FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 16, June, 2026, pp 759 – 771
DOI: https://doi.org/10.33003/fjs-2026-1016-5443

The Weighted Sanni–Lindley Distribution: A Flexible Model for Reliability and Failure Time Analysis

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ABSTRACT
The Lindley distribution has been widely applied in reliability and survival analysis; however, its limited flexibility often restricts its ability to adequately model complex lifetime data. To address this limitation, this study introduces a new three-parameter lifetime model, termed the Weighted Sanni–Lindley Distribution (WSLD), obtained by incorporating a weighting mechanism into the recently proposed Sanni distribution framework. The additional parameter enhances the model's flexibility and enables it to accommodate a broader range of distributional shapes and failure-rate behaviours encountered in reliability studies. Fundamental statistical properties of the proposed distribution are derived, including its probability density function, cumulative distribution function, moments, survival function, hazard rate function, Rényi entropy, and order statistics. Parameter estimation is carried out using the method of maximum likelihood, and the corresponding Fisher information matrix is established to facilitate statistical inference and confidence interval construction. Furthermore, a comprehensive Monte Carlo simulation study is conducted to evaluate the finite-sample performance of the maximum likelihood estimators in terms of bias, mean squared error, and standard deviation. The practical applicability of the model is demonstrated using two real-life datasets involving aircraft windshield failures and glass fibre strengths. Comparative analyses based on log-likelihood, AIC, AICc, and BIC reveal that the proposed WSLD consistently provides a superior fit relative to competing lifetime distributions, thereby offering a robust and flexible framework for reliability and survival data modelling.

Received: 10 Jul. 2026
Accepted: 04 Sep. 2026
Published: 15 Sep. 2026

Keywords: Weighted Distribution; Lindley Distribution; Maximum Likelihood Estimation; Hazard Rate; Goodness-of-Fit; Reliability Analysis

INTRODUCTION

Statistical lifetime distributions, also known as ageing distributions, are widely utilized across various fields for modeling datasets. Their applications range from reliability engineering, such as machine life cycles, to medical sciences, where they help analyse survival times of patients post-surgery. In computer science, these distributions are employed to assess software failure rates, while in insurance, they inform the duration of customer policies without claims. Additionally, marketing and social sciences utilize lifetime distributions to study customer lifetimes and marriage durations, respectively. For further details regarding these applications, we refer to previous works (Lai & Xie, 2006; Pham, 2006; Arai et al., 2009; Lai, 2013; Bemmaor & Glady, 2012). Notably, Aderoju and Adeniyi (2022), Bashiru et al., (2025) and Salau et al. (2025) have advanced the lifetime model by embedding it within a new family of Exponential-Gamma distributions.

One significant distribution in this context is the Lindley distribution, introduced by Lindley (1958) as a one-parameter model characterized by its probability density function (pdf):

$$f(x;\ \theta) = \frac{\theta^2(1+x)e^{-\theta x}}{(1+\theta)}\ ,\ \ x, \theta > 0 \qquad (1)$$

Shanker et al. (2013) and Aderoju (2021), further proposed a two-parameter Lindley distribution and a new two-parameter lifetime distribution, respectively which allows for more nuanced modelling of survival and waiting times.

Statistical lifetime distributions have garnered significant attention across various disciplines due to their applicability in modeling real-world phenomena. The Lindley distribution serves as a foundational model in this domain. Its pdf has been widely studied and extended, leading to various multi-parameter distributions that enhance flexibility and applicability. For example, Shanker et al. (2013) proposed a two-parameter Lindley distribution, which allows for more nuanced modeling of survival and waiting times, making it particularly useful in fields such as healthcare and service operations.

In addition to the Lindley distribution, the Quasi Lindley distribution (QLD), introduced by Shanker and Mishra (2013), provides another avenue for analysis. This two-parameter model retains the characteristics of the Lindley distribution while offering enhanced adaptability to different datasets. Similarly, the Generalized Lindley distribution (GLD) proposed by Zakerzadeh and Dolati (2009) incorporates additional parameters to accommodate a wider range of empirical data, thus making it a powerful tool for researchers.

The introduction of weighted distributions marks a significant advancement in the modelling of lifetime data. The Weighted Lindley distribution (WLD), as proposed by Ghitany et al. (2011), introduces a weighting parameter that modifies the standard Lindley distribution. This approach is particularly useful in scenarios where the data exhibit non-uniform probabilities, allowing researchers to better fit the model to observed data.

Further developments include the Weighted Generalized Quasi Lindley distribution (WGQLD) introduced by Benchiha et al. (2021), which expands on the WLD by incorporating additional parameters for improved flexibility.

The Generalized Weighted Lindley (GWL) distribution proposed by Ramos and Louzada (2016) also represents a significant contribution, offering a three-parameter model that enhances the robustness of the lifetime analysis.

The emergence of the Power Samade distribution (PSD), introduced by Aderoju and Babaniyi (2023), exemplifies the ongoing innovation in this field. By adding a shape parameter to the Samade probability density function, the PSD provides researchers with a more versatile tool for analyzing lifetime data across various contexts.

Moreover, the concept of weighted distributions has been extended to other distributions, such as the length-biased weighted Weibull distribution (Das & Roy, 2011b) and the length-biased weighted generalized Rayleigh distribution (Das & Roy, 2011a). These innovations highlight the importance of adapting traditional models to account for biases in data collection and sampling methods.

In environmental statistics, issues of non-random and non-replicated observations necessitate the use of weighted distributions, as discussed by Ahmad et al. (2016). And Nanuwong et al. (2015). These studies play the critical role that weighted approaches play in enhancing the accuracy and relevance of statistical models in real-world applications.

This research aims to address gaps in the existing literature by developing a three-parameter weighted generalized Lindley distribution that not only extends the capabilities of the new two-parameter exponential-gamma-based distribution but also provides an effective model for various applications, incorporating additional parameters and weighting mechanisms to contribute to the evolving landscape of statistical lifetime distributions.

We introduce the proposed distribution in Section 2 and study its mathematical properties in Section 3. The corresponding Maximum Likelihood estimates of the parameters are discussed in Section 4. In Section 5, we discuss real-life applications, and present conclusion in Section 6.

The Weighted Sanni-Lindley Distribution

In this section, we present the Weighted Sanni-Lindley (WSL) Distribution by introducing a weighted parameter into the probability density function of the New Two-Parameter Exponential-Gamma-Based Distribution of Sanni et al. (2025).

The probability density function (pdf) of the New Two-Parameter Exponential-Gamma-Based Distribution is given by

$$f(x;\ \alpha,\theta) = \frac{\theta^4}{(\theta^3\ +\ 6)}\left(1\ +\ \frac{6\theta^{\alpha-4}x^{\alpha-1}}{\Gamma(\alpha)}\right)e^{-\theta x};\quad x,\theta,\alpha\ >\ 0 \qquad (2)$$

And the cumulative distribution function (cdf) is given by

$$F(x) = \frac{\left(1-e^{-\theta x}\right)\theta^3 + \frac{6x^{\alpha}\theta^{\alpha}(x\theta)^{-\alpha}\left(\Gamma(\alpha)-\Gamma(\alpha,x\theta)\right)}{\Gamma(\alpha)}}{(6+\theta^3)},\ if\ x,\theta,\alpha\ >\ 0 \qquad (3)$$

Suppose X is a non-negative random variable with probability density function g(x). Let w(x) be the non-negative weight function; then the probability density function of the weighted random variable x_w is given by

$$g(x;\ \theta) = \frac{w(x)f(x;\theta)}{E[w(x)]} \qquad (4)$$

Where w(x) is a non-negative weighted function, and E[w(x)] is the expected value of the weighted function. It is assumed that the expected value of the weighted function exists, and it is positive (Liu & Shi, 2025).

The Probability Density Function

Proposition 1. Suppose a random variable (X) follows a Weighted Sanni-Lindley Distribution giving as $X \sim WSLD(\alpha,\theta,\beta)$. The PDF of (X) is given by:

$$fw(X;\ \alpha,\theta,\beta) = \frac{e^{-\theta x}x^{\beta-1}\,\theta^{\beta}\,(6x^{\alpha}\theta^{\ \alpha}x\theta^4\Gamma(\alpha))}{\theta^3\Gamma(\alpha)\Gamma(\beta+1) + 6\Gamma(\alpha+\beta)}, x > 0,\alpha,\theta\ >\ 0,\beta \geq 0$$

Where:

α is the shape parameter

θ is the scale parameter

β is the weight parameter

Γ (α) is the gamma function.

Proof 1. The probability density function of the Weighted Sanni-Lindley Distribution can be expressed as

$$fw(x;\ \alpha,\theta,\beta)\ =\ \frac{W(x)f(x;\alpha,\theta)}{E[W(x)]} \qquad (5)$$

Rather and Ozel (2020) suggest that

$$W(x)\ =\ x^{\beta}$$

$$W(x)f(x;\ \alpha,\theta)\ =\ \frac{x^{\beta}\,e^{-\theta x}\,\theta^4\left(1+\frac{6\theta^{\alpha-4}\,x^{\alpha-1}}{\Gamma(\alpha)}\right)}{(\theta^3+6)} \qquad (6)$$

And $E[W(x)]\ =\int_0^{\infty} x^{\beta}\ f(x;\ \alpha,\theta)\ dx$

$$E[W(x)]\ =\ \frac{\theta-\beta\left(\theta 3\Gamma(\alpha)\Gamma(\beta+1)+ 6\Gamma(\alpha+\beta)\right)}{(\theta^3+6)\Gamma(\alpha)} \qquad (7)$$

Substituting (6) and (7) into (5), the WSL distribution can be expressed as:

$$fw(X;\ \alpha,\theta,\beta) = \frac{e^{-\theta x}\ x^{\beta-1}\theta^{\beta}\left(6x^{\alpha}\theta^{\alpha}+x\theta^4\Gamma(\alpha)\right)}{\theta^3\Gamma(\alpha)\Gamma(\beta+1)+ 6\Gamma(\alpha+\beta)}\ x\ >\ 0,\alpha,\theta\ >\ 0,\beta \geq 0 \qquad (8)$$

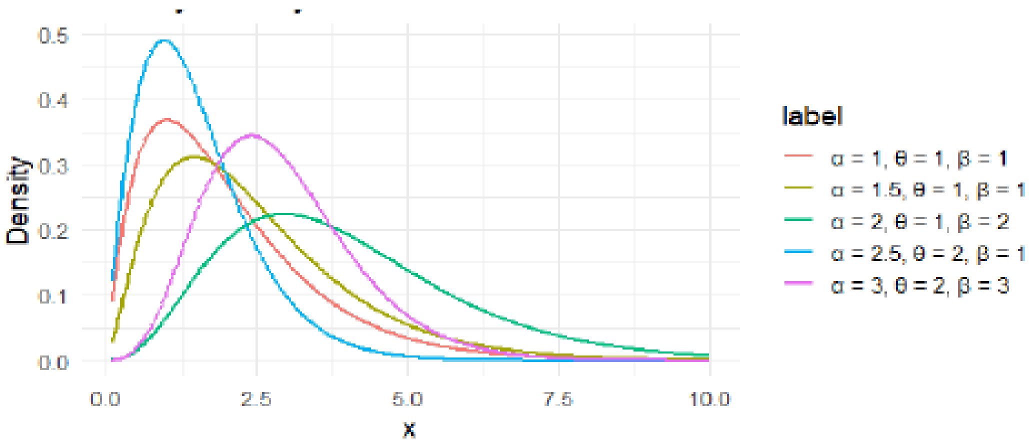

Figure 1: The probability density function of the WSL distribution

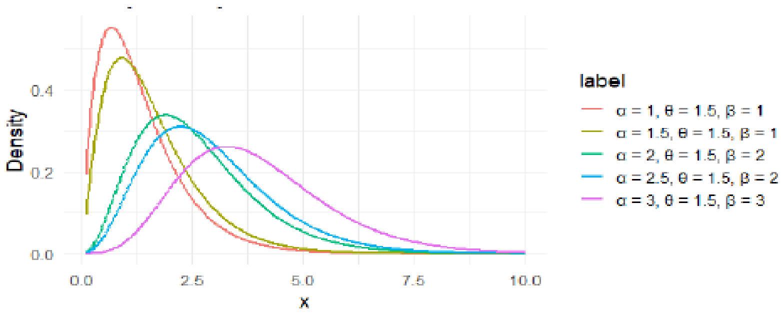

Figure 2: The probability density function of the WSL distribution

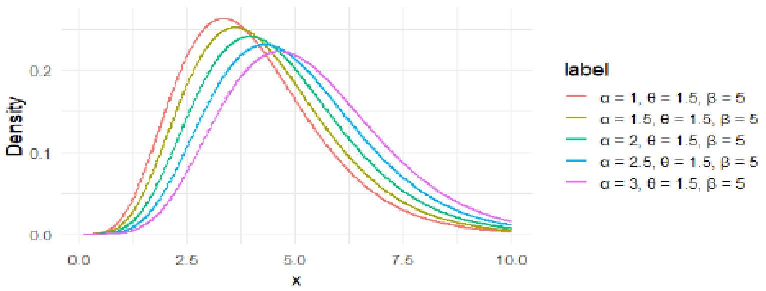

Figure 3: The probability density function of the WSL distribution

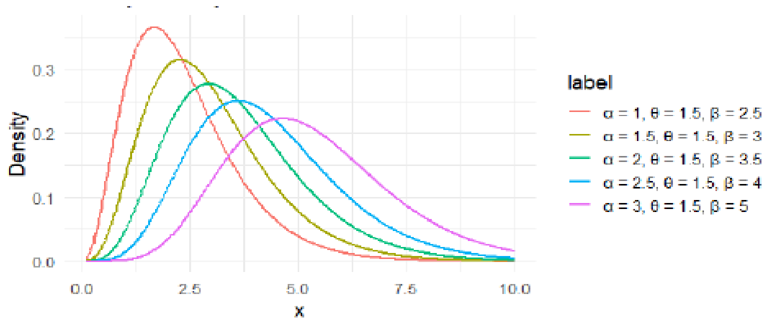

Figure 4: The probability density function of the WSL distribution

The graphs presented in Figures 1, 2, 3, and 4 show the pattern of the pdf of the proposed Weighted Sanni-Lindley Distribution at various values of α, θ, and β. The pdf shows different shapes, which makes it flexible to capture different shapes as may be exhibited by different data fields.

Cumulative Distribution Function

The corresponding cdf of a continuous distribution is obtained by

$$Fw(x;\ \alpha,\theta,\beta)\ = \frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\ \times(\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)\big(\Gamma(1+\beta)-\ \Gamma(1+\beta,x\theta)\big)+\ 6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\ \Gamma(\alpha+\beta,x\theta)))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)\ +\ 6\Gamma(\alpha+\beta)}$$

$$x,\alpha,\theta\ >\ 0,\beta\ \geq\ 0 \qquad (9)$$

Where, $\gamma(\beta+1,\theta x)$ and $\gamma(\beta+\alpha,\theta x)$ are respectively the lower and upper incomplete gamma function.

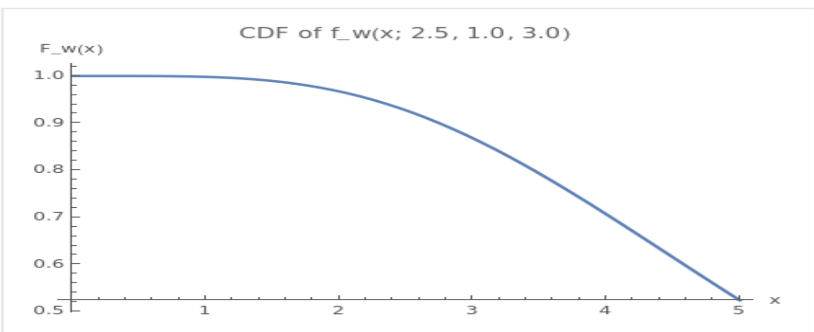

Figure 5: The cdf plot of Sanni Weighted Lindley Distribution

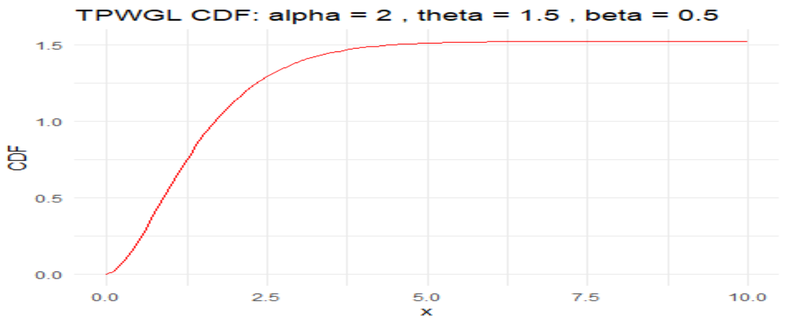

Figure 6: The cdf plot of Sanni Weighted Lindley Distribution

Figures 5 and 6 present the shapes of the cdf of the proposed Weighted Sanni-Lindley Distribution at selected values of α, θ, and β. Just like its pdf, the cdf also shows different shapes, which makes it flexible to capture different data sets.

Statistical Properties

Moment of the weighted Sanni-Lindley Distribution

The moments of distribution are used to describe the characteristics or shape of the distribution.

The n^{th} moment of a random variable X with the Weighted Sanni-Lindley Distribution:

$$E(X^n) = \mu_n = \int_0^\infty x^n f w(x;\alpha,\theta,\beta)dx \qquad (10)$$

Where 'n' is a positive integer.

Hence, the n^{th} moment of a random variable X with the Weighted Sanni-Lindley Distribution is obtained by putting equation (8) into equation (10) and is given by:

$$E(X^n) = \int_0^\infty x^n \frac{e^{-\theta x}x^{\beta-1}\theta^\beta(6x^\alpha\theta^\alpha+x\theta^4\Gamma(\alpha))}{\theta^3\Gamma(\alpha)\Gamma(\beta+1)+6\Gamma(\alpha+\beta)}dx$$

$$E(X^n) = \frac{\theta^{-n}(\theta^3\Gamma(\alpha)\Gamma(1+n+\beta)+6\Gamma(n+\alpha+\beta))}{\theta^3\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)} \qquad (11)$$

The first four moments of the distribution are described below:

First Moment (n = 1)

$$E[X] = \frac{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}{\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta)}$$

Second Moment (n = 2)

$$E[X^2] = \frac{\theta^3\Gamma(\alpha)\Gamma(3+\beta)+6\Gamma(2+\alpha+\beta)}{\theta^5\Gamma(\alpha)\Gamma(1+\beta)+6\theta^2\Gamma(\alpha+\beta)}$$

Third Moment (n = 3)

$$E[X^3] = \frac{\theta^3\Gamma(\alpha)\Gamma(4+\beta)+6\Gamma(3+\alpha+\beta)}{\theta^6\Gamma(\alpha)\Gamma(1+\beta)+6\theta^3\Gamma(\alpha+\beta)}$$

Fourth Moment (n = 4)

$$E[X^4] = \frac{\theta^3\Gamma(\alpha)\Gamma(5+\beta)+6\Gamma(4+\alpha+\beta)}{\theta^7\Gamma(\alpha)\Gamma(1+\beta)+6\theta^4\Gamma(\alpha+\beta)}$$

The Variance (σ^2)

Given the random variable X having the Weighted Sanni-Lindley Distribution with parameters α, θ, β, the variance is given by

$$\sigma^2 = \frac{\theta^3\Gamma(\alpha)\Gamma(3+\beta)+6\Gamma(2+\alpha+\beta)}{\theta^5\Gamma(\alpha)\Gamma(1+\beta)+6\theta^2\Gamma(\alpha+\beta)} - \left(\frac{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}{\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta)}\right)^2$$

The Coefficient of Variation

Given the random variable X having the Weighted Sanni-Lindley Distribution with parameters α, θ, β, the coefficient of variation is given by

$$CV = \frac{\sqrt{\frac{\theta^3\Gamma(\alpha)\Gamma(3+\beta)+6\Gamma(2+\alpha+\beta)}{\theta^5\Gamma(\alpha)\Gamma(1+\beta)+6\theta^2\Gamma(\alpha+\beta)} - \left(\frac{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}{\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta)}\right)^2}}{\frac{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}{\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta)}}$$

$$CV = \frac{(\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta))\sqrt{\frac{(\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta))^2}{(\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta))^2}+\frac{\theta^3\Gamma(\alpha)\Gamma(3+\beta)+6\Gamma(2+\alpha+\beta)}{\theta^5\Gamma(\alpha)\Gamma(1+\beta)+6\theta^2\Gamma(\alpha+\beta)}}}{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}$$

Skewness

Given the random variable X having the Weighted Sanni-Lindley Distribution with parameters α, θ, β, the skewness is given by

$$S_k = \frac{\frac{\theta^3\Gamma(\alpha)\Gamma(4+\beta)+6\Gamma(3+\alpha+\beta)}{\theta^6\Gamma(\alpha)\Gamma(1+\beta)+6\theta^3\Gamma(\alpha+\beta)}}{\left[\frac{\theta^3\Gamma(\alpha)\Gamma(3+\beta)+6\Gamma(2+\alpha+\beta)}{\theta^5\Gamma(\alpha)\Gamma(1+\beta)+6\theta^2\Gamma(\alpha+\beta)} - \left(\frac{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}{\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta)}\right)^2\right]^{3/2}}$$

Kurtosis

Given the random variable X having the Weighted Sanni-Lindley Distribution with parameters α, θ, β, the kurtosis is given by

$$K_s = \frac{\frac{\theta^3\Gamma(\alpha)\Gamma(4+\beta)+6\Gamma(3+\alpha+\beta)}{\theta^7\Gamma(\alpha)\Gamma(1+\beta)+6\theta^4\Gamma(\alpha+\beta)}}{\left[\frac{\theta^3\Gamma(\alpha)\Gamma(3+\beta)+6\Gamma(2+\alpha+\beta)}{\theta^5\Gamma(\alpha)\Gamma(1+\beta)+6\theta^2\Gamma(\alpha+\beta)} - \left(\frac{\theta^3\Gamma(\alpha)\Gamma(2+\beta)+6\Gamma(1+\alpha+\beta)}{\theta^4\Gamma(\alpha)\Gamma(1+\beta)+6\theta\Gamma(\alpha+\beta)}\right)^2\right]^2}$$

Table 1: Summary Statistics of the Moments of the Distribution

Parameter		**Statistic**				
Θ	***β***	***μ***	***σ²***	***CV***	***Sₖ***	***Kₛ***
α = 0.5						
0.5	0.25	1.5266	3.0791	1.1494	5.5621	23.9785
1	0.5	1.1037	1.1448	0.9694	6.1049	23.8615
5	1	0.3977	0.0798	0.7102	8.4331	29.8215
10	10	1.0999	0.1111	0.3015	47.0315	198.5232
20	15	0.8	0.04	0.25	76.4995	363.3719
α = 1.0						
0.5	0.25	2.5	5	0.8944	6.5405	24.8625
1	0.5	1.5	1.5	0.8165	7.1443	26.25
5	1	0.4	0.08	0.7071	8.4853	30
10	10	1.1	0.11	0.3015	47.0358	198.5455
20	15	0.8	0.04	0.25	76.5	363.375

Parameter		**Statistic**				
Θ	***β***	***μ***	***σ^2***	***CV***	***S_k***	***K_s***
α = 5.0						
0.5	0.25	10.398	21.5996	0.445	18.7266	66.4685
1	0.5	5.2463	6.1967	0.4745	16.333	55.6835
5	1	0.5548	0.2109	0.8276	6.9721	24.5438
10	10	1.4429	0.1639	0.2806	56.4193	247.2235
20	15	0.9488	0.0551	0.2473	78.6448	375.1057
α = 10.0						
0.5	0.25	20.3091	44.0181	0.3267	38.2873	152.9202
1	0.5	10.0934	13.5863	0.3652	28.7908	105.4189
5	1	0.9838	0.9068	0.9679	5.3066	15.3198
10	10	1.9984	0.2013	0.2245	102.1617	523.6531
20	15	1.2495	0.0627	0.2004	139.6824	781.0064
α = 20.0						
0.5	0.25	40.1621	93.0498	0.2401	84.5777	407.7071
1	0.5	19.8889	31.1272	0.2805	55.2695	232.7918
5	1	2.2612	4.061	0.8912	5.1002	12.126
10	10	3	0.3	0.1826	181.0864	1090.9842
20	15	1.75	0.0875	0.169	225.1466	1446.1501

Table 1 shows the nature of the Mean (μ_3), variance (σ^2), coefficient of skewness (Sk), and coefficient of kurtosis (Ks) of the WSLD for varying values of the parameters. As the parameter values change, we can observe how the statistics also change, providing insights into the behaviour of the distribution under different conditions. For instance, as θ, α, and β increase, the mean (μ) tends to increase or decrease depending on the specific values, while the variance (σ^2) and kurtosis (Ks) generally increase. Skewness (Sk) exhibits more varied behaviour, sometimes increasing and sometimes decreasing with changes in the parameters.

Reliability Analysis

The reliability analysis of a distribution is determined by the survival and hazard rate function of the distribution. Aldeni et al. (2019) opined that the most commonly used measures for describing the underlying distribution of a lifetime variable are the survival function or the hazard function. These are obtained respectively as follows:

The Survival Function

The survival function is a function that gives the probability that a patient, device, or other object of interest will survive past a certain time by Kleinbaum and Klein (1996). The survival function is the complementary cumulative distribution of the lifetime.

Let the lifetime X be a continuous random variable with cumulative density function F(x) and probability density function f(x) on the interval [0, ∞]. Its survival function is

$$S(x;\ \alpha,\theta,\beta) = 1 - \boldsymbol{F}w(x;\ \alpha,\theta,\beta)$$

This implies

$$S(x) = 1 - \frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta))+6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)} \qquad (12)$$

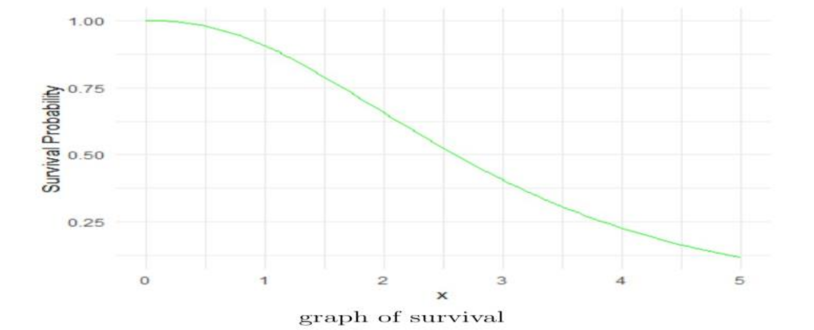

Figure 7: Survival function of the WSLD for selected values of α, θ, and β, illustrating the effect of parameter variation on reliability behaviour

Figure 7 presents the shapes of the survival function of the proposed Weighted Sanni-Lindley Distribution at selected values of $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\beta}$. The plots also show a decreasing rate of survival at time x.

The Hazard Function

The hazard function H(x) of an event is the probability of the failure of the event at time x. It is the probability that an event will fail at a given time x. The Hazard H(x) rate function of the Weighted Sanni-Lindley Distribution is obtained as:

$$H(x;\ \alpha,\theta) = \frac{fw(x;\alpha,\theta,\beta,x)}{1 - Fw(x;\alpha,\theta,\beta,x)}$$

$$H(x;\ \alpha,\theta) = \frac{fw(x;\alpha,\theta,\beta,x)}{Sw(x;\alpha,\theta,\beta,x)}$$

$$H(x;\ \alpha,\theta) = \frac{e^{-\theta x}\, x^{\beta-1}\, \theta^{\beta}\left(6x^{\alpha}\theta^{\alpha}+x\theta^{4}\Gamma(\alpha)\right)}{1 - x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\, \theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)\left(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta)\right)+6x^{\alpha}\theta^{\alpha}\left(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta)\right)} \tag{13}$$

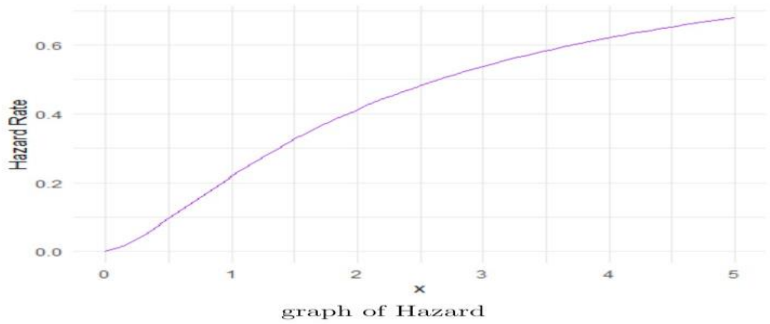

Figure 8: The hazard function of the Weighted Sanni-Lindley Distribution

Figure 8 presents the shapes of the hazard function of the proposed Weighted Sanni-Lindley Distribution at selected values of α̂, θ̂ and β̂. The plots indicate that items are more likely to fail at the early stage of operation, after which the failure rate stabilizes and remains approximately constant. This suggests that failures are more common during the initial period, while failures occurring during the useful life of the item happen randomly at a constant rate.

Rényi's Entropy

Rényi's Entropy is used for quantifying the diversity and randomness of a distribution. According to Rényi (1961), the entropy of a random variable X is a measure of variation of uncertainty. Renyi entropy is defined by

$$R_H(x;\ \alpha,\theta,\beta)\ =\ \frac{1}{(1-P)} log \int_0^{\infty}[f(x;\ \alpha,\theta,\beta)]^2\, dx$$

Where p > 1

Hence, the Renyi's Entropy of the Weighted Sanni-Lindley Distribution is obtained as follows:

$$R_H(x;\ \alpha,\theta,\beta)\ =\ \frac{1}{(1-P)} log \int_0^{\infty}\left[\frac{e^{-\theta x}\, x^{\beta-1}\, \theta^{\beta}\,(6x^{\alpha}\theta^{\alpha}+x\theta^{4}\Gamma(\alpha))}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)\, +\, 6\Gamma(\alpha+\beta)}\right]^{p} dx$$

Let:

$$C\ =\ \frac{\theta^{\beta}}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)+6\Gamma(\alpha+\beta)}$$

$$R_H(x;\ \alpha,\theta,\beta) = \frac{1}{(1-P)} \log C^{p} \int_0^{\infty}\left[6e^{-\theta x}\, x^{\alpha+\beta-1}\, \theta^{\alpha} + e^{-\theta x}\, x^{\beta}\, \theta^{4}\Gamma(\alpha)\right]^{p} dx$$

Order Statistics

If $X_{(1)}, X_{(2)}, ..., X_{(n)}$, denote the order statistics of a random sample $X_1, X_2, ..., X_n$, acquired from a population characterized by a cdf F(x) and the pdf f(x), then the pdf of X_k can be expressed in the following

$$f_{x(k)}(x) =\ \frac{n!}{(k-1)!(n-k)!} f(x)\, [F(x)]^{k-1}[1-F(x)]^{n-k}$$

Hence, the Order Statistics of the Weighted Sanni-Lindley Distribution is obtained as

$$f_{x(k)}(x) = \frac{n!}{(k-1)!(n-k)!} \times\ \frac{e^{-\theta x}\, x^{\beta-1}\, \theta^{\beta}\,(6x^{\alpha}\theta^{\alpha}+x\theta^{4}\Gamma(\alpha))}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)+6\Gamma(\alpha+\beta)} \times$$
$$\left[x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\ (\theta^{3}(x\theta)^{\alpha}\Gamma[\alpha](\Gamma[1+\beta]-\Gamma[1+\beta,x\theta])\ +\ 6x^{\alpha}\theta^{\alpha}(\Gamma-\Gamma[\alpha+\beta,x\theta]))\right]^{k-1} \times$$
$$\left[1\ -\ x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\ (\theta^{3}(x\theta)^{\alpha}\Gamma[\alpha](\Gamma[1+\beta]-\Gamma[1+\beta,x\theta])+6x^{\alpha}\theta^{\alpha}(\Gamma-\Gamma[\alpha+\beta,x\theta]))\right]^{n-k}$$

Where n > 0.

The function of the first and n[th] order statistics of the Weighted Sanni-Lindley Distribution are:

$$f_{x(1)}(x))\ =\ \frac{n!}{(n-1)!} \times\ \frac{e^{-\theta x} x^{\beta-1}\theta^{\beta}(6x^{\alpha}\theta^{\alpha}+x\theta^{4}\Gamma(\alpha))}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)+6\Gamma(\alpha+\beta)}$$
$$\left[\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\,[\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta))+6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)}\right]^{1-1} \times$$
$$\left[1\ -\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\,[\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta))+6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)}\right]^{n-1}$$

$$f_{x(1)}(x)\ =\ \frac{n!}{(n-1)!} \times \frac{e^{-\theta x}\, x^{\beta-1}\, \theta^{\beta}\,(6x^{\alpha}\theta^{\alpha}+x\theta^{4}\Gamma(\alpha))}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)\, +\, 6\Gamma(\alpha+\beta)} \times$$
$$\left[1\ -\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\,[\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta))+6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)}\right]^{n-1}$$

And

$$f_{x(n)}(x) =\ \frac{n!}{(n-1)!\,(n-n)!} \frac{e^{-\theta x}\, x^{\beta-1}\, \theta^{\beta}\left(6x^{\alpha}\theta^{\alpha}+x\theta^{4}\Gamma(\alpha)\right)}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)+\ 6\Gamma(\alpha+\beta)} \times$$
$$\left[\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\,[\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta))+6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)}\right]^{n-1} \times$$
$$\left[1-\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\,[\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta))+6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)}\right]^{n-n}$$

$$f_{x(n)}(x) = \frac{n!}{(n-1)!}\frac{e^{-\theta x}\, x^{\beta-1}\,\theta^{\beta}\,\left(6x^{\alpha}\theta^{\alpha} + x\theta^{4}\Gamma(\alpha)\right)}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1) + 6\Gamma(\alpha+\beta)} \times \left[\frac{x^{\beta}\theta^{\beta}(x\theta)^{-\alpha-\beta}\,[\theta^{3}(x\theta)^{\alpha}\Gamma(\alpha)(\Gamma(1+\beta)-\Gamma(1+\beta,x\theta)) + 6x^{\alpha}\theta^{\alpha}(\Gamma(\alpha+\beta)-\Gamma(\alpha+\beta,x\theta))}{\theta^{3}\Gamma(\alpha)\Gamma(1+\beta)+6\Gamma(\alpha+\beta)}\right]^{n-1}$$

Maximum Likelihood Estimation

Estimation theory is a branch of statistics that deals with estimating the values of parameters based on measured empirical data that has a random component (Joshi & Kumar, 2020). The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. An estimator attempts to approximate the unknown parameters using the measurements. We have considered different estimation procedures for the parameters of the Weighted Sanni-Lindley Distribution.

Maximum likelihood estimate is a method that is mostly used for estimating the parameters of an assumed probability distribution due to the fact that its estimators possess desirable properties such as asymptotic efficiency, consistency and invariance property. To obtain the maximum likelihood estimators of parameters of Weighted Sanni-Lindley Distribution, let X_1, X_2, X_3, X_4, X_n, denote a random sample of size n from Weighted Sanni-Lindley Distribution and define the likelihood function of the random sample as

$$L = \prod_{i=1}^{n} f(x_i) \tag{14}$$

Hence, the Maximum Likelihood of the WSL distribution is obtained as

$$L(\alpha,\theta,\beta;\, x_i) = \prod_{i=1}^{n} f(x_i;\alpha,\theta,\beta)$$

This gives

$$L(\alpha,\theta,\beta;\, x_i) = \prod_{i=1}^{n} \frac{e^{-\theta x_i}\, x_i^{\beta-1}\theta^{\beta}\,(6x_i^{\alpha}\theta^{\alpha} + x_i\theta_4\Gamma(\alpha))}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)+6\Gamma(\alpha+\beta)}$$

Where the log-likelihood function is

$$L(\alpha,\theta,\beta;x) = \sum_{i=1}^{n} In\left(\frac{e^{-\theta x_i}\, x_i^{\beta-1}\theta^{\beta}(6x_i^{\alpha}\theta^{\alpha} + x_i\theta^{4}\Gamma(\alpha))}{\theta^{3}\Gamma(\alpha)\Gamma(\beta+1)+6\Gamma(\alpha+\beta))}\right)$$

This function represents the log of the likelihood that the parameters α, θ and β would result in the observed data points x_i given the assumed distribution.

$$L(\alpha,\theta,\beta;\, x) = \sum_{i=1}^{n}\left[\ln(\theta^{\beta}) + x_i^{\beta-1} + \ln\left(6x_i^{\alpha}\theta^{\alpha} + x_i\theta^{4}\Gamma(\alpha)\right) - \theta x_i - \ln\left(\theta^{3}\Gamma(\alpha)\Gamma(\beta+1) + 6\Gamma(\beta+\alpha)\right)\right]$$

$$L(\alpha,\theta,\beta;x) = \sum_{i=1}^{n}(\theta^{\beta}) + \sum_{i=1}^{n}(x_i^{\beta-1}) + \sum_{i=1}^{n} ln(6x_i^{\alpha}\theta^{\alpha} + x_i\theta^{4}\Gamma(\alpha)) - \sum_{i=1}^{n}\theta x - \sum_{i=1}^{n} ln(\Gamma(\alpha)\Gamma(\beta+1)\theta^{3} + 6\Gamma(\beta+\alpha))$$

$$L(\alpha,\theta,\beta;x) = \sum_{i=1}^{n}(\theta^{\beta}) + \sum_{i=1}^{n}\left(x_i^{\beta-1}\right) + \sum_{i=1}^{n}\ln\left(6x_i^{\alpha}\theta^{\alpha} + x_i\theta^{4}\Gamma(\alpha)\right) - n\theta\bar{x} - nln\left(\Gamma(\alpha)\Gamma(\beta+1)\theta^{3} + 6\Gamma(\beta+\alpha)\right) \tag{15}$$

Differentiating the $log(L(\alpha,\theta,\beta;\, x))$ partially with respect to the associated parameters, we have

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n}\frac{6x_i^{\alpha}\, ln(x_i)\theta^{\alpha} + 6x_i^{\alpha}\, ln(\theta)\theta^{\alpha} + x_i\theta^{4}\Gamma'(\alpha)}{6x_i^{\alpha}\theta^{\alpha} + x_i\theta^{4}\Gamma(\alpha)} - n\frac{\Gamma'(\alpha)\Gamma(\beta+1)\theta^{3} + 6\Gamma'(\beta+\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)\theta^{3} + 6\Gamma(\beta+\alpha)} \tag{16}$$

$$\frac{\partial L}{\partial \theta} = \frac{n\beta}{\theta} + \sum_{i=1}^{n}\frac{\alpha\cdot 6x_i^{\alpha}\theta^{\alpha-1} + 4x_i\theta^{3}\Gamma(\alpha)}{6x_i^{\alpha}\theta^{\alpha} + x_i\theta^{4}\Gamma(\alpha)} - n\bar{x} - \frac{3n\Gamma(\alpha)\Gamma(\beta+1)\theta^{2}}{\Gamma(\alpha)\Gamma(\beta+1)\theta^{3} + 6\Gamma(\beta+\alpha)} \tag{17}$$

$$\frac{\partial L}{\partial \beta} = n\, ln(\theta) + \sum_{i=1}^{n} x_i^{\beta-1}\ln(x_i) - n\frac{\Gamma(\alpha)\Gamma'(\beta+1)\theta^{3} + 6\Gamma'(\beta+\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)\theta^{3} + 6\Gamma(\beta+\alpha)} \tag{18}$$

Where $\Gamma'(\cdot)$ is the derivative of the gamma function and $\alpha > 0, \theta > 0, \beta > 0\ and\ x > 0$

The Maximum Likelihood Estimates, $(\hat{\alpha},\hat{\theta},\hat{\beta})\ of\ (\alpha,\theta,\beta)$, can be obtained by solving $\frac{\partial \ell(\alpha,\theta,\beta)}{\partial \alpha} = 0$, $\frac{\partial \ell(\alpha,\theta,\beta)}{\partial \theta} = 0$ and $\frac{\partial \ell(\alpha,\theta,\beta)}{\partial \beta} = 0$, simultaneously. This is called the score function and is given as;

$$U(\phi) = \left[\frac{\partial \ell(\alpha,\theta,\beta)}{\partial \alpha}, \frac{\partial \ell(\alpha,\theta,\beta)}{\partial \theta}, \frac{\partial \ell(\alpha,\theta,\beta)}{\partial \beta}\right]^{T} = 0$$

It is clear that equations (19)-(21) have no explicit analytical solution, hence, they can be solved numerically using the Newton-Raphson iterative method which is a powerful technique for solving nonlinear systems of equations. The numerical estimates of $\phi = (\hat{\alpha}, \hat{\theta}\ and\ \hat{\beta})$ are easily obtained using R statistical software (R Core Team, 2025).

For the interval estimate of the distribution parameters$\phi = (\hat{\alpha}, \hat{\theta}\ and\ \hat{\beta})$, we obtain the 3×3 observed information matrix whose elements are the second derivatives of the log-likelihood function as

$$I(\alpha,\theta,\beta) = \begin{bmatrix} -E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \theta}\right) & -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right) \\ -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \theta}\right) & -E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) & -E\left(\frac{\partial^2 \ell}{\partial \theta \partial \beta}\right) \\ -E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right) & -E\left(\frac{\partial^2 \ell}{\partial \theta \partial \beta}\right) & -E\left(\frac{\partial^2 \ell}{\partial \beta^2}\right) \end{bmatrix}_{(\hat{\alpha},\hat{\theta},\hat{\beta})=(\alpha,\theta,\beta)}$$

To obtain confidence intervals, the asymptotic normality results were employed. Consequently, the asymptotic distribution of the MLEs is by Zaindin and Sarhan (2009) as,

$$\left(\sqrt{n\,(\hat{\alpha}MLE - \alpha)}, \sqrt{n\,(\hat{\theta}MLE - \theta)}, \sqrt{n\,(\hat{\beta}MLE - \beta)}\ :\right)^{d} \rightarrow N_3(0, I^{-1}(\alpha,\theta,\beta))$$

$I(\alpha,\theta,\beta)$, the Fisher information matrix of the unknown parameters $(\alpha, \theta\ and\ \beta)$ was defined as the variance-covariance matrix of the Weighted Sanni-Lindley Distribution is given by;

$$I(\alpha,\theta,\beta) = \begin{bmatrix} var(\hat{\alpha}^2) & cov(\hat{\alpha},\hat{\theta}) & cov(\hat{\alpha},\hat{\beta}) \\ cov(\hat{\alpha},\hat{\theta}) & cov(\hat{\theta}^2) & cov(\hat{\theta},\hat{\beta}) \\ cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\theta},\hat{\beta}) & var(\hat{\beta}^2) \end{bmatrix}$$

The elements of the Fisher information matrix are the second derivatives of parameters α, θ and β.
Now, the approximate $100(1-\gamma)\%$ confidence intervals of the parameters α, θ and β of the Weighted Sanni-Lindley Distribution take the forms

$\hat{\alpha} \pm Z_{\gamma/2} \sqrt{var(\hat{\alpha})}$

$\theta \pm Z_{\gamma/2} \sqrt{var(\theta)}$

$\beta \pm Z_{\gamma/2} \sqrt{var(\beta)}$

Where $Z_{\gamma/2}$ is the upper $\gamma/2$ percentile of the standard normal distribution.

Simulation Study

A simulation study was conducted to evaluate the performance of the maximum likelihood estimators (MLEs) of the parameters of the proposed Weighted Sanni–Lindley (WSL) distribution. Simulation studies provide a controlled framework for examining the finite-sample behavior of estimators under known parameter settings, allowing assessment of their bias, variability, and efficiency. This approach is particularly valuable when the analytical properties of estimators are complex or when closed-form expressions for moments are intractable.

In this study, random samples were generated from the WSL distribution using the inverse transform method for various sample sizes. The estimation of parameters was carried out using the method of maximum likelihood, and the optimization was performed through the L-BFGS-B algorithm to ensure convergence and parameter stability. The empirical performance of the estimators was assessed based on three key metrics; bias, mean squared error (MSE), and standard deviation (SD), computed across several replications.

The simulation results provide insights into the consistency and precision of the MLEs as the sample size increases. Specifically, the patterns of bias, MSE, and SD across different sample sizes serve as indicators of estimator efficiency and the adequacy of the proposed model for practical applications.

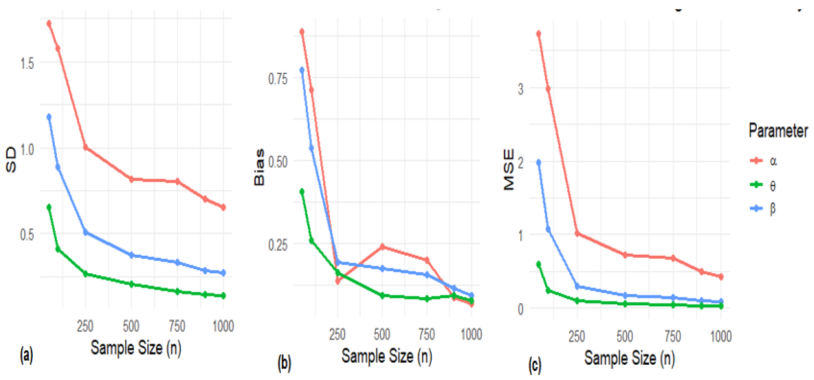

Figure 9: SD, Bias and MSE plots of Simulation study of MLEs for Weighted Sanni–Lindley Distribution at α=1.5, θ=2.0, β=1.2

Figure 9 presents the plots of the standard deviation (SD), bias, and mean squared error (MSE) of the maximum likelihood estimators (MLEs) for the parameters of the Weighted Sanni–Lindley distribution when α = 1.5, θ = 2.0, and β = 1.2. The SD plot shows a steady decline in variability for all parameters as the sample size increases, confirming that the precision of the estimators improves with larger samples. Among the parameters, α exhibits the highest level of variability, whereas θ remains the most stable across all sample sizes. The bias plot reveals that all estimators become nearly unbiased as the sample size increases, with slight bias observed at small sample sizes that quickly diminishes as n grows. Similarly, the MSE plot decreases consistently with increasing sample size, indicating improved estimator accuracy and efficiency. Overall, the results demonstrate that the MLEs are consistent, asymptotically unbiased, and efficient, with their performance improving notably as the sample size increases.

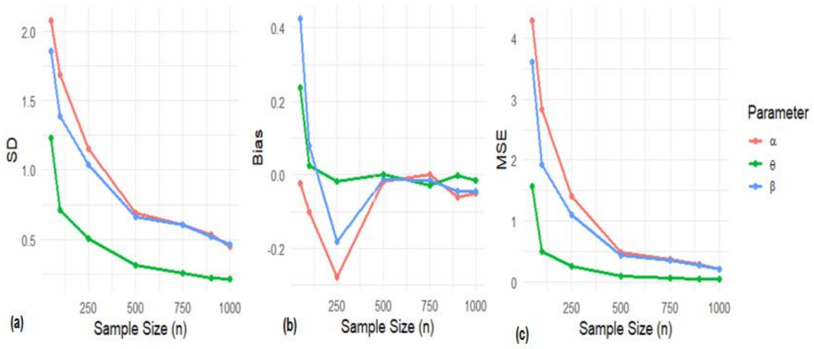

Figure 10: SD, Bias and MSE plots of Simulation study of MLEs for Weighted Sanni–Lindley Distribution at $\alpha = 5.0, \theta = 7.5, \beta = 5.0$

Figure 10 displays the SD, bias, and MSE plots of the MLEs for the Weighted Sanni–Lindley distribution at α = 5.0, θ = 7.5, and β = 5.0. Similar to Figure 9, the SD plot shows a pronounced reduction in estimator variability as the sample size increases, signifying enhanced estimation stability with larger n. The bias plot fluctuates slightly around zero, particularly for β, but generally remains small and converges toward zero as the sample size increases, suggesting asymptotic unbiasedness of the estimators. The MSE plot shows a consistent downward trend across all parameters, reinforcing the consistency and efficiency of the MLEs. Notably, α tends to have higher MSE values at smaller sample sizes but converges with β and θ as the sample size reaches 1000. In summary, the results confirm that the MLEs of the Weighted Sanni–Lindley distribution retain their desirable statistical properties; decreasing SD, bias, and MSE, as sample size increases, even for larger parameter values.

Model Diagnostic Method

The model selection criteria considered in this paper are the AIC (Akaike Information Criterion) by Akaike (1973), AICC (Corrected Akaike Information Criterion) by Hurvich and Tsai (1989), BIC (Bayesian Information Criterion) by Schwarz (1978), KS Statistic (Kolmogorov–Smirnov) by Kolmogorov (1933) and Smirnov (1948), AD Statistic (Anderson–Darling) by Anderson (1952) and Darling (1954), and CvM Statistic (Cramér–von Mises) by Cramér (1928) and von Mises (1931).

Where the AIC, AICC, BIC, KS, AD, and CvM are calculated as follows:

$$AIC = 2k - 2\log L(\hat{\alpha},\theta)$$

$$AICC = AIC + \frac{k(k-1)}{(n-k-1)}$$

$$BIC = k\log(n) - 2\log L(\hat{\alpha},\theta)$$

$$D_n = \sup_x |F_n(x) - F(x)|$$

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[In\, F\left(x_{(i)}\right) + In\left(1 - F\left(x_{(n+1-i)}\right)\right)\right]$$

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F\left(x_{(i)}\right) - \frac{2i-1}{2n}\right]^2$$

Where k is the number of parameters in the distribution and n is the number of observations.

Table 2: Distributions Considered

Name of distributions	Probability density functions	Introducers / Authors
New Two-Parameter Exponential-Gamma-Based Distribution (NTPEGD)	$f(x) = \frac{\theta^4}{(\theta^3+6)}\left(1+\frac{6\theta\alpha - 4x\alpha - 1}{\Gamma(\alpha)}\right)e^{-\theta x}$	Sanni et al. (2025)
Exponential-Gamma (EGD)	$f(x) = \frac{x^{\alpha-1}\lambda^{\alpha+1}}{\Gamma(\alpha)} e^{-2\lambda x}$	Ogunwale et al. (2019)
Three-Parameter Lindley Distribution (TPLD)	$f(x) = \frac{\theta 2}{(\theta\alpha+\beta)}(\alpha+\beta x)e^{-\theta x}$	Shanker et al. (2017)

Application

In this section, we consider two data on Aircraft Windshields and Glass Fibres.

Data Set 1: Aircraft Windshields

These data are on the Aircraft windshield failures (thousands of hours) reported in Murthy et al. (2004) and these data were recently studied by Ramos et al. (2013)

Data Set 2: Glass Fibres

The data set represents the strength of 1.5 cm glass fibers measured at the National Physical Laboratory, England. Unfortunately, the units of measurements are not given in the paper, and they are taken from Smith & Naylor (1987).

Table 3: Parameter Estimates and Model Selection Criteria for dataset 1

Distribution	Parameter	LogLik.	AIC	AICc	BIC	KS	AD	CvM
WSLD	α=5.2263	−130.3854	266.7709	267.0709	274.0633	0.05953	0.5686	0.05782
	θ=2.4435							
	β=1.7023							
	P-Value					0.9097	0.6777	0.8290
NTPEGD	α=3.1452	−138.4124	280.8248	280.9729	285.6864	0.9860	322.1008	27.2738
	β=1.0681							
Exponential-Gamma	α=1.5717	−316.9373	637.8746	638.0228	642.7362	0.9999	710.0691	27.9928
	λ=0.5028							
TPLD	α=0.7804	−143.0174	292.0349	292.3349	299.3273	0.9989	563.4504	27.9546
	θ=0.0182							
	β=3.4087							

Table 3 presents the parameter estimates and model selection criteria for the fitted distributions using Dataset 1 (failure times of 84 aircraft windshields). Among the competing models, the proposed WSLD achieved the highest log-likelihood (−130.3854) and the lowest AIC (266.7709), AICc (267.0709), and BIC (274.0633), indicating the best balance between goodness of fit and model complexity. In contrast, the NTP-EGD, TPLD, and Exponential–Gamma distributions produced relatively poorer fits, with the Exponential–Gamma model performing the worst. These results suggest that the proposed WSLD provides the most appropriate representation of Dataset 1.

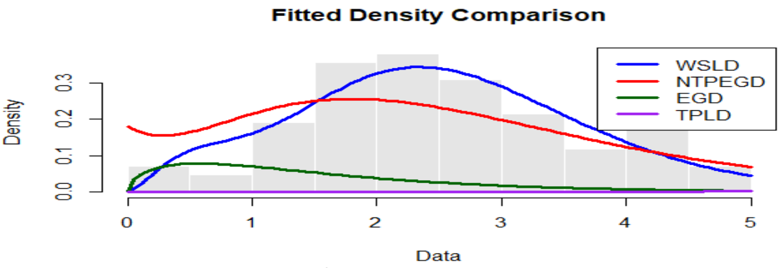

Figure 11: Density Comparison of all the Models (dataset1)

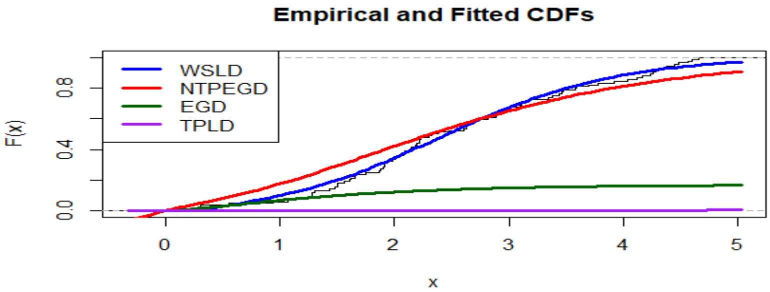

Figure 12: Empirical and CDFs Plots (Dataset1)

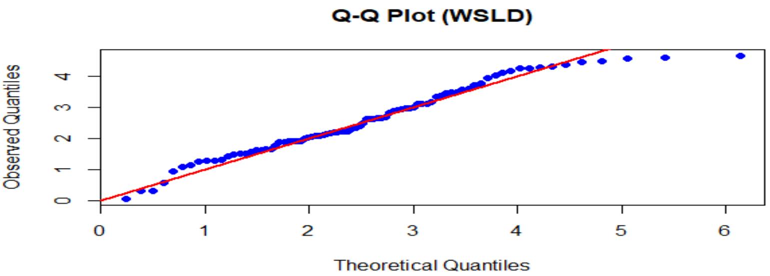

Figure 13: Q-Q Plot of WSLD (Dataset1)

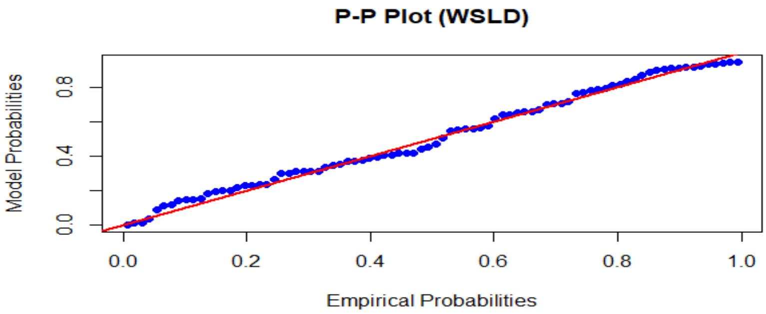

Figure 14: P-P Plot of WSLD (Dataset1)

Table 4: Parameter Estimates and Model Selection Criteria for dataset 2

Distribution	Parameter	LogLik	AIC	AICc	BIC	KS	AD	CvM
WSLD	α=1.9263							
	θ=11.6007	−23.9501	53.9002	54.3069	60.3296	0.5924	189.0503	11.6221
	β=16.4313							
	P-Value					0.0001	7.1428e^{-6}	0.0001
NTPEGD	α=2.3002	−71.9826	147.9653	148.1653	152.2515	0.3160	12.9756	2.5324
	β=1.2262							
Exponential-Gamma	α=1.8612	−165.6444	335.2889	335.4889	339.5751	0.7550	45.7928	9.8328
	λ=0.9494							
TPLD	α=1.3273	−66.3173	138.6346	139.0413	145.064	0.9998	497.5072	20.9911
	θ=0.0100							
	β=30549.9							

Table 4 summarizes the parameter estimates and model selection criteria for the fitted distributions using Dataset 2 (strength of 1.5 cm glass fibers). Consistent with the results for Dataset 1, the proposed WSLD attained the highest log-likelihood (−23.9501) and the smallest AIC (53.9002), AICc (54.3069), and BIC (60.3296) values among the competing models, indicating superior model fit. The TPLD and NTP-EGD showed comparatively weaker performance, while the Exponential–Gamma distribution exhibited the poorest fit. Overall, these findings demonstrate that the WSLD provides the most suitable model for Dataset 2.

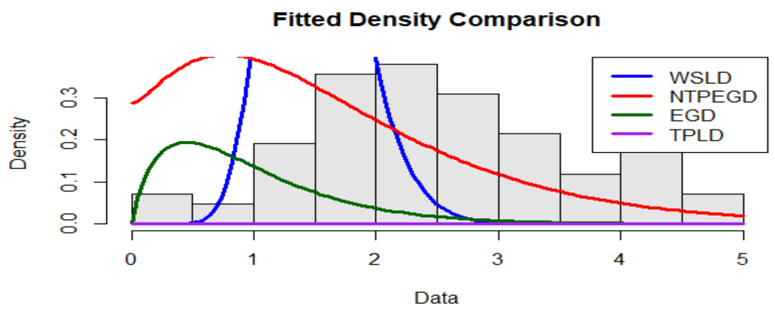

Figure 15: Density Comparison of all the Models (Dataset2)

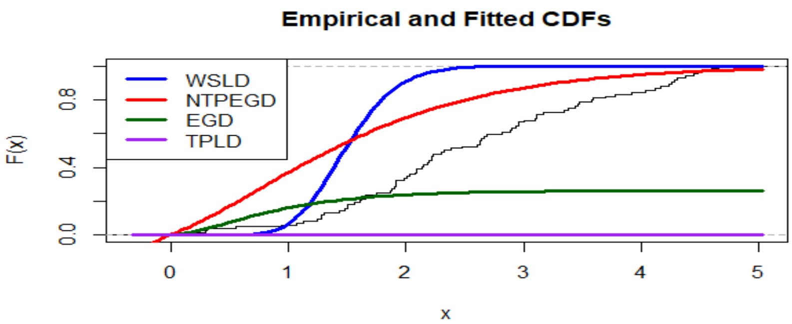

Figure 16: Empirical and CDFs Plots (Dataset2)

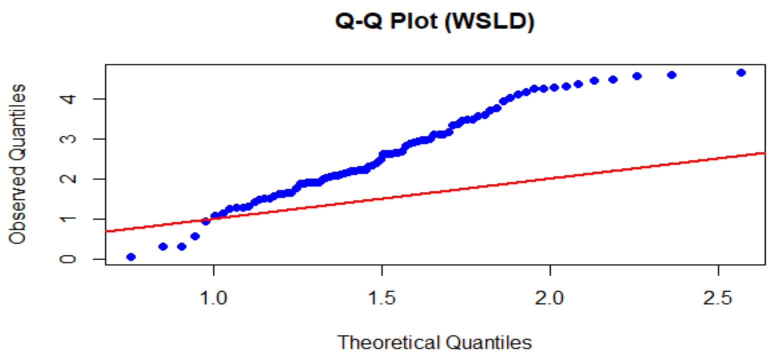

Figure 17: Q-Q Plot 0f WSLD (Dataset2)

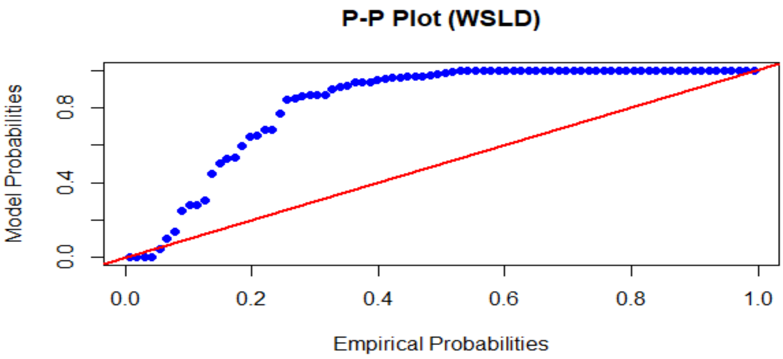

Figure 18: P-P Plot 0f WSLD (Dataset2)

CONCLUSION

The proposed Weighted Sanni–Lindley distribution extends the model by Sanni et al. (2025) through the introduction of an additional shape parameter (β), resulting in a three-parameter distribution characterized by parameters α (shape), θ (scale), and β (flexibility). This generalization significantly enhances its ability to model various forms of lifetime and reliability data. The parameters were estimated using the Maximum Likelihood Estimation (MLE) method, and the model's performance was evaluated through both simulation studies and real data applications. From the empirical analysis using two benchmark datasets, the failure times of aircraft windshields and the strength of glass fibres, the WSLD consistently produced the highest log-likelihood and the lowest AIC, AICc, and BIC values compared to other competing models, such as the NTP-EGD, Exponential–Gamma, and TTPLD distributions. These results clearly demonstrate that the Weighted Sanni–Lindley distribution provides an improved and more flexible framework for analyzing reliability data. The proposed model is therefore recommended as a viable alternative for modelling lifetime and survival data where flexibility and goodness-of-fit are crucial.

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