025933