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Some Fixed Point Theorems Involving Akram-Type Contractive Condition in Banach Spaces

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ABSTRACT
This paper introduces a novel Akram-type contractive condition together with a new two-step iterative scheme, referred to as the Ishikawa-Mann iterative process. Using the proposed iterative process, alongside certain existing iterative processes, we establish convergence results in Banach spaces for operators satisfying the newly formulated contractive condition. In particular, we prove the existence and uniqueness of fixed points for such operators. The results presented in this study extend and generalize several recent related results available in the literature.

Keywords: Akram-type contractive condition, Fixed point, Ishikawa-Mann iterative

INTRODUCTION

Let D be a complete metric space and J a self map of D. Any point $u \in D$ such that $Ju = u$ is called the fixed point of J. The set of fixed points of J is denoted by $F_J = \{p \in D | Jp = p\}$. It was established in Banach (1922) that mappings satisfying inequality

$$d(Ju, Jv) \leq a d(u, v), \ \forall\, u, v \in D, \qquad (1)$$

where $a \in [0,1)$, has a unique fixed point in D. Inequality (1) is called Banach's contraction condition, and any operator satisfying it is called an $a-$contractive map. Several generalizations and extensions of Banach's contraction condition have been introduced by various authors in the literature.

Rakotch (1962) generalized inequality (1) by introducing a monotone decreasing function: Let $\gamma: (0, \infty) \to [0,1)$ be a monotone decreasing function, for all $u, v \in D$ a self map J of D such that

$$d(Ju, Jv) \leq \gamma(d(u, v)), \qquad (2)$$

has a fixed point in D. Kannan (1968), Chatterjea (1972) and Zamfirescu (1972) generalized (1) by employing contractive conditions that capture mappings that are not necessarily continuous. Kannan (1968) employed the following inequality condition: Let J be a self map of D satisfying

$$d(Ju, Jv) \leq \sigma[d(u, Ju) + d(v, Jv)], \forall\, u, v \in D, \qquad (3)$$

where $\sigma \in [0,1/2)$. Then, J has a fixed point in D. Reich (1971), Reich and Kannan (1971) obtained a generalization of Banach (1922) and Kannan (1968) by employing the following inequality relations: Let J be a self map of D and a, b, c are real constants satisfying $a + b + c < 1$. Then, J has a fixed point in D if the following inequality is satisfied

$$d(Ju, Jv) \leq a d(u, Ju) + b d(v, Jv) + c d(u, v) \ \ \forall\, u, v \in D. \qquad (4)$$

Zamfirescu employed a more general contrative condition which generalized Banach (1922), Kannan (1968) and Chatterjea (1972): For a self map J of D there exists real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < 1/2$ and $0 \leq c < 1/2$ such that for every $u, v, w \in D$ at least one of the following inequalities is satisfied,

$$\begin{cases} d(Ju, Jv) \leq a d(u, v) \\ d(Ju, Jv) \leq b[d(u, Ju) + (v, Jv)] \\ d(Ju, Jv) \leq c[d(u, Jv) + d(v, Ju)]. \end{cases} \qquad (5)$$

Equivalently, (5) can be written as

$$d(Ju, Jv) \leq 2\eta d(u, Ju) + \eta d(u, v), \qquad (6)$$

where $0 \leq \eta < 1$ and $\eta = max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$. Meanwhile, Biachini (1972) and Khan (1978/79) employed conditions that are independent of conditions of Kannan (1968), Chatterjea (1972) and Zamfirescu (1972) to extend Banach's contraction condition (1). Biachini (1972) employed the following inequality: A self map J on a metric space D is said to be $B-$contractive if there exists a number $b \in [0,1)$ such that

$$d(Ju, Jv) \leq b\, max\{ d(u, Ju), d(v, Jv)\}, \ \forall\, u, v \in D, \qquad (7)$$

while inequality condition

$$d(Ju, Jv) \leq h\sqrt{d(Ju, u) d(Jv, v)}\ \forall\, u, v \in D \qquad (8)$$

was employed by Khan (1978/79) Several other authors have introduced contractive conditions that are more general than (6) and employed them in various directions. See Rhoades (1990), Osilike (1995), Imoru and Olatinwo (2003), Olatinwo (2016) among others. Akram $et\ al$ (2008) introduced a general class of contractions known as $A-$contraction.

Definition 1 Akram et al. (2008): A self map J of a metric space D is said to be an $A-$contraction if for all $u, v \in D$ and some $\mu \in A$, the following inequality is satisfied

$$d(Ju, Jv) \leq \mu(d(u, v), d(u, Ju), d(v, Jv)), \qquad (9)$$

where A is the set of all functions $\mu: \mathscr{R}_+^3 \to \mathscr{R}^+$ satisfying: (i) μ is continuous on the set of set of $\mathscr{R}_+^3$ (with respect to euclidean metric on $\mathscr{R}^3$); (ii) There exits $k \in [0,1)$ such that $a \leq kb$ whenever any of $a \leq \mu(a, b, b)$ or $a \leq \mu(b, a, b)$ or $a \leq \mu(b, b, a)$ holds for all a, b.

It was shown in [1] that $A-$contraction is a generalization of contractive conditions (3), (4), (6), (7) , (8) and several others in the literature. Olatinwo and Omidire (2016) and Omidire $et\ al$ (2025) recently introduced some generalization of $A-$contraction.

The Picard iterative process,

$$u_{n+1} = Ju_n, \ \ n = 0,1,2,3, \cdots \qquad (10)$$

has been employed to approximate the fixed point of operators satisfying contractive conditions (1)-(9). We state some of the iterative processes that generalize Picard iterative

process (10) in Banach's space settings as follows. Let $u_0 \in D$ be an initial guess in D, the sequence $\{u_n\}_{n=0}^{\infty}$ defined by
$u_{n+1} = (1 - \sigma_n)u_n + \sigma_n Ju_n,\ n = 0,1,2,\cdots,$ (11)
where $\{\sigma_n\}_{n=0}^{\infty} \subset [0,1]$, is called Mann iterative process. Iterative process (11) is employed in Mann (1953) to approximate fixed point of operators that are not necessarily a −contractive.
Let $u_0 \in D$ be an initial guess in D, the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} u_{n+1} = (1 - \sigma_n)u_n + \sigma_n Js_n \\ s_n = (1 - \beta_n)u_n + \beta_n Ju_n,\ n = 0,1,2,\cdots, \end{cases} \quad (12)$$

where $n \geq 1$, $\{\sigma_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in [0,1]$ is called Ishikawa iterative process. Eqn. (12) was introduced and employed in Ishikawa (1974). Khan $et\ al$ (2011) introduced Picard-Mann iterative process

$$\begin{cases} u_{n+1} = Jz_n \\ z_n = (1 - b_n)u_n + b_n Ju_n, n = 0,1,2,\cdots. \end{cases} \quad (13)$$

For more on iterative processes, interested reader can see Olatinwo and Omidire (2023), Obarhua et al. (2026) and references therein.
Motivated by the results in Akram et al (2008), and the iterative processes mentioned in this paper, we intend to introduce a more robust and novel contractive condition that is larger than the class of A-contraction by using M^* −function. We also introduce Ishikawa-Mann iterative process which generalized Picard, Mann, Picard-Mann, Ishikawa and several other iterative processes in literature. Further, convergence of Picad, Mann and the newly introduced Ishikawa-Mann iterative process to the unique fixed point of the nobel iterative condition shall be established.
In the next section, we state our new contractive condition, new fixed point iterative processes and some definitions that shall be employed in the sequel.

Preliminaries
We define and employ our new generalized Akram-type contractive conditions, and fixed point iterative processs.

Definition 2 *: Let (D, d) be a metric space. A map $J: D \to D$ is said to be M^* − contractive map if it satisfies the following condition: $\forall\, u, v \in D$ and some $\sigma \in M^*$,*
$d(Ju, Jv) \leq \sigma(d(u, v), d(u, Ju) + d(u, Jv), d(v, Jv) + d(v, Ju)),$ (14)
where M^* is the set of all functions $\sigma: \mathscr{R}_+^3 \to \mathscr{R}_+$ satisfying,
(i) σ is continuous on the set $\mathscr{R}^3$ (with respect to the Euclidean metric on $\mathscr{R}^3$);
(ii) if the condition $r \leq \sigma(r, s, s + t)$, or, $r \leq \sigma(s, r, s + t)$, or, $r \leq \sigma(s, s + t, r)$, or, $r \leq \sigma(r, s + t, s)$, or, $r \leq \sigma(s + t, r, s)$, or, $r \leq \sigma(s + t, s, r)$ holds for some $r, s, t \in \mathscr{R}^+ \cup \{0\}$, there exists $k \in [0,1)$ such that $r \leq ks$.

Remark 1 *(i) Contractive condition (14) of definition (2) reduces to A −contraction (9) of Definition (1) when $t = 0$, $u = Jv$ and $v = Ju$.*

Definition 3 *Let D be a non empty set. For any $u_0 \in D$, some sequence $\{u_n\} \subset D$ defined by*

$$\begin{cases} u_{n+1} = (1 - a_n)(1 - b_n)u_n + b_n(1 - a_n)Ju_n + a_n Jr_n \\ r_n = (1 - c_n)(1 - d_n)u_n + d_n(1 - c_n)Ju_n + c_n Ju_n, n = 0, 1, 2, \cdots, \end{cases} \quad (15)$$

will be called Ishikawa-Mann iterative process where the parameter sequences $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [0,1]$.

Remark 2 (i) Iterative process (15) reduces to Ishikawa iterative process (12) if $b_n = d_n = 0$. Iterative process (12) was employed in Ishikawa (1974) and more recently Pragadeeswarar and Gopi (2023).
(ii) Iterative process (15) reduces to Mann iterative process (11) if $a_n = 0$. Iterative process (11) was employed in Mann (1953).
(iii) Iterative process (15) reduces to Picard iterative process (10) if $a_n = 1$ and $c_n = d_n = 0$. Iterative process (10) was employed in Akram $et\ al$ (2008).
(iv) Iterative process (15) reduces to Picard-Mann iterative process (13) if $a_n = 1$ and $d_n = 0$. Picard-Mann iterative process (13) was employed in Khan $et\ al$(2011).

RESULTS AND DISCUSSION

Theorem 1 *Let $(D, \|\cdot\|)$ be a Banach space and $J: D \to D$ a map of D. Suppose J is an M^* −contractive operators satisfying condition (14). Then,*
(i) J possesses a unique fixed point $p \in D$
(ii) Sequence $\{u_n\}_{n=0}^{\infty} \subset D$ defined $u_{n+1} = Ju_n, n = 0,1,2,\cdots$, converges to the unique fixed point p of J given any initial guess $u_0 \in D$.
Proof. First, we prove existence of the fixed point. Given that $\{u_n\}_{n=1}^{\infty}$ is a sequence in the space. From inequality (14), we have

$$\begin{aligned} \| u_2 - u_1 \| &= \| Ju_1 - Ju_0 \| \\ &\leq \sigma(\| u_1 - u_0 \|, \| u_1 - Ju_1 \| + \| u_1 - Ju_0 \|, \| u_0 - Ju_0 \| + \| u_0 - Ju_1 \|) \\ &= \sigma(\| u_1 - u_0 \|, \| u_1 - u_2 \|, \| u_0 - u_1 \| + \| u_0 - u_2 \|) \end{aligned}$$

Choose $r = \| u_1 - u_2 \|$, $s = \| u_1 - u_0 \|$ and $t = \| u_2 - u_0 \|$. That is, $r \leq \sigma(s, r, s + t)$. It follows directly from Property (ii) of M^* in Definition (2) 1 that,

$$\| u_2 - u_1 \| \leq k \| u_1 - u_0 \|. \quad (16)$$

For $n = 2$, we have

$$\begin{aligned} \| u_3 - u_2 \| &= \| Ju_2 - Ju_1 \| \\ &\leq \sigma(\| u_2 - u_1 \|, \| u_2 - Ju_2 \| + \| u_2 - Ju_1 \|, \| u_1 - Ju_1 \| + \| u_1 - Ju_2 \|) \\ &= \sigma(\| u_2 - u_1 |, \| u_2 - u_3 \|, \| u_1 - u_2 \| + \| u_1 - u_3 \|) \end{aligned}$$

Choose $r = \| u_2 - u_3 \|$, $s = \| u_2 - u_1 \|$ and $t = \| u_3 - u_1 \|$. That is, $r \leq \sigma(s, r, s + t)$. Applying condition (ii) of M^*, we get;

$$\begin{aligned} \| u_3 - u_2 \| &\leq k \| u_2 - u_1 \| \\ &\leq k^2 \| u_1 - u_0 \|, \end{aligned}$$

Generally, we have

$$\| u_{n+1} - u_n \| \leq k^n \| u_1 - u_0 \|. \quad (17)$$

Let $n, m \in \mathcal{N}$. Using inequality (17) and triangle inequality repeatedly we have,
$\| u_n - u_{n+m} \| \leq \| u_n - u_{n+1} \| + \| u_{n+1} - u_{n+2} \| + \cdots + \| u_{n+m-1} - u_{n+m} \|$
$\leq k^n \| u_0 - u_1 \| + k^{n+1} \| u_0 - u_1 \| + \cdots + k^{n+m-1} \| u_0 - u_1 \|$
$\leq \left(\frac{k^n}{1-k}\right) \| u_0 - u_1 \|.$ (18)

Since $k \in [0,1)$, the right hand side of inequality (18) goes to 0 as n goes to ∞ which shows that sequence $\{u_n\}$ is a Cauchy sequence. It follows from completeness property of the space that there exists $p \in D$ such that u_n goes to p as n goes to ∞.
We show, in the next lines that p is the fixed point of J.
$\| p - Jp \| \leq \| p - Ju_n \| + \| Ju_n - Jp \|$
$\leq \| p - u_{n+1} \| + \sigma(\| u_n - p \|, \| u_n - Ju_n \| + \| u_n - Jp \|, \| p - Jp \| + \| p - Ju_n \|)$
$\leq \| p - u_{n+1} \| + \sigma(\| u_n - p \|, \| u_n - u_{n+1} \| + \| u_n - Jp \|, \| p - Jp \| + \| p - u_{n+1} \|)$
$= \sigma(0, \| p - Jp \|, \| p - Jp \| + 0) \ as\, n \ \to \infty.$
Choose, $r = \| p - Jp \| = t$ and $s = 0$. We obtain; $r \leq \sigma(s, r, s + t) = \sigma(0, r, r)$. Consequent upon condition (ii) of Definition (2), we have $r \leq ks$. That is, $\| p - Jp \| \leq 0$, which implies that $Jp = p$.
Further, we show uniqueness of p. Suppose there exists $q \in D$ distinct from $p \in D$ such that $Jq = q$ and $\| q - p \| > 0$. Then,
$0 < \| q - p \| = \| Jq - Jp \|$
$\leq \sigma(\| q - p \|, \| q - Jq \| + \| q - Jp \|, \| p - Jp \| + \| p - Jq \|)$
$\leq \sigma(\| p - q \|, \| p - q \|, \| p - q \|).$
Choosing $r = \| p - q \| = s$ and $t = 0$, then $r \leq \sigma(r, s, s + t) = \sigma(s, s, s)$ which implies that
$r \leq ks = k \| p - q \|$.
That is, $\| p - q \| \leq k \| p - q \|$, from which it follows that $(1 - k) \| p - q \| \leq 0$.
Since $(1 - k) > 0$ then $\| p - q \| \leq 0$. But norm is non negative, therefore $\| p - q \| = 0$. This is a contradiction to the supposition that $\| p - q \| > 0$. Hence, $\| p - q \| = 0$. Thus, p is unique.
Next, we give an example to show that the class of M^* −contraction maps is larger than the class of A −contraction maps employed by Akram et al (2008).

Example 1: *Let* $D = [0,2]$ *be a metric space with usual metric*
$d(u, v) = |u - v|\ u, v \in D$. Define $J: D \to D$ by

$$Ju = \begin{cases} 1, when\ u \in [0, \frac{1}{2}] \\ 2, when\ u \in (\frac{1}{2}, 2] \end{cases} \tag{19}$$

Let $\sigma: \mathscr{R}_+^3 \to \mathscr{R}_+$ be defined by
$l\sigma(p, q, r) = \frac{p+q+\sqrt{qr}}{3}$
Then, (i) J is not an A −contraction map (ii) J is an M^* −contraction map.
Proof. At $u = 1/2$ and $v = 1$, we have the following values:
$d(u, v) = d(\frac{1}{2}, 1) = |1/2 - 1| = 1/2,$
$d(u, Ju) = |u - Ju| = |1/2 - 1| = 1/2,$
$d(v, Jv) = |v - Jv| = |1 - 2| = 1,$
$d(Ju, Jv) = |Ju - Jv| = |1 - 2| = 1,$
$d(u, Jv) = |u - Jv| = |1/2 - 2| = 3/2$, and
$d(v, Ju) = |v - Ju| = |1 - 1| = 0.$
(i) A −contraction requires that,
$d(Ju, Jv) \leq \sigma(d(u, v), d(u, Ju), d(v, Jv)) \quad \forall\, u, v \ \in D.$
Now,

$$\sigma(d(u, v), d(u, Ju), d(v, Jv)) = \sigma\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$
$$= \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2} + \sqrt{(\frac{1}{2})(1)}\right)$$
$$= \frac{1}{3}\left(1 + \frac{1}{\sqrt{2}}\right)$$
$$= \frac{2+\sqrt{2}}{6} = 0.5690 < 1 = d(Ju, Jv)$$

i.e. $d(Ju, Jv) > \sigma(d(u, v), d(u, Ju), d(v, Jv))$. Thus, J is not an A −contraction map.
(ii) M^* −contraction requires that,
$d(Ju, Jv) \leq \sigma(d(u, v), d(u, Ju) + d(u, Jv), d(v, Jv) + d(v, Ju))\ \forall\, u, v \in D.$
Indeed,

$$\sigma(d(u, v), d(u, Ju) + d(u, Jv), d(v, Jv) + d(v, Ju)) = \sigma\left(\frac{1}{2}, \frac{1}{2} + \frac{3}{2}, 1 + 0\right)$$
$$= \sigma\left(\frac{1}{2}, 2, 1\right)$$
$$= \frac{1}{3}\left(\frac{1}{2} + 2 + \sqrt{2}\right)$$
$$= \frac{5+2\sqrt{2}}{6} = 1.3047 > 1 = d(Ju, Jv)$$

i.e. $d(Ju, Jv) \leq \sigma(d(u, v), d(u, Ju) + d(u, Jv), d(v, Jv) + d(v, Ju))$
for $u = 1/2$ and $v = 1$.
Furthermore, for any two distinct points $u, v \in [0, \frac{1}{2}]$, $d(Ju, Jv) = 0$, $d(u, v) \neq 0$, $d(u, Ju) \neq 0$ and $d(v, Jv) \neq 0$, therefore,

$d(Ju,Jv) \leq \sigma(d(u,v), d(u,Ju) + d(u,Jv), d(v,Jv) + d(v,Ju))\ \forall u,v \in (0,\frac{1}{2}]$.
Similarly, for any distinct points $u,v \in (\frac{1}{2},2]$, $d(Ju,Jv) = 0$, $d(u,v) \neq 0$, $d(u,Ju) \neq 0$ and $d(v,Jv) \neq 0$, it follows that
$d(Ju,Jv) \leq \sigma(d(u,v), d(u,Ju) + d(u,Jv), d(v,Jv) + d(v,Ju))\ \forall u,v \in (\frac{1}{2},1]$
Lastly, for distinct points u,v such that $u \in (0,\frac{1}{2}]$ and $v \in (\frac{1}{2},2]$, it is clear to see that $d(Ju,Jv) = 1$, $d(u,v) > 0$, $d(u,Ju) + d(u,Jv) \geq 2$, $d(v,Jv) + d(v,Ju) \geq 1$. The same results is obtained for $v \in (0,\frac{1}{2}]$ and $u \in (\frac{1}{2},2]$.
Choose $p = d(u,v) > 0$, $q = d(u,Ju) + d(u,Jv) \geq 2$ and $r = d(v,Jv) + d(v,Ju) \geq 1$,

$$\begin{aligned}\sigma(p,q,r) &= \frac{p+q+\sqrt{qr}}{3}\\ &> \frac{0+2+\sqrt{2\times 1}}{3}\\ &= \frac{0+2+\sqrt{2}}{3} > 1 = d(Ju,Jv)\end{aligned}$$

It follows that $d(Ju,Jv) \leq \sigma(d(u,v), d(u,Ju) + d(u,Jv), d(v,Jv) + d(v,Ju))\ \forall\ u \in [0,\frac{1}{2}]$ and $v \in (\frac{1}{2},2]$.
Therefore,

$$d(Ju,Jv) \leq \sigma(d(u,v), d(u,Ju) + d(u,Jv), d(v,Jv) + d(v,Ju))\ \forall\ u,v \in (0,2]$$

Hence, J is an M^* −contraction map. The set of fixed point of J is a singleton, $F_J = \{2\}$.

Theorem 2 *Let $\sigma \in M^*$ and $\{J_n\}_{=1}^{\infty}$ be a sequence of self maps on a complete metric space D Suppose J satisfies the following condition:*

$$d(J_iu,J_jv) \leq \sigma(d(u,v), d(u,J_iu) + d(u,J_jv), d(v,J_jv) + d(v,J_iu)), \tag{20}$$

for all $u,v \in D$. Then $\{J_n\}_{=1}^{\infty}$ possesses a unique common fixed point in D.
Proof. For any initial guess $u_0 \in D$ and each $n \in \mathcal{N}$, we define Picard iteration by
$u_n = J_n u_{n-1}$. Using inequality (3.5)we get,

$$\begin{aligned}d(u_1,u_2) &= d(J_1u_0,J_2u_1)\\ &\leq \sigma(d(u_0,u_1), d(u_0,J_1u_0) + d(u_0,J_2u_1), d(u_1,J_2u_1) + d(u_1,J_1u_0))\\ &= \sigma(d(u_0,u_1), d(u_0,u_1) + d(u_0,u_2), d(u_1,u_2))\end{aligned}$$

Choose $r = d(u_1,u_2)$, $s = d(u_0,u_1)$ and $t = d(u_0,u_2)$. i.e. $r \leq \sigma(s, s+t, r)$. Then, by Property (ii) of M^* in Definition (2) 1, there exists $k \in [0,1)$ such that
$r \leq ks$. That is,
$d(u_1,u_2) \leq kd(u_0,u_1)$.
Following similar argument, we obtain
$d(u_2,u_3) \leq k^2 d(u_0,u_1)$,
and by induction, we get

$$d(u_n,u_{n+1}) \leq k^n d(u_0,u_1). \tag{21}$$

Using inequality (21) and applying triangle inequality repeatedly, for any positive integers m,n we obtain

$$d(u_n,u_{n+m}) \leq \left(\frac{k^n}{1-k}\right) d(u_0,u_1). \tag{22}$$

By allowing n to go to infinity, inequality (22) becomes $lim_{n\to\infty} d(u_n,v_{n+m}) \leq 0$. Since norm is non-negative, this yields $lim_{n\to\infty} d(u_n,v_{n+m}) = 0$. Hence, sequence u_n is a Cauchy sequence. Since the space is complete, there exists a point $p \in D$ such that sequence u_n converges to p. We claim that p is the common fixed point of sequence of operators $\{J_n\}_{n=0}^{\infty}$. Clearly,

$$\begin{aligned}d(J_np,p) &\leq d(J_np,u_{t+1}) + d(u_{t+1},p)\\ &= d(J_np,J_{t+1}u_t) + d(u_{t+1},p)\\ &\leq \sigma(d(p,u_t), d(p,J_np) + d(p,J_{t+1}u_t), d(u_t,J_{t+1}u_t) + d(u_t,J_np)) + d(u_{t+1},p)\\ &= \sigma(d(p,u_t), d(p,J_np) + d(p,u_{t+1}), d(u_t,u_{t+1}) + d(u_t,J_np)) + d(u_{t+1},p)\\ &= \sigma(0, d(p,J_np), d(p,J_np)) \quad as \quad t\to\infty.\end{aligned}$$

Let $r = d(p,J_np) = t$ and $s = 0$. i.e. $r \leq \sigma(s,r,s+t) = \sigma(s,r,r)$. By Property (ii) of M^* there exists $k \in [0,1)$ such that $r \leq ks = 0$. That is, $d(p,J_np) = 0$. Hence, $J_np = p$. Lastly, we show that p is unique. Suppose, there exists $p^* \in D$ distinct from p such that $d(p^*,J_np^*) = 0$ and $d(p,p^*) > 0$. Then,

$$\begin{aligned}0 < d(p,p^*) &= d(J_ip,J_jp^*)\\ &\leq \sigma(d(p,p^*), d(p,J_ip) + d(p,J_jp^*), d(p^*,J_jp^*) + d(p^*,J_ip))\\ &= \sigma(d(p,p^*), d(p,p^*), d(p^*,p))\end{aligned}$$

Choosing $r = d(p,p^*) = s$ and $t = 0$. i.e. $r \leq \sigma(s,r,s+t) = \sigma(s,s,s)$. Applying property (ii) of M^* there exists $k \in [0,1)$ such that $r \leq ks$. That is, $d(p,p^*) \leq kd(p,p^*)$. This is a contradiction to the supposition that $d(p,p^*) > 0$. Hence, $p = p^*$. This completes the proof.

Theorem 3: *Let $(D,\|\cdot\|)$ be a Banach space and $J: D \to D$ be an M^* −contractive map satisfying contractive condition (14). Suppose J has a fixed point $u \in D$, then (i) $u \in D$ is unique (ii) The Mann iterative procedure*
$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Jx_n$
converges to u.
Proof. (i) Suppose $u,w \in D$ such that $u = Ju$, $w = Jw$ and $\| u - w \| > 0$. Then

$$\begin{aligned}0 < \| u - w \| &= \| Ju - Jw \|\\ &\leq \sigma(\| u - w \|, \| u - Ju \| + \| u - Jw \|, \| w - Jw \| + \| w - Ju \|)\\ &= \sigma(\| u - w \|, \| u - w \|, \| w - u \|).\end{aligned}$$

Choosing $r = \| u - w \| = s$ and $t = 0$, by Property (ii) of M^*, we obtain,
$\| u - w \| \leq k \| u - w \|$.
That is, $\| u - w \| \leq k \| u - w \|$, which implies that $\| u - w \| = 0$. This is a contradiction to the supposition that $\| u - w \| > 0$. Hence, $u = w$.
(ii) Let sequence $\{x_n\}_{n=0}^{\infty} \subset D$ be defined by $x_{n+1} = (1 - \lambda_n)x_n + \lambda_n J x_n$. We shall establish that the sequence converges to u. Using triangle inequality, we have,

$$\begin{aligned} \| u_{n+1} - u \| &= \| (1 - \lambda_n)u_n + \lambda_n J u_n - u \| \\ &= \| (1 - \lambda_n)u_n + \lambda_n J u_n - ((1 - \lambda_n)u + \lambda_n u) \| \\ &\leq (1 - \lambda_n) \| u_n - u \| + \lambda_n \| J u_n - u \| \\ &= (1 - \lambda_n) \| u_n - u \| + \lambda_n \| J u_n - J u \|. \end{aligned} \tag{23}$$

Using contractive condition (14) we have,

$$\begin{aligned} \| J u_n - J u \| &\leq \sigma(\| u_n - u \|, \| u_n - J u_n \| + \| u_n - J u \|, \| u - J u \| + \| u - J u_n \|) \\ &= \sigma(\| u_n - u \|, \| u_n - J u_n \| + \| u_n - J u \|, \| u - J u_n \|) \\ &= \sigma(\| u_n - u \|, \| u_n - J u_n \| + \| u_n - u \|, \| J u - J u_n \|), \end{aligned}$$

choosing $r = \| J u - J u_n \|$, $s = \| u_n - u \|$ and $t = \| u_n - J u_n \|$, we get, $r \leq \sigma(s, s + t, r)$ which on applying Property (ii) of M^* −contraction, gives
$r \leq ks$.
That is,

$$\| J u_n - J u \| \leq k \| u_n - u \|. \tag{24}$$

Using inequality (24) in (23) yields,

$$\begin{aligned} \| u_{n+1} - u \| &\leq (1 - \lambda_n) \| u_n - u \| + \lambda_n k \| u_n - u \| \\ &= (1 - (1 - k)\lambda_n) \| u_n - u \| \\ &= \beta_n \| u_n - u \|. \end{aligned}$$

In other words,

$$\begin{aligned} \| u_{n+1} - u \| &\leq \beta_n \| u_n - u \| \\ &\leq \beta_n \cdot \beta_{n-1} \| u_{n-1} - u \| \\ &\leq \beta_n \cdot \beta_{n-1} \cdot \beta_{n-2} \| u_{n-2} - u \| \\ &\vdots \\ &\leq \beta_n \cdot \beta_{n-1} \cdot \beta_{n-2} \cdots \beta_3 \cdot \beta_2 \cdot \beta_1 \cdot \beta_0 \| u_0 - u \| \\ &= \Pi_{r=0}^{n} \beta_r \| u_0 - u \|, \end{aligned} \tag{25}$$

where $\beta_n = 1 - (1 - k)\lambda_n$. Since $\beta_r \in (0,1), r = 0,1,2, \cdots, n$, taking the limit of both sides of inequality (25) as $n \to \infty$, we obtain,

$$\lim_{n \to \infty} P u_{n+1} - u P = 0 \tag{26}$$

which implies that, $lim_{n \to \infty} x_n = u$. Hence, the result.

Theorem 4: *Let* $(D, \|\cdot\|)$ *be a Banach space and* $J: D \to D$ *an* M^* *−contractive map satisfying contractive condition* (14). *Suppose* J *has a fixed point* $p \in D$. *Then,*
(i) Ishikawa-Mann iterative process defined by

$$\begin{cases} u_{n+1} = (1 - a_n)(1 - b_n)u_n + b_n(1 - a_n)J u_n + a_n J r_n \\ r_n = (1 - c_n)(1 - d_n)u_n + d_n(1 - c_n)J u_n + c_n J u_n, \ n = 0,1,2, \ldots \end{cases}$$

converges to the fixed point p of J for any initial guess $u_0 \in D$.
(ii) p is unique.

Proof. (i). Applying triangle inequality,

$$\| u_{n+1} - p \| \leq (1 - a_n)(1 - b_n) \| u_n - p \| + b_n(1 - a_n) \| J u_n - p \| + a_n \| J r_n - p \|. \tag{27}$$

Recall that,

$$\begin{aligned} \| J u_n - p \| &= \| J u_n - J p \| \\ &\leq \sigma(\| u_n - p \|, \| u_n - J u_n \| + \| u_n - J p \|, \| p - J p \| + \| p - J u_n \|) \\ &= \sigma(\| u_n - p \|, \| u_n - J u_n \| + \| u_n - p \|, \| p - J u_n \|). \end{aligned} \tag{28}$$

Choosing $r = \| J u_n - p \|$, $s = \| u_n - p \|$ and $t = \| u_n - J u_n \|$, we have,
$r \leq \sigma(s, s + t, r)$
By property (ii) of contractive condition (14) this implies that
$r \leq ks$.
That is,

$$\| J u_n - p \| \leq k \| u_n - p \|. \tag{29}$$

Similarly,

$$\| J r_n - p \| \leq k \| r_n - p \|. \tag{30}$$

Using inequality (29) and inequality (30) in (27), we get,

$$\| u_{n+1} - p \| \leq (1 - a_n)(1 - b_n) \| u_n - p| + k b_n(1 - a_n) \| u_n - p \| + k a_n \| r_n - p \| = [(1 - a_n)(1 - (1 - k)b_n)] \| u_n - p \| + k a_n \| r_n - p \|. \tag{31}$$

Where,

$$\begin{aligned} \| r_n - p \| &\leq (1 - c_n)(1 - d_n) \| u_n - p \| + d_n(1 - c_n) \| J u_n - p \| + c_n \| J u_n - p \| \\ &= (1 - c_n)(1 - d_n) \| u_n - p \| + (d_n(1 - c_n) + c_n) \| J u_n - J p \| \\ &\leq (1 - c_n)(1 - d_n) \| u_n - p \| + (d_n(1 - c_n) + c_n)k \| u_n - p \\ &= [(1 - c_n)[1 - (1 - k)d_n] + k c_n] \| u_n - p \|. \end{aligned} \tag{32}$$

Using inequality (32) in inequality (31) we have,

$$\begin{aligned}\| u_{n+1} - p \| &\leq [(1-a_n)(1-(1-k)b_n)] \| u_n - p \| \\ &+ka_n[(1-c_n)[1-(1-k)d_n] + kc] \| u_n - p \| \\ &= \{(1-a_n)(1-(1-k)b_n) + a_n k[(1-c_n)(1-(1-k)d_n) + kc_n]\} \| u_n - p \| \\ &= \beta_n \| u_n - p \| \\ &\leq \beta_n \cdot \beta_{n-1} \| u_{n-1} - p \| \\ &\leq \beta_n \cdot \beta_{n-1} \cdot \beta_{n-2} \| u_{n-2} - p \| \\ &\vdots \\ &\leq \Pi_{r=0}^{n} \beta_r \| u_0 - u \|. \end{aligned} \tag{33}$$

Where $\beta_n = (1-a_n)(1-b_n(1-k)) + a_n k[(1-c_n)(1-d_n(1-k)+c_n k] \in (0,1)$. Since $\beta_n \in (0,1)$, the limit of both sides of inequality (33) yields $lim_{n\to\infty} Pu_{n+1} - p\, P \leq 0$ as n goes to ∞ . Since norm is non-negative, we accept $lim_{n\to\infty} Pu_{n+1} - p\, P = 0$ as $n \to \infty$, which implies that $lim_{n\to\infty} u_n = p$

CONCLUSION

We established in this paper, existence, uniqueness and convergence of fixed point of mappings satisfying a novel Akram-type contractive condition that is larger than the class of A-contraction of Akram et al. (2008) An iterative process that generalized those of Khan (1978/79), Mann (1953), Ishikawa (1974) and several other iterative processes were employed to achieve convergence results for unique fixed points and unique common fixed points. Our results generalized those of Akram et al. (2008) several other related results in literature.

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