FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 9, May, 2026, pp 282 – 287
DOI: https://doi.org/10.33003/fjs-2026-1009-5419

Development of Third Derivatives Falkner-Type for the Solution of Second Order Differential Equations

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ABSTRACT
Second-order ordinary differential equations frequently arise in scientific and engineering applications and efficient numerical methods are often required when analytical solutions are difficult or impossible to obtain. In this paper, a Falkner type method for k = 2 with two off-step point were derived for the numerical solution of second order initial value problems. The idea of collocation and interpolation techniques was adopted in the derivation of the schemes. The basic properties of numerical methods were analysed and the methods were found to be consistent, zero stable and hence convergent. Numerical experiments were carried out on three (3) problems of second order initial value problem (IVP). The results obtained for the proposed methods in comparison with the exact solutions and some existing methods from the literature show the efficiency and reliability of the proposed schemes.

Received: 15 June 2026
Accepted: 24 June 2026
Published: 30 June 2026

Keywords: Falkner-type methods, Third-derivative methods, Second-order ODEs, Initial value problems (IVPs), Numerical methods

INTRODUCTION

Differential equations of the form

$$y''(x) = f(x, y(x), y'(x)), y(a) = y_0, y'(a) = y'_0 \qquad (1)$$

where $x \in [a,b], y: [a,b] \to \mathbb{R}$ and $f: [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are sufficiently differentiable functions, are widely used to model numerous real-life problems such as chemical kinetics, orbital dynamics, circuit theory, control systems, and Newton's second law of motion. However, in most cases, the differential equations arising from these applications do not possess closed-form analytical solutions.

Consequently, one of the practical approaches to handling such problems is to consider a discrete domain rather than a continuous one. Hence, for practical purposes in science and engineering, numerical approximations to the exact solution are often sufficient. Although it is possible to integrate equation (1) by reducing it to a system of first-order differential equations and applying existing numerical methods, this approach increases computational effort and may reduce efficiency.

It is therefore more natural and efficient to develop numerical methods that solve the problem directly without reduction. In particular, the second-order differential equation (1) can be transformed into a third-derivative Falkner-type formulation by differentiating with respect to the independent variable (x). This transformation leads to a third-derivative representation of the form

$$y'''(x) = g(x, y(x), y'(x), y''(x)) \qquad (2)$$

which forms the basis of the third-derivative Falkner-type method. This approach enhances the development of higher-order numerical schemes by incorporating additional derivative information, thereby improving accuracy and computational efficiency. (Omar & Suleiman, 2021; Ramos & Vigo-Aguiar, 2000).

Scholars have proposed numerous numerical methods for approximating initial value problems such as (1); these methods range from discrete schemes (Lambert, 1973; Butcher, 2008; Fatunla, 1988) to predictor–corrector methods (Onumanyi *et al.*, 1994; Fatunla, 1994; Awoyemi and Idowu, 2005; Areo and Adeniyi, 2013; Omar and Kuboye, 2015; Ndanusa and Tafida, 2016) and block methods (Badmus and Yahaya, 2009; Jator and Li, 2012; Mohammed, 2011; Mohammed and Adeniyi, 2014; Badmus *et al.*, 2015; Akinfenwa *et al.*, 2013; Omar and Adeyeye, 2016; Akinfenwa *et al.*, 2017). Recent investigations have continued to improve the development of hybrid block and Falkner-type methods for solving higher-order ordinary differential equations directly. Ramos and Singh (2022) developed a third-derivative hybrid block integrator for second-order boundary value problems with improved convergence and computational accuracy. Rufai (2022) proposed an efficient third-derivative hybrid block method based on interpolation and collocation techniques for second-order boundary value problems. In 2023, Hussain *et al.*, (2023) developed a two-step block method with third and fourth derivatives for solving second-order fuzzy ordinary differential equations, demonstrating enhanced stability and accuracy properties. Furthermore, Akinnukawe and Okunuga (2024) proposed a one-step optimised hybrid block scheme with hybrid points for numerical integration of second-order ordinary differential equations, showing improved error minimisation and efficiency. Recent studies in 2025 by Abdelrahim *et al.*, (2025). introduced a three-step hybrid block method with generalised off-step points for directly solving third-order ordinary differential equations, while Kayode *et al.*, (2025) proposed a three-step hybrid block approach for direct integration of third-order initial value problems with enhanced convergence and stability characteristics. These developments indicate the growing interest in higher-derivative and Falkner-type numerical schemes due to their improved computational performance and reduced cost in solving complex differential systems.

In this paper, we present a hybrid block method based on third-derivative Falkner-type formulas, in which two generalized off-step points are considered within the interval $0 \leq x \leq 2$ to increase the number of function evaluations and improve the accuracy of the method.

MATERIALS AND METHODS

In this section, we derive some linear multistep methods based on a third-derivative Falkner-type formulation for the solution of second-order ordinary differential equations.

Considering the transformed in equation (2):

Following the standard multistep approach, we seek approximate solutions in the form:

$$y'_{n+1} = y'_n + hy''_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j g_n \qquad (3)$$

$$y''_{n+1} = y''_n + h \sum_{j=0}^{k-1} \gamma_j \nabla^j g_n \qquad (4)$$

where h is the step size, $y_n, y'_n \; and \; y''_n$, are numerical approximations to the exact solution and its derivatives at the grid point $x_n = a + nh; n = 0,1,2,3,\dots,N,\; h = \frac{(b-a)}{N}$ and

$$g_n = g(x_n, y_n, y'_n, y''_n) \qquad (5)$$

While $\nabla^j g_n$ denotes the backward differential operator.

We then construct the continuous approximation by imposing the following conditions

$$\left.\begin{aligned} y(x) &= y_n \\ y'(x) &= z_n \\ y''(x) &= f_{n+j} \\ y'''(x) &= g_{n+j} \\ j &= 0,\dots,1 \end{aligned}\right\} \qquad (6)$$

Equation (6) leads to a system of equations and unknowns written in the form AU=B

$$U = \begin{pmatrix} 1 & x_n & {x_n}^2 & {x_n}^3 & {x_n}^4 & {x_n}^5 & {x_n}^6 & \cdot\;\cdot\;\cdot & {x_n}^m \\ 0 & 1 & 2x_n & {3_x}^2 & 4{x_n}^3 & 5{x_n}^4 & 6{x_n}^5 & \cdot\;\cdot\;\cdot & m{x_n}^{m-1} \\ 0 & 0 & 2 & 6x_n & 12{x_n}^2 & 20{x_n}^3 & 30{x_n}^4 & \cdot\;\cdot\;\cdot & m(m-1){x_n}^{m-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & 2 & 6x_{n+1} & 12{x_{n+1}}^2 & 20{x_{n+1}}^3 & 30{x_{n+1}}^4 & \cdot\;\cdot\;\cdot & m(m-1){x_{n+1}}^{m-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & 2 & 6x_{n+2} & 12{x_{n+2}}^2 & 20{x_{n+2}}^3 & 30{x_{n+2}}^4 & \cdot\;\cdot\;\cdot & m(m-1){x_{n+2}}^{m-2} \\ 0 & 0 & 0 & 6 & 24x_n & 60{x_n}^2 & 120{x_n}^3 & \cdot\;\cdot\;\cdot & m(m-1)(m-2){x_n}^{m-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60{x_{n+1}}^2 & 120{x_{n+1}}^3 & \cdot\;\cdot\;\cdot & m(m-1)(m-2){x_{n+1}}^{m-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60{x_{n+2}}^2 & 120{x_{n+2}}^3 & \cdot\;\cdot\;\cdot & m(m-1)(m-2){x_{n+2}}^{m-3} \end{pmatrix}$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}, B = \begin{bmatrix} y_n \\ y'_n \\ f_n \\ f_{n+1} \\ f_{n+2} \\ g_n \\ g_{n+1} \\ g_{n+2} \end{bmatrix} \qquad (7)$$

Solving (7) using matrix inversion method to obtain α_j and then substituted into (3) to get the continuous scheme of the form:

$$y(x_{n+2}) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{k} \alpha'_j(x) hy'_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} + h^3 \sum_{j=0}^{k} \delta_j(x) g_{n+j} \qquad (8)$$

where $\alpha_j(x), \beta_j(x) \; and \; \delta_j(x)$ are continuous coefficients. We observe that (6) includes first derivative and can be obtained by substituting the coefficients of α_j into the first derivative of (4) to yield

$$hy'(x_{n+2}) = \sum_{j=0}^{1} \alpha_j(x) hy'_{n+j} + h^2 \sum_{j=0}^{2} \beta_j(x) f_{n+j} + h^3 \sum_{j=0}^{2} \gamma_j(x) g_{n+j} \qquad (9)$$

the main and additional methods can be obtained from (8) and (9). Both methods are called Hybrid Falkner-Type Block methods (HFBM);

Two – step method with $\frac{1}{2}, \frac{3}{2}$ off-step points

In the derivation of a continuous for two step method, two off-step points $\frac{1}{2}, \frac{3}{2}$ are considered with these specifications: $r = 2, k = 2, s = 5, p = 5$. The continuous form of the method is of the form:

$$y(x) = \alpha_0 y_n + h\alpha'_0 z_n + h^2 \left[\beta_0 f_n + \beta_1 f_{n+\frac{1}{2}} + \beta_2 f_{n+1} + \beta_3 f_{n+\frac{3}{2}} + \beta_4 f_{n+2}\right] + h^3 \left[\gamma_0 g_n + \gamma_1 g_{n+\frac{1}{2}} + \gamma_2 g_{n+1} + \gamma_3 g_{n+\frac{3}{2}} + \gamma_4 g_{n+2}\right] \qquad (10)$$

Evaluating (10) above at point $x = x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}} \; and \; x_{n+2}$ gives the following four discrete scheme that form the block method

$$y_{n+\frac{1}{2}} = \frac{2602339}{38320128} h^2 f_n + \frac{148231}{11975040} h^2 f_{n+\frac{1}{2}} + \frac{1807}{80640} h^2 f_{n+1} + \frac{243193}{11975040} h^2 f_{n+\frac{3}{2}} + \frac{382169}{191600640} h^2 f_{n+2} + \frac{28343}{9123840} h^3 g_n \frac{-551}{24948} h^3 g_{n+\frac{1}{2}} - \frac{32027}{1774080} h^3 g_{n+1} - \frac{3959}{798336} h^3 g_{n+\frac{3}{2}} - \frac{14339}{63866880} h^3 g_{n+2} + \frac{1}{2} hz_n + y_n \qquad (11)$$

$$y_{n+1} = \frac{140}{891} h^2 f_n + \frac{784}{4455} h^2 f_{n+\frac{1}{2}} + \frac{1}{10} h^2 f_{n+1} + \frac{272}{4455} h^2 f_{n+\frac{3}{2}} + \frac{26}{4455} h^2 f_{n+2} + \frac{1277}{166320} h^3 g_n \frac{-41}{693} h^3 g_{n+\frac{1}{2}} - \frac{40}{693} h^3 g_{n+1} - \frac{17}{1155} h^3 g_{n+\frac{3}{2}} - \frac{109}{166320} h^3 g_{n+2} + hz_n + y_n \qquad (12)$$

$$y_{n+\frac{3}{2}} = \frac{39015}{157696} h^2 f_n + \frac{18531}{49280} h^2 f_{n+\frac{1}{2}} + \frac{3159}{8960} h^2 f_{n+1} + \frac{6813}{49280} h^2 f_{n+\frac{3}{2}} + \frac{8469}{788480} h^2 f_{n+2} + \frac{9747}{788480} h^3 g_n \frac{-4509}{49280} h^3 g_{n+\frac{1}{2}} - \frac{19197}{197120} h^3 g_{n+1} - \frac{9}{308} h^3 g_{n+\frac{3}{2}} - \frac{27}{22528} h^3 g_{n+2} + \frac{3}{2} hz_n + y_n \qquad (13)$$

$$y_{n+2} = \frac{6353}{18711} h^2 f_n + \frac{55808}{93555} h^2 f_{n+\frac{1}{2}} + \frac{208}{315} h^2 f_{n+1} + \frac{34304}{93555} h^2 f_{n+\frac{3}{2}} + \frac{3457}{93555} h^2 f_{n+2} + \frac{538}{31185} h^3 g_n \frac{-3712}{31185} h^3 g_{n+\frac{1}{2}} - \frac{80}{693} h^3 g_{n+1} - \frac{128}{4455} h^3 g_{n+\frac{3}{2}} - \frac{20}{6237} h^3 g_{n+2} + 2hz_n + y_n \qquad (14)$$

The following schemes are obtained by differentiating equation (10) and evaluating at points $x = x_{n+\frac{1}{2}}, , x_{n+1}, x_{n+\frac{3}{2}}$,and x_{n+2} *to give the following*

$$hy'_{n+\frac{1}{2}} = \frac{1539551}{8709120} hf_n + \frac{89371}{544320} hf_{n+\frac{1}{2}} + \frac{103}{1260} hf_{n+1} + \frac{38341}{544320} hf_{n+\frac{3}{2}} + \frac{59681}{8709120} hf_{n+2} + \frac{26051}{2903040} h^2 g_n \frac{-31207}{362880} h^2 g_{n+\frac{1}{2}} - \frac{81}{1280} h^2 g_{n+1} - \frac{1243}{72576} h^2 g_{n+\frac{3}{2}} - \frac{2237}{2903040} h^2 g_{n+2} + hz_n \qquad (15)$$

$$hy'_{n+1} = \frac{24463}{136080}hf_n + \frac{3308}{8505}hf_{n+\frac{1}{2}} + \frac{104}{315}hf_{n+1} + \frac{788}{8505}hf_{n+\frac{3}{2}} + \frac{1153}{136080}hf_{n+2} + \frac{421}{45360}h^2g_n \frac{-38}{567}h^2g_{n+\frac{1}{2}} - \frac{1}{10}h^2g_{n+1} - \frac{62}{2835}h^2g_{n+\frac{3}{2}} - \frac{43}{45360}h^2g_{n+2} + hz_n \quad (16)$$

$$hy'_{n+\frac{3}{2}} = \frac{6501}{35840}hf_n + \frac{921}{2240}hf_{n+\frac{1}{2}} + \frac{81}{140}hf_{n+1} + \frac{711}{2240}hf_{n+\frac{3}{2}} + \frac{411}{35840}hf_{n+2} + \frac{339}{35840}h^2g_n \frac{-279}{4480}h^2g_{n+\frac{1}{2}} - \frac{81}{1280}h^2g_{n+1} - \frac{183}{4480}h^2g_{n+\frac{3}{2}} - \frac{9}{7168}h^2g_{n+2} + hz_n \quad (17)$$

$$hy'_{n+2} = \frac{1601}{8505}hf_n + \frac{4096}{8505}hf_{n+\frac{1}{2}} + \frac{208}{315}hf_{n+1} + \frac{4096}{8505}hf_{n+\frac{3}{2}} + \frac{1601}{8505}hf_{n+2} + \frac{29}{2835}h^2g_n \frac{-128}{2835}h^2g_{n+\frac{1}{2}} - \frac{1}{10}h^2g_{n+1} + \frac{128}{2835}h^2g_{n+\frac{3}{2}} - \frac{29}{2835}h^2g_{n+2} + hz_n \quad (18)$$

Analysis of the Methods

In this section, we discuss in general the order and error constants, consistency, zero stability and the convergence of the proposed method

Order and Error Constants

Let the linear difference operator L associated with k-step method be defined as

$$L[y(x_n);h] = \sum_{j=0}^{k}\left(\alpha_j y(x_n + jh) - h\beta y'(x_0) - h^2\gamma_{vj} f(x_n + jvh)\right) \quad (19)$$

and

$$L[y'(x_n);h] = \sum_{j=0}^{k}\left(h\overline{\beta}_j y'(x_n + jvh) - h^2\overline{\gamma}_{vj} hf(x_n + jvh)\right) \quad (20)$$

respectively. Assuming that y (x_n) and $y'(x_n)$ are sufficiently differentiable, we can expand the terms in (19) and (20) as Taylor series about the point x_n to obtain the expression

$$L[y(x_n);h] = C_0 y(x_n) + C_1 hy'(x_n) + \ldots + C_q h^q y^q(x_n) + .. \quad (21)$$

and

$$L[y'(x_n);h] = \overline{C}_0 y'(x_n) + \overline{C}_1 hy'(x_n) + \ldots + \overline{C}_q h^q y^{(q+1)}(x_n) + \ldots \quad (22)$$

Respectively;

Where the constants C_q and $\overline{C}_q$, $q = 0,1,..$ are given as follows

$$\left.\begin{aligned} C_0 &= \sum_{j=0}^{k}\alpha_j \\ C_1 &= \sum_{j=1}^{k} j\alpha_j \\ C_2 &= \frac{1}{2!}\sum_{j=1}^{k}(j)^2\alpha_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-1}\beta_j \\ C_q &= \frac{1}{q!}\sum_{j=1}^{k}(j)^q\alpha_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-1}\beta_j \end{aligned}\right\} \quad (23)$$

q= 2, 3, …

$$\left.\begin{aligned} \bar{C}_0 &= \sum_{j=0}^{k}\bar{\alpha}_j \\ \bar{C}_1 &= \sum_{j=1}^{k} j\bar{\alpha}_j - \sum_{j=0}^{k} j\bar{\beta}_j \; C_2 = \frac{1}{2!}\sum_{j=1}^{k}(j)^2\alpha_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-1}\beta_j \\ \bar{C}_2 &= \frac{1}{2!}\sum_{j=1}^{k}(j)^q\bar{\alpha}_j - \frac{1}{(q-2)!}\sum_{j=1}^{k} j^{q-1}\bar{\beta}_j \\ C_q &= \frac{1}{q!}\sum_{j=1}^{k}(j)^q\bar{\alpha}_j - \frac{1}{(q-1)!}\sum_{j=1}^{k} j^{q-1}\bar{\beta}_j \end{aligned}\right\}. \quad (24)$$

The methods (23) and (24) are of order p if $C_0 = C_1$, C… C_p= C_{p+2}= 0, C_{p+2} ≠0 and $C_{p+2}$is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ the principal truncation error at the point x_n

Zero Stability

This is the concept concerning the behavior of a numerical method as $h \to 0$, the system of equation (9) becomes

$$\left.\begin{aligned} y_{n+1} &= y_{n+k-r} y_{n+2} = y_{n+k-r} \\ y_{n-2} &= y_{n+k-r} \\ y_{n+k-1} &= y_n \\ y_{n+k} &= y_{n+k-r} \end{aligned}\right\} \quad (25)$$

which can be written in matrix form as

$$A^0\overline{Y}_\mu - A^I\overline{Y}_{\mu-1} = 0 \quad (26)$$

Where $\overline{Y}_\mu = (y_{n+1}, y_{n+2}, \ldots, y_{n+k})^T, \overline{Y}_{\mu-1} = (y_n, y_{n+1}, \ldots, y_{n+k-r})^T$,

A^0 is the idendity matrix of dimension K and A^I is a matrix of dimension K

Consistency

Each of the methods is consistent as they all have order > 1.

Convergence

The convergence of the proposed methods is considered in the light of the basic properties in conjunction with the fundamental theorem of Dahlquist (1956), (Henrichi 1962) for linear multistep methods. We state here the Dahlquist theorem without proof.

Theorem

The necessary and sufficient condition for a multistep method to be convergent is for it to be consistent and zero-stable.

RESULTS AND DISCUSSION

Numerical Experiments

In this section, some standard second-order initial value problems of ordinary differential equations are solved using the proposed Falkner-type method in order to demonstrate its efficiency and accuracy. The implementation is carried out as a block (self-starting) method, in which the continuous form of the scheme generates both the main and additional discrete Falkner formulas simultaneously at each integration step within the interval of integration.

The absolute errors produced by the proposed method are compared with those obtained from existing methods available in the literature. The results of these comparisons are presented in tabular form for ease of analysis and validation of the performance.

Problem 1: (Source: Alhassan and Mohammed (2021))

Consider the nonlinear homogenous problem given by:

$y" = -1001\,, y' = 100y, y(0) = 1, y'(0) = -1$

$0 \le x \le 1,\; h = 0.05$

$Exact\ solution: y(x) = e^{-x}$

Table 1: Comparison of Absolute Errors for Problem 1

X	New method h=0.1	Alhassan and Mohammed (2021). h=0.1	Mohammed, *et al*., 2021 h=0.1
0.1	2.423721941115e-23	1.005 e-16	1.400 e-19
0.2	1.6297628465361e-22	9.642 e-17	8.300 e-19
0.3	2.6119073636853 e-22	4.795 e-16	1.400 e-19
0.4	4.5099965195365 e-22	4.530 e-16	4.200 e-19
0.5	5.5916773417768 e-22	8.329 e-16	7.400 e-19
0.6	7.4305156956890 e-22	7.743 e-16	2.600 e-19
0.7	8.3191920508133 e-22	1.080 e-15	1.250 e-18
0.8	9.8474022419874 e-22	9.960 e-16	7.800 e-19
0.9	1.04376060098732 e-21	1.223 e-15	1.580 e-18
1.0	1.15783105897925 e-21	1.122 e-15	1.120 e-18

Table 1 shows the comparison of the performance of the proposed New method with some existing methods for problem 1. It is shown that the new method yields higher accurate results than the existing methods

Problem 2: (Source: Areo *et al.,* (2020))

Consider the linear homogenous problem given by:

$y'' = y', y(0) = 1, y'(0) = -1$

$0 \leq x \leq 1, \qquad h = 0.01$

$Exact\ solution: y(x) = 1 - e^{-x}$

Table 2: Comparison of Absolute Errors for Problem 2

X	New method h=0.1	Areo *et al.*, (2020) h=0.1.	Alhassan and Mohammed (2021). h=0.1	Okuonghae, and Ozobokeme, (2024). h=0.1
0.1	8.2529631990832e-22	2.8033131e-15		3.619 e-12
0.2	2.38006899755559 e-21	1.4460655 e-14	1.063 e-14	3.999 e-12
0.3	5.94950726089020 e-21	3.8025139 e-14	2.272 e-14	4.420 e-12
0.4	1.068046904160224 e-20	7.4662498 e-14	3.786 e-14	4.885 e-12
0.5	1.816991745134147 e-20	1.2800871 e-13	6.090 e-14	5.399 e-12
0.6	2.740627689998542 e-20	1.9995117 e-13	8.853 e-14	5.966 e-12
0.7	4.03749676649794 e-20	2.9531932 e-13	1.268 e-13	6.594 e-12
0.8	5.58809345409729 e-20	4.1633363 e-13	1.717 e-13	7.287 e-12
0.9	7.63899535963153 e-20	5.6954441 e-13	2.307 e-13	8.054 e-12
1.0	1.004861626801172 e-19	7.5828233 e-13	2.992 e-13	8.900 e-12

Table 2 shows the comparison of the performance of the proposed New method with some existing methods for problem 2. It is shown that the new method yields higher accurate results than the existing methods

Problem 3: (Source: Enoch and Alakofa, (2024))

Consider a highly stiff initial value problem given by:

$y'' = 100y\ , y(0) = 1, y'(0) = -10$

$0 \leq x \leq 1,\ \ h = 0.01$

$Exact\ solution: y(x) = e^{-10x}$

Table 3: Comparison of Absolute Errors for Problem 3

x	New method h= 0.01	Enoch and Alakofa, 2024. h=0.01	Areo. *et al* 2020 h=0.01
0.01	6.550e-22	1.791916191e-9	1.7239432e-11
0.01	1.8168e-21	4.313791577e-9	4.5028758e-11
0.03	4.1790 e-21	6.867877733 e-9	6.9175443e-11
0.04	6.9925 e-21	8.831792069 e-9	1.1053702e-10
0.05	1.08440e-20	2.2928577760e-8	1.7659828e-09
0.06	1.51393e-20	3.9259581933 e-8	3.4460591e-09
0.07	2.03784e-20	5.7797474233e-8	5.1578813e-09
0.08	2.60962e-20	7.8555166365e-8	6.9324004e-09
0.09	3.27237e-20	1.01584213652e-8	9.8575126e-09
0.1	3.99039e-20	1.26973767780e-8	1.2885945e-08

Table 3 shows the comparison of the performance of the proposed New method with some existing methods for problem 3. It is shown that the new method yields higher accurate results than the existing methods

CONCLUSION

In this study, a third-derivative Falkner-type block hybrid method was developed for the direct numerical solution of second-order initial value problems in ordinary differential equations. The method was formulated using interpolation and collocation techniques with off-step points to enhance computational accuracy and efficiency. Analysis of the proposed scheme established its consistency, zero-stability, and convergence, with a ninth-order accuracy. Numerical experiments conducted on three test problems, including stiff and nonlinear equations, revealed that the method produced significantly smaller absolute errors than existing methods reported in the literature. These results demonstrate that the developed Falkner-type method is accurate, stable, efficient, and reliable for solving second-order ordinary differential equations. Consequently, it provides an effective numerical tool for a wide range of scientific and engineering applications and serves as a foundation for further research into higher-order hybrid block methods.

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