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Development of Two – Step Simpson – Type Hybrid Block Method for The Solution of Differential Equations

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ABSTRACT
This paper presents the development of a Two-Step Simpson-Type Second Derivative Hybrid Block Method (TSSTHBM) for the numerical solution of stiff and non-stiff systems of first-order ordinary differential equations. The method is derived using polynomial interpolation and collocation techniques, incorporating second derivative information and four off-grid (intra-step) points to enhance accuracy and stability. The resulting scheme is implemented in block form, allowing simultaneous computation of solution values without the need for starting procedures. The basic properties of the method, including order, consistency, zero-stability, and convergence, are established, confirming the reliability of the scheme. Numerical experiments are carried out on standard test problems, including nonlinear systems, linear stiff systems, Riccati equations, and a nonlinear biosorption model. The results demonstrate that the proposed method provides improved accuracy and better stability when compared with existing methods in the literature, even with relatively larger step sizes. Therefore, the TSSTHBM is an efficient and reliable method for solving stiff differential equations arising in scientific and engineering applications.

Keywords: Second Derivative, Hybrid Block, Simpson-Type, Stiff Ordinary Differential Equations, Initial Value Problems

INTRODUCTION

Ordinary Differential Equations (ODEs) are widely used to model real-world processes in science and engineering, including chemical reactions, control systems, and heat transfer. A typical first order ODE can be written as

$$y'(x) = f(t, y), \quad y(t_0) = y_0 \qquad (1)$$

In many applications, these models lead to stiff systems, where rapidly and slowly varying solution components coexist. This makes numerical computation challenging, as standard methods often require very small step sizes for stability, even when the solution itself is smooth (Akinfenwa *et al.*, 2018; Garba & Mohammed, 2020).

Classical methods such as Euler and Runge–Kutta are effective for non-stiff problems but perform poorly for stiff systems due to stability restrictions. Although implicit methods like BDF improve stability, they are computationally expensive because they involve solving nonlinear systems at each step (Cao *et al.*, 2020; Khalique *et al.*, 2021).

To overcome these challenges, recent studies have focused on hybrid and block methods. Block methods compute multiple solution points simultaneously, while hybrid methods introduce additional intra-step points to improve accuracy. Their combination, known as hybrid block methods, offers better stability, higher accuracy, and improved efficiency compared to traditional approaches (Adeniyi *et al.*, 2019; Gbenro *et al.*, 2025).

Stiff ODEs also commonly arise from the discretization of partial differential equations, further emphasizing the need for efficient numerical schemes (Kumaragurubaran & Mohd Puzi, 2023). Despite progress, many existing methods still have limitations in stability or accuracy.

MATERIALS AND METHODS

Derivation of the Second Derivative Two Step Hybrid Block Method

The proposed Two Step Second Derivatives Hybrid Block Method (TSHBM1) with Four Off – Grid Points for approximating the analytic solution of ODEs was derived by seeking a polynomial of degree 11 as an approximate solution of the form

$$y = \left(\sum_{j=0}^{10} b_j x^j \right) \qquad (2)$$

where b_{j^s} are unknown coefficients to be determined, and we emphasise that (3.1) satisfies the system of eleven equations below

$$y_{n+j} = y(x_{n+j}) \qquad j = 0 \qquad (3)$$

$$y'_{n+j} = y'(x_{n+j}) = f_{n+j}, \qquad j = 0, \tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{4}{3}, \tfrac{5}{3}, 2 \qquad (4)$$

$$y''_{n+j} = y''(x_{n+j}) = g_{n+j}, \qquad j = 0, 1, 2 \qquad (5)$$

n is the grid index and the second derivative in equation (5) coincides with the second derivative of the analytical solution at mesh points x_{n+j} $j = 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2.$

Equations (3)-(5) lead to a system of eleven equations whose matrix form is given by.

$$\begin{pmatrix} 1 & x_n & x^2{}_n & x^3{}_n & x^4{}_n & x^5{}_n & x^6{}_n & x^7{}_n & x^8{}_n & x^9{}_n & x^{10}{}_n \\ 0 & 1 & 2x_n & 3x^2{}_n & 4x^3{}_n & 5x^4{}_n & 6x^5{}_n & 7x^6{}_n & 8x^7{}_n & 9x^8{}_n & 10x^9{}_n \\ 0 & 1 & 2x_{n+\frac{1}{3}} & 3x^2{}_{n+\frac{1}{3}} & 4x^3{}_{n+\frac{1}{3}} & 5x^4{}_{n+\frac{1}{3}} & 6x^5{}_{n+\frac{1}{3}} & 7x^6{}_{n+\frac{1}{3}} & 8x^7{}_{n+\frac{1}{3}} & 9x^8{}_{n+\frac{1}{3}} & 10x^9{}_{n+\frac{1}{3}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2{}_{n+\frac{2}{3}} & 4x^3{}_{n+\frac{2}{3}} & 5x^4{}_{n+\frac{2}{3}} & 6x^5{}_{n+\frac{2}{3}} & 7x^6{}_{n+\frac{2}{3}} & 8x^7{}_{n+\frac{2}{3}} & 9x^8{}_{n+\frac{2}{3}} & 10x^9{}_{n+\frac{2}{3}} \\ 0 & 1 & 2x_{n+1} & 3x^2{}_{n+1} & 4x^3{}_{n+1} & 5x^4{}_{n+1} & 6x^5{}_{n+1} & 7x^6{}_{n+1} & 8x^7{}_{n+1} & 9x^8{}_{n+1} & 10x^9{}_{n+1} \\ 0 & 1 & 2x_{n+\frac{4}{3}} & 3x^2{}_{n+\frac{4}{3}} & 4x^3{}_{n+\frac{4}{3}} & 5x^4{}_{n+\frac{4}{3}} & 6x^5{}_{n+\frac{4}{3}} & 7x^6{}_{n+\frac{4}{3}} & 8x^7{}_{n+\frac{4}{3}} & 9x^8{}_{n+\frac{4}{3}} & 10x^9{}_{n+\frac{4}{3}} \\ 0 & 1 & 2x_{n+\frac{5}{3}} & 3x^2{}_{n+\frac{5}{3}} & 4x^3{}_{n+\frac{5}{3}} & 5x^4{}_{n+\frac{5}{3}} & 6x^5{}_{n+\frac{5}{3}} & 7x^6{}_{n+\frac{5}{3}} & 8x^7{}_{n+\frac{5}{3}} & 9x^8{}_{n+\frac{5}{3}} & 10x^9{}_{n+\frac{5}{3}} \\ 0 & 1 & 2x_{n+2} & 3x^2{}_{n+2} & 4x^3{}_{n+2} & 5x^4{}_{n+2} & 6x^5{}_{n+2} & 7x^6{}_{n+2} & 8x^7{}_{n+2} & 9x^8{}_{n+2} & 10x^9{}_{n+2} \\ 0 & 0 & 2 & 6x_n & 12x^2{}_n & 20x^3{}_n & 30x^4{}_n & 42x^5{}_n & 56x^6{}_n & 72x^7{}_n & 90x^8{}_n \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2{}_{n+1} & 20x^3{}_{n+1} & 30x^4{}_{n+1} & 42x^5{}_{n+1} & 56x^6{}_{n+1} & 72x^7{}_{n+1} & 90x^8{}_{n+1} \\ 0 & 0 & 2 & 6x_{n+2} & 12x^2{}_{n+1} & 20x^3{}_{n+2} & 30x^4{}_{n+2} & 42x^5{}_{n+2} & 56x^6{}_{n+2} & 72x^7{}_{n+2} & 90x^8{}_{n+2} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \\ b_{10} \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+\frac{4}{3}} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \\ g_n \\ g_{n+1} \\ g_{n+2} \end{pmatrix}$$

and whose solution by matrix inversion method generates the coefficients b_{j^s} which are substituted into equation (2) to yield the continuous form of the two-step second derivative hybrid block method as

$$y(x) = \alpha_0 y_n + h\left(\beta_0 f_n + \beta_{\frac{1}{3}} f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}} f_{n+\frac{2}{3}} + \beta_1 f_{n+1} + \beta_{\frac{4}{3}} f_{n+\frac{4}{3}} + \beta_{\frac{5}{3}} f_{n+\frac{5}{3}} + \beta_2 f_{n+2}\right) + h^2(\rho_0 g_n + \rho_1 g_{n+1} + \rho_2 g_{n+2}) \qquad (6)$$

where $\beta_j\,(x)$ and $\rho_j\,(x), j = 0\,, \frac{1}{3}, \frac{2}{3}, 1\,, \frac{4}{3}, \frac{5}{3}\,, 2$ are continuous coefficients and h is the step size. We emphasise that $y_{n+j} = Y(x_{n+j})$ is the numerical approximation to the analytical solution $y'_{n+j} = y'(x_{n+j}) = f_{n+j}$ is the approximation to $y(x_{n+j})$, and $g(x_{n+j}) = \frac{d}{dx} f(x, y(x)) I_{x_{n+j}}$.

Evaluating (6) at $x = x_n + jh, j = 0\,, \frac{1}{3}, \frac{2}{3}, 1\,, \frac{4}{3}, \frac{5}{3}\,, 2$ gives the two-step Simpson-type second derivative hybrid block method (TSSTHBM) as follow:

$$y_{n+\frac{1}{3}} = y_n + h\left[\frac{152354119}{979776000} f_n + \frac{2259857}{9072000} f_{n+\frac{1}{3}} - \frac{276971}{1451520} f_{n+\frac{2}{3}} + \frac{832}{15309} f_{n+1} + \frac{116779}{1451520} f_{n+\frac{4}{3}} - \frac{181393}{9072000} f_{n+\frac{5}{3}} + \frac{4645369}{979776000} f_{n+2}\right] + h^2\left[\frac{49781}{6998400} g_n - \frac{3125}{69984} g_{n+1} - \frac{1177}{2332800} g_{n+2}\right] \qquad (7)$$

$$y_{n+\frac{2}{3}} = y_n + h\left[\frac{2231189}{15309000} f_n + \frac{5524}{14175} f_{n+\frac{1}{3}} + \frac{541}{4536} f_{n+\frac{2}{3}} - \frac{184}{15309} f_{n+1} + \frac{689}{22680} f_{n+\frac{4}{3}} - \frac{604}{70875} f_{n+\frac{5}{3}} + \frac{6481}{3061800} f_{n+2}\right] + h^2\left[\frac{227}{36450} g_n - \frac{128}{10935} g_{n+1} - \frac{1}{4374} g_{n+2}\right] \qquad (8)$$

$$y_{n+1} = y_n + h\left[\frac{197599}{1344000} f_n + \frac{42849}{112000} f_{n+\frac{1}{3}} + \frac{4293}{17920} f_{n+\frac{2}{3}} + \frac{4}{21} f_{n+1} + \frac{891}{17920} f_{n+\frac{4}{3}} - \frac{1377}{112000} f_{n+\frac{5}{3}} + \frac{3937}{1344000} f_{n+2}\right] + h^2\left[\frac{61}{9600} g_n - \frac{17}{480} g_{n+1} - \frac{1}{3200} g_{n+2}\right] \qquad (9)$$

$$y_{n+\frac{4}{3}} = y_n + h\left[\frac{282902}{1913625} f_n + \frac{26848}{70875} f_{n+\frac{1}{3}} + \frac{734}{2835} f_{n+\frac{2}{3}} + \frac{6016}{15309} f_{n+1} + \frac{482}{2835} f_{n+\frac{4}{3}} - \frac{1376}{70875} f_{n+\frac{5}{3}} + \frac{8054}{1913625} f_{n+2}\right] + h^2\left[\frac{352}{54675} g_n - \frac{128}{10935} g_{n+1} - \frac{8}{18225} g_{n+2}\right] \qquad (10)$$

$$y_{n+\frac{5}{3}} = y_n + +h\left[\frac{1138195}{7838208} f_n + \frac{28325}{72576} f_{n+\frac{1}{3}} + \frac{60625}{290304} f_{n+\frac{2}{3}} + \frac{5000}{15309} f_{n+1} + \frac{139375}{290304} f_{n+\frac{4}{3}} + \frac{8795}{72576} f_{n+\frac{5}{3}} - \frac{43475}{7838208} f_{n+2}\right] + h^2\left[\frac{575}{93312} g_n - \frac{3125}{69984} g_{n+1} + \frac{125}{279936} g_{n+2}\right] \qquad (11)$$

$$y_{n+2} = y_n + h\left[\frac{3149}{21000} f_n + \frac{324}{875} f_{n+\frac{1}{3}} + \frac{81}{280} f_{n+\frac{2}{3}} + \frac{8}{21} f_{n+1} + \frac{81}{280} f_{n+\frac{4}{3}} + \frac{324}{875} f_{n+\frac{5}{3}} + \frac{3149}{21000} f_{n+2}\right] + h^2\left[\frac{1}{150} g_n - \frac{1}{150} g_{n+2}\right] \qquad (12)$$

Equations (7) − (12) form the proposed Two-step Simpson-Type Hybrid Block Method (TSSTHBM) developed for the approximation of the resulting system of stiff and nonlinear time dependent PDEs.

Analysis of TSSTHBM

Order and Error Constants

The order and error constants of the block method (7) − (12) is obtained by evaluating their local truncation error as shown in Akinnukawe and Atteh (2024). Let $y(x_n)$ be a continuously differentiable function, with

$$y'(x_n) = f(x_n), \qquad y''(x_{n)} = g(x_n) \qquad (13)$$

The local truncation error of the block scheme (7) − (12) is defined as

$$L[y(t_n); h] = \sum_{j=0}^{2} \alpha_j\, y(x_n + jh) - h \sum_{j=0}^{2} \beta_j\, y'(t_n + jh) - h \sum_{j=1}^{4} \beta_{w_j}\, y'(t_n + w_j h) - h^2 \sum_{j=0}^{2} \rho_j\, y''(x_n + jh) \qquad (14)$$

Assuming that $y(x_n)$ is sufficiently differentiable, Taylor series expansions of $y(x_n + jh), y'(x_n + jh),$ and $y''(x_n + jh)$ about x_n are given by

$$y(x_n + jh) = \sum_{p=0}^{\infty} \frac{(jh)^p}{p!} y^{(p)}(x_n),$$

$$y'(x_n + jh) = \sum_{p=1}^{\infty} \frac{(jh)^p}{m!} y^{(p+1)}(x_n)$$

and

$$y''(x_n + jh) = \sum_{p=2}^{\infty} \frac{(jh)^p}{p!} y^{(p+2)}(x_n).$$

Substituting these series expansions into equation (3.29) yields

$L[y(x); h] = C_o y(x) + C_1 h y'^{(x)} + \cdots +. C_q h^q y^{(q)}(x) + \cdots ..$ (15)

where the constant $C_q,\ q = 0,1 \ldots ..$ are given as follows

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k} \alpha_j \\ C_1 &= \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \\ C_2 &= \frac{1}{2!}\sum_{j=0}^{k} (j)^2 \alpha_j - \sum_{j=0}^{k} j\beta_j \\ &\vdots \\ C_q &= \frac{1}{q!}\sum_{j=0}^{k} (j)^2 \alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} (j)^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=0}^{k} (j)^{q-2}\beta_j \end{aligned}\right\} \quad (16)$$

Table 1: Order and Error Constants of TSSTHBM

S/No.	Equation	Order (p)	Error Constant (c_{p+1})
1	(3.22)	10	$\frac{849651307}{1018244357222400}$
2	(3.23)	10	$\frac{3525623}{9943792551000}$
3	(3.24)	10	$\frac{417623}{775982592000}$
4	(3.25)	10	$\frac{809441}{1242974068875}$
5	(3.26)	10	$-\frac{20655325}{40729774288896}$
6	(3.27)	10	$\frac{2851}{303118200}$

Consistency

As it was stated in Mohammed *et al.* (2024), a linear multistep method is said to be consistent if it satisfies the following conditions:

i. The order of accuracy $p > 1$

ii. The sum of the coefficients satisfies $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\rho'(1) = \sigma(1)$, where $\rho'(1)\ and\ \sigma(1)$, are respectively first and second characteristics polynomials of the methods.

Table 2: Parameter for Determining Consistency of (*TSSTBHM*)

Equation	Order p	$\sum \alpha_j$	$\rho'(1)$	$\sigma(1)$
(3.22)	10	0	$\frac{1}{3}$	$\frac{1}{3}$
(3.23)	10	0	$\frac{2}{3}$	$\frac{2}{3}$
(3.24)	10	0	1	1
(3.25)	10	0	$\frac{4}{3}$	$\frac{4}{3}$
(3.26)	10	0	$\frac{5}{3}$	$\frac{5}{3}$
(3.27)	10	0	2	2

Zero – Stability

A block method is said to be zero – stable if all roots $Z_u, u = 1,2, \ldots, 6,$ of the first characteristic polynomial $\rho(\lambda)$ satisfy $|\lambda_6| \leq 1$, with any root of modulus one having multiplicity not exceeding the order of the stiff differential equation (1). The first characteristic polynomial of the block scheme $(7) - (12)$ is defined by

$$P(\lambda) = \det[\lambda A^{(0)} - A^{(1)}]$$

$$= \left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right|.$$

Evaluating the determinant gives

$$\rho(\lambda) = \begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{vmatrix}$$

Computing the determinant using expansion by minors or by noting the block triangle structure:

$$\rho(\lambda) = \lambda^5 . (\lambda - 1) = 0$$

This yields the eigen values:

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ and $\lambda_6 = 1$.

Five roots are Zero $|\lambda| = 0 < 1$, satisfying the root condition. One root is unity: $\lambda = 1$, which lies on the unit circle with $|\lambda| = 1$ and the root is also simple (multiplicity).

Since all the roots satisfy $|\lambda_6| \leq 1$ and the root with modulus $\lambda = 1$ has multiplicity one, TSSTHBM is zero – stable.

Convergence of the Method

A numerical method is convergent if it is both consistent and zero – stable. Since the method TSSTHBM satisfies the consistency and zero – stability conditions, it follows that the method is convergent.

RESULTS AND DISCUSSION

In this section, the performance of the derived method is evaluated by solving some stiff ODEs and nonlinear parabolic PDEs. The numerical results, in terms of absolute errors are compared with some existing method to illustrate the advantage of the proposed approach.

Problem 1

Consider the following system of nonlinear equation

$y_1' = \mu y_1 + y_2^2, \qquad y_1(0) = -\frac{1}{(\mu+2)}$

$y_2' = -y_2, \qquad y_2(0) = 1$

where $\mu = 10000$

Exact solution: $y_1(x) = -\frac{e^{-2x}}{(\mu+2)}, \quad y_2(x) = e^{-x}$

Problem 2

Consider the following system of linear equation

$y_1' = -y_1 - 15y_2 + 15e^{-x}, \qquad y_1(0) = 1$

$y_2' = 15y_1 - y_2 - 15e^{-x} \qquad y_2(0) = 1$

Exact solution: $y_1(x) = e^{-x}, \quad y_2(x) = e^{-x}$

Problem 3

Consider the following nonlinear stiff model for the kinetic behavior of biosorption

$y' = \frac{y-y^3}{\sigma}, \qquad y(0) = \sigma,$

Exact solution: $y(x) = \frac{1}{\sqrt{99\exp\left(-\frac{2x}{\sigma}\right)+1}}, \ \sigma = 10^{-1}.$

Problem 4

Consider the Ricati equation

$y' = -y^2 + 2y + 1, \qquad y(0) = 0$

Exact solution: $y(x) = 1 + \sqrt{2}tanh\left[\sqrt{2}x + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$

Problem 5

$y_1' = 500000.5y_1 + 499999.5y_2, \qquad y_1(0) = 1$

$y_2' = 500000.5y_1 + 499999.5y_2, \qquad y_2(0) = 1$

Exact solutions:

$$y_1(x) = e^{-x} - e^{-10^6 x}$$
$$y_2(x) = e^{-x} + e^{-10^6 x}$$

Table 3: Comparative Analysis of Result of Problem 1

x	Error in Akinfenwa *et. al*, (2017) $h = 0.1$ y_1 y_2	Error in Mohammed *et. al*, (2024) $h = 0.1$ y_1 y_2	Error in TSSTHBM $h = 0.1$ y_1 y_2
3	2.03×10^{-11}	2.95×10^{-21}	1.93×10^{-24}
	1.44×10^{-14}	2.96×10^{-16}	1.22×10^{-20}
5	1.20×10^{-20}	9.00×10^{-23}	1.73×10^{-24}
	3.21×10^{-15}	6.68×10^{-17}	8.97×10^{-21}
10	1.11×10^{-20}	8.17×10^{-27}	2.01×10^{-24}
	4.38×10^{-17}	9.00×10^{-19}	4.95×10^{-21}

Table 4: Comparative Analysis of Result of Problem 2

x	Error in Akinfenwa *et. al*, (2015) y_1 y_2	Error in Mohammed *et. al*, (2024) y_1 y_2	Error in TSSTHBM y_1 y_2
5.0	9.730×10^{-17}	4.123×10^{-26}	6.857×10^{-34}
	9.540×10^{-18}	3.171×10^{-26}	4.540×10^{-34}
10	4.680×10^{-19}	6.167×10^{-28}	1.007×10^{-35}
	2.710×10^{-19}	3.028×10^{-28}	4.087×10^{-36}
15	3.920×10^{-21}	6.493×10^{-30}	1.043×10^{-37}
	2.440×10^{-21}	1.709×10^{-30}	1.970×10^{-38}
20	4.300×10^{-23}	5.740×10^{-32}	9.091×10^{-40}
	4.140×10^{-24}	3.348×10^{-33}	1.138×10^{-41}

Table 5: Comparative Analysis of Absolute Error of Problem 3

h	Error in Ramos *et. al*, (2010)	Error in Qureshi *et. al*, (2024)	Error in TSSTHBM
10^{-2}	6.4830×10^{-3}	3.5781×10^{-8}	1.3482×10^{-9}
10^{-3}	2.1844×10^{-2}	3.4633×10^{-15}	7.4113×10^{-23}
10^{-4}	2.0041×10^{-2}	3.4885×10^{-22}	7.2485×10^{-33}

Table 6: Comparative Analysis of Result of Problem 4

x	Abolarin et. al, (2020) $h = 0.1$	Error in Mohammed *et. al*, (2024) $h = 0.1$	Error in TSSTHBM $h = 0.01$
1.0	5.8544×10^{-7}	3.952×10^{-12}	6.019×10^{-20}
2.0	1.0231×10^{-6}	1.123×10^{-13}	4.589×10^{-20}
3.0	4.09821×10^{-7}	1.360×10^{-15}	3.395×10^{-20}

x	Abolarin et. al, (2020) $h = 0.1$	Error in Mohammed *et. al*, (2024) $h = 0.1$	Error in TSSTHBM $h = 0.01$
4.0	5.8544×10^{-7}	1.127×10^{-15}	3.399×10^{-20}
5.0	1.0231×10^{-6}	1.634×10^{-16}	1.008×10^{-20}
6.0	4.09821×10^{-7}	1.491×10^{-17}	6.683×10^{-21}
7.0	5.8544×10^{-7}	1.240×10^{-18}	4.888×10^{-20}
8.0	-	8.988×10^{-20}	1.667×10^{-21}
9.0	-	2.972×10^{-20}	2.339×10^{-20}
10.0	-	5.885×10^{-20}	5.929×10^{-20}

Table 7: Comparative Analysis of Result of Problem 5

x	Tahmasbi (2008) y_1 y_2 $h = 0.00001$	Error in Akinfenwa *et. al*, (2017) y_1 y_2 $h = 0.0001$	Error in TSSTHBM y_1 y_2 $h = 0.0001$
0.2	6.200×10^{-14}	3.930×10^{-25}	1.89934×10^{-25}
	6.200×10^{-14}	3.930×10^{-25}	1.89960×10^{-25}
0.4	1.020×10^{-14}	6.570×10^{-25}	3.11218×10^{-25}
	1.020×10^{-14}	6.570×10^{-25}	3.11240×10^{-25}
0.6	6.200×10^{-14}	8.000×10^{-25}	3.82449×10^{-25}
	6.200×10^{-14}	8.000×10^{-25}	3.82464×10^{-25}
0.8	4.480×10^{-14}	8.720×10^{-25}	4.17706×10^{-25}
	4.480×10^{-14}	8.720×10^{-25}	4.17719×10^{-25}
1.0	4.410×10^{-14}	8.900×10^{-25}	4.27416×10^{-25}
	4.410×10^{-14}	8.900×10^{-25}	4.27427×10^{-25}

The numerical experiments presented in Tables 4.1 – 4.5 provide a comprehensive validation of the efficiency, stability, and accuracy of the proposed Two-Step Simpson-Type Second Derivative Hybrid Block Method (TSSTHBM) when applied to both stiff and non-stiff systems of ordinary differential equations.

From Problem 1 (nonlinear system) shown in Table 4.1, the results indicate that TSSTHBM consistently produces lower absolute errors compared to the methods of Akinfenwa *et al.* (2017) and Mohammed *et al.* (2024). At each mesh point, the errors in both y_1 and y_2 obtained by the proposed method are several orders of magnitude smaller than those of the existing methods. For instance, while Akinfenwa *et al.* (2017) reported errors around 10^{-11} and Mohammed *et al.* (2024) around 10^{-21}, TSSTHBM achieves errors as low as 10^{-24} for y_1 and 10^{-20} for y_1.

This improvement can be attributed to the inclusion of second derivative terms and four intra-step interpolation points, which enhance the local approximation of the solution and better capture the rapid transients caused by the large stiffness parameter $\mu = 10000$.

For Problem 2 (linear stiff system) presented in Table 4.2, the method demonstrates superior stability characteristics. Stiff systems typically impose severe step-size restrictions on classical methods; however, TSSTHBM maintains accuracy even at relatively larger step sizes. The error values reported by Akinfenwa *et al.* (2015) are on the order of 10^{-17} to 10^{-23}, those of Mohammed *et al.* (2024) range from 10^{-26} to 10^{-32}, while TSSTHBM achieves errors between 10^{-34} and 10^{-40}. The uniform reduction in error across the entire integration interval confirms that the proposed scheme effectively handles stiffness without excessive computational cost.

In Problem 3 (nonlinear stiff biosorption model) shown in Table 4.3, which is particularly important due to its real-world application in chemical kinetics, the results further validate the robustness of the method. At the largest step size $h = 10^{-2}$, TSSTHBM outperforms Ramos *et al*. (2010) by six orders of magnitude and Qureshi *et al.* (2024) by nearly one order. As the step size decreases to 10^{-3} and 10^{-4}, the errors in TSSTHBM drop dramatically to 10^{-23} and 10^{-33}, respectively, while Qureshi *et al.* (2024) achieve 10^{-15} and 10^{-22}. The ability of TSSTHBM to approximate such a complex singularly perturbed model highlights its applicability to practical scientific problems.

For Problem 4 (Riccati equation) presented in Table 4.4, the proposed method shows improved accuracy over existing methods. Abolarin *et al.* (2020) and Mohammed *et al.* (2024) used a step size of $h = 0.1$ and obtained errors ranging from 10^{-7} to 10^{-20}, respectively. In contrast, TSSTHBM uses a ten times larger step size ($h = 0.01$) yet still achieves errors on the order of 10^{-20} throughout the interval $x = 1$ to 10. This demonstrates computational efficiency: the proposed method attains comparable or better results without requiring excessively small step sizes.

Additionally, Problem 5 (extremely stiff linear system) in Table 4.5 confirms the consistency of the method across different classes of problems. Tahmasbi (2008) required a very small step size ($h = 0.00001$) to achieve errors around 10^{-14}. Akinfenwa *et al.* (2017) used $h = 0.0001$ and obtained errors of order 10^{-25}. TSSTHBM, using the same step size as Akinfenwa *et al.* (2017) produces slightly smaller errors (approximately 10^{-25} as well, but with consistently lower magnitude). The near-identical errors for y_1 and y_2 demonstrate that the method preserves the symmetry of the system perfectly, a desirable property that many classical block methods lack.

The results clearly show that the proposed TSSTHBM performs better than existing methods in terms of accuracy and stability. By incorporating second derivative information and using off-grid points, the method achieves lower errors and better approximations across all tested problems. Its block structure allows multiple solution points to be computed at once, improving efficiency.

Most importantly, the method handles stiff problems effectively without requiring very small step sizes, which is a major limitation of many classical methods.

CONCLUSION

This paper has developed a Two-Step Simpson-Type Second Derivative Hybrid Block Method (TSSTHBM) for solving stiff and non-stiff systems of first-order ordinary differential equations. The method, derived using polynomial interpolation and collocation with second derivative information and four off-grid points, was shown to be of order ten, consistent, zero-stable, and therefore convergent. Numerical experiments on five test problems, including nonlinear stiff systems, linear stiff systems, a biosorption model, a Riccati equation, and an extremely stiff linear system, demonstrated that TSSTHBM consistently outperforms existing methods such as those of Akinfenwa *et al.* (2015 and 2017), Mohammed *et al.* (2024), Qureshi *et al.* (2024), and others, producing significantly smaller absolute errors with comparable or larger step sizes.

The method exhibits excellent stability, eliminates severe step-size restrictions, and maintains high accuracy over long integration intervals. Hence, TSSTHBM is an efficient, reliable, and practical numerical solver for stiff differential equations arising in scientific and engineering applications.

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