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A THREE-STEP INTERPOLATION TECHNIQUE WITH PERTURBATION TERM FOR DIRECT SOLUTION OF THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
In this paper, we developed a new three-step method for numerical solution of third order ordinary differential equations. Interpolation and collocation methods were used by choosing interpolation points at $s=3$ steps points using power series, while collocation points at $r=(k-1)$ step points, using a combination of powers series and perturbation terms gotten from the Legendre polynomials, giving rise to a polynomial of degree $r+s-2$ and $r+s$ equations. All the analysis on the method derived shows that it is zero-stable, convergent and the region of stability is absolutely stable. Numerical examples were provided to test the performance of the method. Results obtained when compared with existing methods in the literature, shows that the method is accurate and efficient.

Keywords: Three-step Interpolation technique; Legendre Polynomial; Perturbation Term; Third-order ODEs; Convergent; Power Series; Absolutely stable.

INTRODUCTION

Countless real life problems in sciences, and engineering are model of third-order ordinary differential equations of initial value problems. Interestingly, some differential equations emanating from the modelling of physical phenomena, often lacks analytic solutions, henceforth the development of numerical method to obtain approximation solutions becomes indispensable. (see Ehigie *et al.,* 2010). This manuscript examines the numerical solutions of third-order ordinary differential equations with initial conditions of the form

$$y'''(x)=f(x,y,y',y''),\ y(a)=\eta_0,\ y'(a)=\eta_1,\ y''(a)=\eta_2 \quad (1)$$

In the past, equation (1) is solved by method of reducing it to its equivalent system of first order ordinary differential equations and thereafter appropriate numerical method for first order ODEs would be applied to solve the systems. However, the reduction of higher order ordinary differential equations to a system of first order has a lot of misfortunes which includes; utilization of human effort, computational weigh down and non-economization of computer time as discussed by the following authors; Awoyemi (1999), Awoyemi (1999), Awoyemi (2001), Fatunla (1998), Lambert (1973), Gout et al., (1973), Bruguano and Trigiante (1998) just to mention few.

In strive to cater for the difficulties encountered in reduction method and also bring about upgrading on numerical methods. The authors are Anake al., (2012), Omar and Suleiman (2003), Omar and Suleiman (2005), Ogunware et al., (2015), Abhulimen and Aigbiremhon (2018), Aigbiremhon and Ukpebor (2019), Badmus and Yahaya (2009) developed block methods for solving higher order ODEs in a straight line which the accuracy is better than when it is reduced to system of first order ordinary differential equations.

Different approach of Linear multistep method for solving equation (1) directly have been developed by some erudite researchers such as Adoghe al., (2016), Olabode (2007), Mohammed and Adeniyi (2014), Adesanya (2013), Olabode B.T. (2009), Abualnaja (2015), Jator (2007), Lambert (1991), Henrici (1962) and Ogunware & Omole (2020).

They proposed direct methods and implemented in block mode for the solution of third-order ordinary differential equations. In the light of this, we projected a three-step method using power series as the interpolation equation and combination of power series with Legendre polynomial as the perturbation term as the collocation equation to solve equation (1) directly. The perturbation terms help in minimising the error in the problems, thereby increase the accuracy of the new method. In the next section, the development or derivation of the method is specified;

DEVELOPMENT OF THE METHOD

In this segment, we present the derivation of discrete method to solve (1) at a series of nodal points $x_n = x_0 + nh$, where $h > 0$ is the step length or grid size defined by $h = x_{n+1} - x_n$ and $y(x)$ denotes the true solution to (1) while the approximate solution is denoted by the point series.

$$y_{(x)} = c_0 x_n^0 + c_1 x_n^1 + c_2 x_n^2 + \ldots + c_k x_n^k \tag{2}$$

The proposed method depends on the perturbed collocation method with respect to the power series with the Legendre polynomials as the perturbation term. Interpolation and collocation approach were used by choosing interpolation point at $s = 3$ grid points and collocation points at $r = (k-1)$ step points. We have a polynomial of degree $r + s - 2$ and $(r+s)$ equations.

In the first place, we consider the approximation solution of (1) in the power series.

$p_i(x) = x^i,\ i = 0,1,\ldots,k$

Hence (2) becomes

$$y_k(x) = c_i p_i(x) = \sum_{i=0}^{k} c_i x^i \tag{3}$$

The third derivatives of (3) is given as

$$y_k'''(x) = c_i p_i'''(x) = \sum_{i=0}^{R} i(i-1)(i-2) c_i x^{i-3} \tag{4}$$

Combining equation (1) and (4), with the perturbation term, we have

$$\sum_{i=n} c_i p_i'''(x) = f\left(x, y, y', y''\right) + \lambda L_k(x_{n+i}), i = 2(1)k \tag{5}$$

Where $L_k(x)$ is the Legendre polynomial of degree k, valid in $x_n \le x \le x_{n+k}$ and λ is a perturbed parameter.

In particular, we deal with case k = 3 (three-step points), where equation (3) and (5) are the interpolation and collocation equations correspondingly.

The well-known Legendre polynomials are generated using the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{1}{dx^n} \left[(x^2-1)^n\right]$$, where

$L_0(x) = 1, L_1(x) = x$. The rest are computed using the recurrence formula.

$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), i = 1, 2, \ldots$$

giving: $L_2(x) = \frac{1}{2}\left(3x^2 - 1\right),\ L_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$

$$L_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right),\ L_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \text{ etc.} \tag{6}$$

In order to use these polynomials in the interval $[x_n, x_{n+k}]$, we describe the shifted Legendre polynomials by introducing the change of variables.

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)} \quad \text{(see Abualnaja, 2015)} \tag{7}$$

Interpolating (3) at s grid points and collocating (5), at k-1 grid points respectively leads to the following systems of equation.

$$\sum_{i=0}^{s} c_i p_i(x) = y_{x+s}, s = 0,1,2 \tag{8}$$

$$\sum_{i=0}^{k} c_i p_i'''(x) = f_{n+j} + \lambda L_k(x_{n+j}), j = 2(1)k \tag{9}$$

Now, we take the polynomial $L_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$ from equation (6) and use (7) to obtain value for $L_3(x_{n+2})$ and $L_3(x_{n+3})$ to be $-\frac{11}{27}$ and 1 respectively.

In addition, from (4), $c_0 p_0'''(x)=0, c_1 p_1'''(x)=0, c_2, p_2'''(x)=0,\ c_3 p_3'''(x)=6c_3$, then (9) will reduced to the form

$$0+0+0+6c_3 = f\left(x,y,y',y''\right)+\lambda L_3\left(x_{n+i}\right), i=2 and 3 \tag{10}$$

We now collocate equation (10) at $x_{n+i}, i=2$ and 3 and interpolate equation (2) at $x_{n+i}, i=0,1,2$ to produce a system of 5 equations with $c_i, i=0,1,2,3 and\ \lambda$ which in matrix from is

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & 0 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & 0 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & 0 \\ 0 & 0 & 0 & 6 & 11/27 \\ 0 & 0 & 0 & 6 & -1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \lambda \end{pmatrix}=\begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{11}$$

Equation (11) is solved by Gaussian elimination method to obtain the value of the unknown parameters, $c_i, (i=0,1,2,3)$ and λ, which are substituted into (2) to yield a continuous implicit three steps method in the form of a continuous linear multistep method describe by the formula

$$y_{(x)} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + h^3 \sum_{j=2}^{3} \beta_j(x) f_{n+j},\ j=2(1)k \tag{12}$$

where

$$\begin{aligned} \alpha_0(t) &= \frac{1}{2}t + \frac{1}{2}t^2 \\ \alpha_1(t) &= -t^2 - 2t \\ \beta_2(t) &= \frac{9}{76}h^3t^3 + \frac{27}{76}h^3t^2 + \frac{9}{38}h^3t + \frac{29}{38} \\ \beta_3(t) &= \frac{11}{228}h^3t^3 + \frac{11}{76}h^3t^2 + \frac{11}{114}h^3t - \frac{27}{38} \end{aligned} \tag{13}$$

as the continuous functions of t with $t=\dfrac{x-x_{n+2}}{h}$, as the transformation equation.

Using (13) for $x = x_{n+3}$, at $t=1$, equation (12) reduces to

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{19}\left[-8 f_{n+3} + 27 f_{n+2}\right] \tag{14}$$

Differentiating (13) yield

$$\begin{aligned} \alpha_0^1(t) &= t + 1/2 \\ \alpha_1^1(t) &= -2t - 2 \\ \alpha_2^1(t) &= t + 3/2 \\ \beta_2^1(t) &= \frac{27}{76}h^3t^2 + \frac{27}{38}h^3t + \frac{9}{38}h^3 \\ \beta_3^1(t) &= \frac{11}{76}h^3t^2 + \frac{11}{38}h^3t + \frac{11}{114}h^3 \end{aligned} \tag{15}$$

Evaluating (15) at $x = x_n, x_{n+1}, x_{n+2}\ and\ x_{n+3}$

Where $t=-2,-1,0 and 1$, (12) yield the following discrete methods respectively.

$$\begin{aligned} 114hy_n^1 + 57y_{n+2} - 228y_{n+1} + 171y_n &= h^3\left[11f_{n+3} + 27f_{n+2}\right] \\ 228hy_{n+1}' - 114y_{n+2} + 114y_n &= h^3\left[-11f_{n+3} - 27f_{n+2}\right] \\ 114hy_{n+2}' - 171y_{n+2} + 228y_{n+1} - 57y_n &= h^3\left[11f_{n+3} + 27f_{n+2}\right] \end{aligned} \tag{16}$$

$$228hy'_{n+3}-570y_{n+2}+912y_{n+1}-342y_n=h^3\left[121f_{n+3}+297f_{n+2}\right]$$

Furthermore, differentiating (13) twice, we have

$$\alpha''_0(t)=1,\qquad \alpha''_1(t)=-2,\qquad \alpha''_2(t)=1$$

$$\beta''_2(t)=\frac{27}{38}h^3(1+t),\beta''_3(t)=\frac{11}{38}h^3(1+t) \tag{17}$$

Evaluating (17) at $x=x_n,x_{n+1},x_{n+2}$ *and* x_{n+3} where $t=-2,-1,0$ and 1, (12) yield the following discrete method respectively

$$38h^2y''_n-38y_{n+2}+76y_{n+1}-38y_n=h^3\left[-11f_{n+3}-27f_{n+2}\right]$$

$$h^2y''_{n+1}-y_{n+2}+2y_{n+1}-y_n=0$$

$$38h^2y''_{n+2}-38y_{n+2}+76y_{n+1}-38y_n=h^3\left[11f_{n+3}+27f_{n+2}\right] \tag{18}$$

$$19h^2y''_{n+3}-19y_{n+2}+38y_{n+1}-19y_n=h^3\left[11f_{n+3}+27f_{n+2}\right]$$

Now we obtained the modified block formula form (14), (16) and (18) as;

$$\begin{pmatrix} 57 & -57 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \\ -228 & 57 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -114 & 0 & 228h & 0 & 0 & 0 & 0 & 0 \\ 228 & -171 & 0 & 0 & 114h & 0 & 0 & 0 & 0 \\ 912 & -570 & 0 & 0 & 0 & 228h & 0 & 0 & 0 \\ 76 & -38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & h^2 & 0 & 0 \\ 76 & -38 & 0 & 0 & 0 & 0 & 0 & 38h^2 & 0 \\ 38 & -19 & 0 & 0 & 0 & 0 & 0 & 0 & 19h^2 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y'_{n+1} \\ y'_{n+1} \\ y'_{n+3} \\ y''_{n+1} \\ y''_{n+2} \\ y''_{n+3} \end{pmatrix} = \begin{pmatrix} 19 & 0 & 0 \\ -171 & -114h & 0 \\ -114 & 0 & 0 \\ 57 & 0 & 0 \\ 342 & 0 & 0 \\ 38 & 0 & 38h^2 \\ 1 & 0 & 0 \\ 38 & 0 & 0 \\ 19 & 0 & 0 \end{pmatrix} \begin{bmatrix} y_n \\ y'_n \\ y''_n \end{bmatrix}$$

$$+\begin{bmatrix} 27h^3 & -8h^3 \\ 27h^3 & 11h^3 \\ 27h^3 & -11h^3 \\ 27h^3 & 11h^3 \\ 297h^3 & 121h^2 \\ 27h^3 & -11h^3 \\ 0 & 0 \\ 27h^3 & 11h^3 \\ 27h^3 & 11h^3 \end{bmatrix} \begin{bmatrix} f_{n+2} \\ f_{n+3} \end{bmatrix} \tag{19}$$

Taking the normalized version of (19), we obtained the block solution

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y'_{n+1}\\ y'_{n+2}\\ y'_{n+3}\\ y''_{n+1}\\ y''_{n+2}\\ y''_{n+3} \end{pmatrix}=\begin{pmatrix} 1&h&\frac{h^2}{2}\\ 1&2h&2h^2\\ 1&3h&\frac{9}{2}h^2\\ 0&1&h\\ 0&1&2h\\ 0&1&3h\\ 0&0&1\\ 0&0&1\\ 0&0&1 \end{pmatrix}\begin{pmatrix} y_n\\ y'_n\\ y''_n \end{pmatrix}+\begin{pmatrix} \frac{9h^3}{76}&\frac{11h^3}{228}\\ \frac{18h^3}{19}&\frac{22h^3}{57}\\ \frac{297h^3}{76}&\frac{45h^3}{76}\\ \frac{27h^2}{76}&\frac{11h^2}{76}\\ \frac{27h^2}{19}&\frac{11h^2}{19}\\ \frac{243h^2}{76}&\frac{99h^2}{76}\\ \frac{27h}{38}&\frac{11h}{38}\\ \frac{27h}{19}&\frac{11h}{19}\\ \frac{81h}{38}&\frac{33h}{38} \end{pmatrix}\begin{pmatrix} f_{n+2}\\ f_{n+3} \end{pmatrix} \tag{20}$$

To simultaneously obtain values for $y_{n+1}, y_{n+2}, y_{n+3}, y'_{n+1}, y'_{n+2}, y'_{n+1}, y''_{n+1}, y''_{n+2}\ and\ y''_{n+3}$. Equation (20) can be written explicitly as

$$\begin{aligned}
y_{n+1} &= y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{228}\left[27f_{n+2} + 11f_{n+3}\right]\\
y_{n+2} &= y_n + 2hy'_n + 2h^2y''_n + \frac{h^3}{57}\left[54f_{n+2} + 22f_{n+3}\right]\\
y_{n+3} &= y_n + 3hy'_n + \frac{9}{2}h^2y''_n + \frac{h^3}{76}\left[297f_{n+2} + 45f_{n+3}\right]\\
y'_{n+1} &= y'_n + hy''_n + \frac{h^2}{76}\left[27f_{n+2} + 11f_{n+3}\right]\\
y'_{n+2} &= y'_n + 2hy''_n + \frac{h^2}{19}\left[27f_{n+2} + 11f_{n+3}\right]\\
y'_{n+3} &= y'_n + 3hy''_n + \frac{h^2}{76}\left[243f_{n+2} + 99f_{n+3}\right]\\
y''_{n+1} &= y''_n + \frac{h}{38}\left[27f_{n+2} + 11f_{n+3}\right]\\
y''_{n+2} &= y''_n + \frac{h}{19}\left[27f_{n+2} + 11f_{n+3}\right]\\
y''_{n+3} &= y''_n + \frac{h}{38}\left[18f_{n+2} + 33f_{n+3}\right]
\end{aligned} \tag{21}$$

CHARACTERISTICS OF THE METHOD

Properties of the method are investigated to establish its validity. These properties aid to show the nature of convergence of the method. These properties include order, error constant, consistency, zero stability and stability domain. All these put together reveal the nature of convergence of the method. However, a brief beginning of these properties is made for a better understanding of the section.

Order and error constant of the method:

Suppose the linear difference operator L associated with the continuous multi-step method (12) be defined as

$$L[y(x),h]=\sum_{j=0}\{\alpha_j y(x_n+jh)-h^3\beta.y'''(x_n+jh)\};\ j=0,1,2-k \quad \text{see Lambert (1973)} \qquad (22)$$

Where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a,b], and α_0 and β_0 are both non – zero.

Expanding $y(x_n+jh)$ and $y'''(x_n+jh),\ j=0,1,2,3,\ldots,k$ in Taylor's series about x_n and collecting like terms in h and y gives.

$$L[y(x).h]=C_0y(x)+C_1hy'(x)+C_2h^2y^2(x)+\ldots+Cph^py^p(x)+\ldots$$

Definition 1

The difference operator L and the associated implicit multi-step method (12) are said to be of order p, if in (23)

$$c_0=c_1=c_2=\ldots c_p=c_{p+1}=c_{p+2}=0,c_{p+3}\neq 0 \qquad (23)$$

Then c_{p+3} is called the error constant and it implies that the local truncation error is given by

$$t_{n+k}=c_{p+3}h^{p+3}y^{(p+3)}_{(x_n)}+0\left(h^{p+4}\right). \text{ (see Lambert, 1973)} \qquad (24)$$

Using part of the block in (21) i.e.

$$y_{n+1}=y_n+hy'_n+h^2y''_n+\frac{h^3}{228}[27f_{n+2}+11f_{n+3}]$$

$$y_{n+2}=y_n+2hy'_n+2h^2y''_n+\frac{h^3}{57}[54f_{n+2}+22f_{n+3}]$$

$$y_{n+3}=y_n+3hy'_n+\frac{9}{2}h^2y''_n+\frac{h^3}{76}[297f_{n+2}+45f_{n+3}]$$

as

$$y_{n+1}-y_n-hy'_n-h^2y''_n-h^3\left[\frac{9}{76}f_{n+2}+\frac{11}{228}f_{n+3}\right]=0$$

$$y_{n+2}-y_n-2hy'_n-2h^2y''_n-h^3\left[\frac{18}{19}f_{n+2}+\frac{22}{57}f_{n+3}\right]=0$$

$$y_{n+3}-y_n-3hy'_n-\frac{9}{2}h^2y''_n-h^3\left[\frac{297}{76}f_{n+2}+\frac{45}{76}f_{n+2}\right]=0 \qquad (25)$$

And using Taylor's series expansion on (25) and collecting terms in h and y, lead to the following

$$C_n=\frac{(1)^n.1}{n!}-\frac{1}{(n-3)!}\left[\frac{9}{76}(2)^{n-3}+\frac{11}{228}(3)^{n-3}\right]$$

$$C_n=\frac{(2)^n.1}{n!}-\frac{1}{(n-3)!}\left[\frac{18}{19}(2)^{n-3}+\frac{22}{57}(3)^{n-3}\right]$$

$$C_n=\frac{(3)^n.1}{n!}-\frac{1}{(n-3)!}\left[\frac{297}{76}(2)^{n-3}+\frac{45}{76}(3)^{n-3}\right]$$

On evaluating at n = 0, 1, 2, and 3

$$\therefore c_0=c_1=c_2=c_3=0$$

But $C_4 = \begin{bmatrix} -\frac{155}{456} \\ -\frac{136}{57} \\ -\frac{945}{152} \end{bmatrix} \therefore C_4 = C_{P+3} \therefore P = 1$

Hence the method is of order P = 1

With error constant $C_{p+3} = \left[-\frac{155}{456}, \frac{136}{57}, \frac{945}{152} \right]^T$ (26)

Consistency

Given a continuous implicit multi-step method (12), the first and second characteristics polynomials are defined as:

$$\rho(z) = \sum_{J=0}^{k} \alpha_j Z^j \quad (27)$$

$$\sigma(z) = \sum_{J=0}^{k} \beta_J Z^J \quad (28)$$

Where Z is the principle root, $\alpha_k \neq 0$ and $\alpha_0^3 + \beta_0^3 \neq 0$

Definition 2

The continuous implicit multi-step method (12) is said to be consistent if it satisfies the following conditions

i. The order $P \geq 1$

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\rho(l) = \rho'(l)$

iv. $\rho'''(l) = 3!\sigma(l)$ see Lambert (1973), Henrici (1972)

Remark:

Condition (i) is sufficient for the associated block method to be consistent i.e. $P \geq 0$.

Jator (2007).

Recall the main method; (14)

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{19}\left[-8f_{n+3} + 27f_{n+2}\right]$$

The first and second characteristics polynomial of the method are given by

$$\rho(z) = z^3 - 3z^2 + 3z - 1 \text{ and } \sigma(z) = \frac{-8z^3 + 27z^2}{19} \text{ respectively}$$

By definition 2, the method (14) is consistent since it satisfies the following.

i. The order of the method is P = 1 ≥ 1

ii. $\alpha_0 = -1,\ \alpha_1 = 3,\ \alpha_2 = -3,\ \alpha_3 = 1$

Thus $\sum_{j=0}^{3} \alpha_j,\ j = 0,1,2,3, \sum_{j=0}^{3} \alpha_J = -1 + 2 - 3 + 1 = 0$

iii. $\rho(z) = z^3 - 3z^2 + 3z - 1$

$$\rho(1) = (1)^3 - 3(1)^2 + 3(1) - 1 = 0$$

$$\rho^1(1) = 3(1)^2 - 6(1) + 3 = 0$$

$$\therefore \rho(1) = \rho^1(1) = 0$$

iv. $\rho'''(1) = 3!\sigma(1)$

Recall $\rho^1(z) = 3z^2 - 6z + 3$

$$\rho''(z) = 6z - 6$$

$$\rho'''(z) = 6$$

$$\therefore \rho'''(1) = 6$$

Recall $\sigma(z) = \dfrac{-8z^3 + 27z^2}{19}$

$$\therefore \sigma(1) = \frac{-8(1)^3 + 27(1)^2}{19} = 1$$

$\therefore 3!\sigma(1) = 6 \times 1 = 6$ and also $\rho'''(1) = 3!\sigma(1) = 6$

The conditions (i - iv) are satisfied, hence the method is consistence. Similarly, the block method (20) is consistent since the order of each method in the block method is greater than 1, as shown in equation (26).

Zero Stability of the main method

Definition 3

The continuous implicit multi-step method (12) is said to be zero-stable if no root of the first characteristics polynomial $\rho(z)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than three (see Lambert, 1991).

Definition 4

The implicit block method (20) is said to be zero stable if the roots $Z_s, S = 1,\ldots,n$ of the first characteristics polynomial $\overline{P}(z)$, defined by

$$\overline{P}(z) = \det(Z\overline{A} - \overline{E}) \qquad (29)$$

Satisfies $|Z_s| \leq 1$ and every root with $|Z_s| = 1$ has multiplicity not exceeding three in the limit as $h \to 0$

Zero stability of the block method

From (20), using definition as $h \to 0$

$\overline{P}(z) = \det[Z\overline{A} - E]$ gives $Z^8(Z-1)$, while when solved gives: $z_2 = z_3 = z_4 \ldots z_8 = 0, z_1 = 1$

Hence the block method is stable.

Zero stability of the main method.

Recall the first characteristics polynomials of (14) given by

$$P(z) = z^3 - 3z^2 + 3z - 1 \qquad (30)$$

Equating (30) to zero and solving for z, gives

$$(z-1)(z-1)(z-1) = 0$$

$$\therefore z_1 = z_2 = z_3 = 1$$

The roots of z of (30) for $|z| = 1$ is simple, hence the method is zero stable as $h \to 0$ as defined by (3) and by the stability of the block method (20).

Convergence

The convergence of the continuous implicit multi-step method (12) is considered in the light of the basic properties, in conjunction with the fundamental theorem of Dahlquist, Henrici (1962), for linear multistep method. In what follows, we state Dahlquist's theorem without proof.

Theorem 3.1: Dahlquist theorem (Lambert, 1973)

The necessary and sufficient condition for a linear multi-step method to be convergent is for it to be consistent and zero stable.

Remark: The numerical method derived here are considered to be convergent by theorem 3.1 as $h \to 0$. Following theorem 3.1, the method (14) is convergence since it satisfies the necessary and sufficient conditions of consistency and zero stability.

Region of absolute stability of the method:

Definition 5

If the first and second characteristics polynomials of linear multi-step the method are ρ and σ respectively, then the polynomial equation can be written as

$$\pi(r,\bar{h}) \Rightarrow \rho(r) - \bar{h}\,\sigma(r) = 0 \tag{31}$$

Where $\bar{h} = (\lambda h)^3$, then $\pi(r,\bar{h})$ is called the stability polynomial of the method defined by p and σ and $\bar{h} = (\lambda h)^3$ is the test equation. (see Abhulimen & Aigbiremhon (2018), Aigbiremhon & Ukpebor, 2019).

So, to get the graph of the stability region

We make $\bar{h}$ the subject of the formular form (31) to get

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \tag{32}$$

Which is then plotted in MATLAB environment to produce the required absolute stability region of the method that will be plotted in a graph.

Using definition 5, and expressing the first and second characteristics polynomial of equation (14) as

$$\rho(r) = r^3 - 3r^2 + 3r - 1 \text{ and } \sigma(r) = \frac{-8r^3 + 27r^2}{19}$$

Using the boundary locus method (Lambert, 1973).

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} = \frac{19(r^3 - 3r^2 + 3r - 1)}{-8r^3 + 27r^2}$$

where $\bar{h} = (\lambda h)^3$

By setting $r = \ell^{i\theta}$, where $\ell^{i\theta} = \cos\theta + i\sin\theta$, we have

$$\bar{h}(\theta) = \frac{19\left[(\cos 3\theta - 3\cos 2\theta + 3\cos\theta - 1) + i(\sin 3\theta - 3\sin 2\theta + 3\sin\theta - 1)\right]}{-8\cos 3\theta + 27\cos\theta + i(-8\sin 3\theta + 27\sin 2\theta)} \tag{33}$$

This is simplified to the form

$$x(\theta) + iy(\theta)$$

using MATLAB mathematical tool to plot (33), which produces the required region of absolute stability region of the method as shown above:

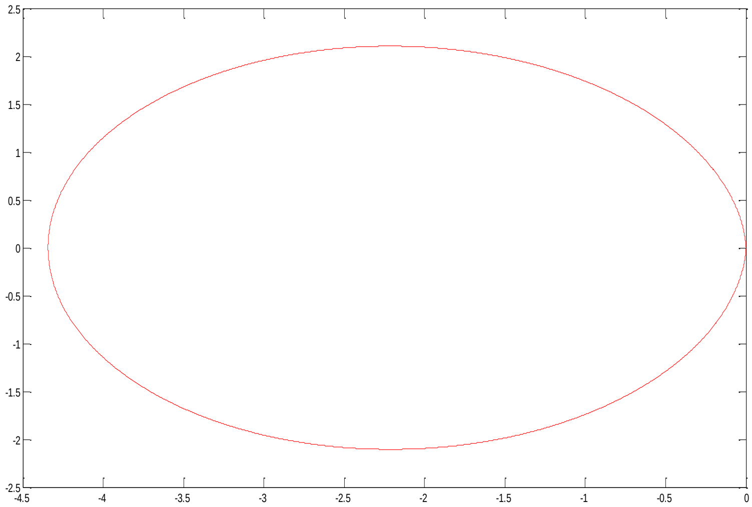

Figure 1: shows the region at which the method is absolutely stable

NUMERICAL EXAMPLES

In order to study the competence of the developed method, we present some numerical examples with the following four problems. The continuous implicit multi-step method 3SM was applied to solve the following test problems

Problem 1

$$y''' = 3\sin x, y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1, \tag{34}$$

Exact solution: $y(x) = 3\cos x + \left(\frac{x^2}{2}\right) - 2$

Source: Olabode (2013) and later solved by Ogunware and Omole (2020)

Problem 2

$$y''' = y'(2xy'' + y'), y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0 \tag{35}$$

$h = 0.1$. Exact solution: $y(x) = 1 + \frac{1}{2} In\left[\frac{2+x}{2-x}\right]$

Source: Adoghe al., (2016)

Problem 3

$$y''' = x - 4y', y(0) = 0, y'(0) = 0, y''(0) = 1 \; h = 0.1. \tag{36}$$

Exact solution: $y(x) = -\frac{3}{16}\cos(2x) + \frac{3}{16} + \frac{x^2}{8}$

Source: Olabode (2007)

Problem 4

$$y''' - y'' + y' - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1 \tag{37}$$

h =0.01. Exact solution: $y(x) = \cos x$

Source: Mohammed and Adeniyi (2014)

Table 1: Showing the exact solution and the computed results from the proposed method for problem one and its comparison with production- corrector method of order eight in Olabode (2013).

x-value	Exact solution	3SM	Error in 3SM	Error in predictor/correction method of order 8 Olabode (2013)
0.1	0.990012495834	0.990012497417	1.5834e-009	4.1723e-009
0.2	0.960199733523	0.960199753068	1.9544e-008	9.5785e-008
0.3	0.911009467376	0.911009557800	9.0423e-008	3.9916e-007
0.4	0.843182982008	0.843183254683	2.7267e-007	1.0369e-006
0.5	0.757747685671	0.757748331895	6.4622e-007	2.1285e-006
0.6	0.656006844729	0.656008156213	1.3115e-006	3.7895e-006
0.7	0.539526561853	0.539528950238	2.3884e-006	6.1301e-006
0.8	0.410120128041	0.410124143448	4.0154e-006	9.2537e-006
0.9	0.269829904811	0.269836252674	6.3479e-006	1.3257e-005
1.0	0.120906917604	0.120916474049	9.5564e-006	1.8228e-005

It could be observed in table 1, that the three-step block multi-step method proposed in this work is more accurate than predictor corrector method of order eight in Olabode (2013).

Table 2: Showing the exact solution and the computed results from the proposed method for problem two and its comparism with a non – linear problem in Adoghe et al., (2016)

x-value	Exact solution	3SM	Error in 3SM	Error in Adoghe et al., (2016)
0.1	1.050041729278	1.050041716947	1.2331e-008	1.9315e-008
0.2	1.100335347731	1.100335259307	8.8424e-008	5.6083e-007
0.3	1.151140435936	1.151140142127	2.9381e-007	3.7551e-006
0.4	1.202732554054	1.202731843891	7.1016e-007	1.3403e-005
0.5	1.255412811882	1.255411364551	1.4473e-006	3.2591e-005
0.6	1.309519604203	1.309516943720	2.6605e-006	5.8165e-005
0.7	1.365443754271	1.365439176882	4.5774e-006	7.1524e-005
0.8	1.423648930193	1.423641385682	7.5445e-006	2.5648e-005
0.9	1.484700278594	1.484688174561	1.2104e-005	1.7092e-004
1.0	1.549306144334	1.549287010982	1.9133e-005	6.7064e-004

It is very clear from table 2, that the three step block multi-step method proposed in this work, is more accurate than the non-linear problem in Adoghe et al., (2016).

Table 3: Showing the exact solution and the computed results from the proposed method for problems three and its comparism with problem in Olabode (2007)

x-value	Exact solution	3SM	Error in 3SM	Error in Olabode (2007)
0.1	0.004987516654	0.004987511793	4.8615e.009	1.6655e-008
0.2	0.019801063624	0.019800994694	6.8948e-008	3.8096e-007
0.3	0.043999572204	0.043999241748	3.3046e-007	1.5665e-006
0.4	0.076867491997	0.076866490944	1.0011e.006	3.9866e-006
0.5	0.117443317649	0.117440968308	2.3493e-006	7.9597e-006
0.6	0.164557921035	0.164553243610	4.6774e-006	1.3680e-005
0.7	0.216881160706	0.216872865762	8.2949e-006	2.1196e-005
0.8	0.272974910431	0.272961418970	1.3491e-005	3.596e-005
0.9	0.331350392754	0.331329886304	2.0506e-005	4.1009e.005
1.0	0.390527531852	0.390498029396	2.9502e-005	5.2605e-005

It could be witnessed in table 3, that the three step block multistep method is better than Olabode (2007).

Table 4: Showing the exact solution and the computed results from the proposed method for problem four and its comparism with problem in Mohammed and Adeniyi (2014).

x-value	Exact solution	3SM	Error in 3SM	Error in Mohammed and Adeniyi (2014)
0.1	0.999950000416	0.999950000417	7.1076e-013	6.7200e-007
0.2	0.999800006666	0.999800006671	4.7071e-012	1.3441e-006
0.3	0.999550033748	0.999550033762	1.3033e-011	2.0170e-006
0.4	0.999200106660	0.999200106688	2.7412e-011	2.6884e-006
0.5	0.998750260394	0.998750260449	5.4703e-011	3.3594e-006
0.6	0.998200539935	0.998200540032	9.7557e-011	NA
0.7	0.997551000253	0.997551000411	1.5823e-010	NA
0.8	0.996801706302	0.996801706551	2.4843e-010	NA
0.9	0.995927330119	0.995952733384	3.7250e-010	NA
1.0	0.995004165278	0.995004165811	5.3320e-010	NA

It should be noted that NA means not available. It could be observed in table 4, that the three-step block multi-step method is more accurate than Mohammed and Adeniyi (2014).

CONCLUSION

In this study, we have successful developed and implemented a continuous implicit multi-step method and used to solve general third-order ordinary differential equations, namely linear problem, non-linear, special and variable coefficient problem. The method is consistent, convergent, zero stable. The method derived, efficiently solved third-order initial value problem as can be seen in tables 1 - 4. In terms of accuracy, our method performs better than the existing methods compared with despite that our method has lesser order of accuracy. the perturbation terms introduced also help in minimising error associated with the problems. Hence the method proposed in this article is computational reliable and efficient in handling any form of third-order ordinary differential equations. Our future research will focus on numerical solution of higher order initial and boundary value problems using the same approach presented here.

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