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THE ODD TEISSIER KUMARASWAMY DISTRIBUTION: A NEW DISTRIBUTION WITH INTERVAL BOUND

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ABSTRACT

This study introduces a new probability distribution on a unit interval called the Odd Teissier Kumaraswamy (OTKw) distribution, and the aim is to improve the flexibility of the Teissier distribution in modeling lifetime data. The new distribution exhibits exceptional flexibility in modelling data with increasing and bathtub hazard rates. The key statistical properties of the proposed distribution such as the moments, moment generating function, probability weighted moments, quantile, Renyi entropy, and order statistics are derived. The parameters of the proposed distribution are estimated using the maximum likelihood estimation (MLE) method. A simulation study is carried out to examine the performance of the MLEs concerning their biases, standard errors, and root mean square errors (RMSE). Finally, to illustrate the practical importance and flexibility of the proposed distribution in modelling real data applications, three data sets were considered, and the results showed that the new distribution performs better than some other known existing distributions.

Keywords: Teissier, Odd Teissier Kumaraswamy, Unit Interval, Simulation, Flexibility

INTRODUCTION

Probability distribution models are used for analyzing the behaviors of some random phenomena and it is of great importance in the process of decision-making in various disciplines, particularly data concerning the time to the occurrence of an event, which is called lifetime data. Many classical distributions have been used extensively to model data in several areas of study over the years, such as biomedical sciences, insurance, finance, engineering, etc. Observations show the need to develop extended forms of classical distributions to achieve more flexibility in modelling real data. Hence, numerous generalized (families of) distributions have been generated in the literature by adding one or more parameters.

The addition of parameters is proven to be useful in exploring tail properties and also for improving the goodness-of-fit of the generated family. Some of the well-known generators include beta-G by Eugene et al. (2002), Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011), McDonald-G (Mc-G) by Alexander et al. (2012), exponentiated generalized-G by Cordeiro et al. (2013), transformed-transformer (T-X) family by Alzaatreh et al. (2013), logistic-G by Torabi and Montazari (2014), Kumaraswamy Odd log-logistic-G by Alizadeh et al. (2015a), Kumaraswamy Marshall-Olkin by Alizadeh et al. (2015b), Kumaraswamy Weibull-G by Hassan and Elgarhy (2016), odd log-logistic family by Cordeiro et al. (2017), the T-Kumaraswamy Family of Distributions by Osatohanmwen et al. (2020), The Generalized Distributions on the Unit Interval based on the T-Topp-Leone Family of Distributions by Pupe et al. (2022), A generating family of unit-Garima distribution: Properties, likelihood inference, and application by Sirinapa and Winai (2024), amongst others.

The Teissier distribution was first introduced by a French biologist Teissier (1934) to study the mortality of some domestic animals as a result of pure ageing. The Teisier distribution has been given little consideration by researchers, not minding the fact that it is preferable to the famous one-parameter exponential distribution for modelling data because of its increasing hazard rate function (hrf) property.Laurent (1975) introduced a location version of Teissier's distribution and studied its characterization based on life expectancy by demonstrating its applications with demographic data. Muth (1977) studied the properties of the Teissier distribution to discover that it has a heavy tail which makes it useful like other well-known heavy tail distributions like gamma, lognormal, and Weibull. Recently, this discovery has attracted the interest of researchers to study the Teissier distribution. A continuous random variable Y is said to follow the Teissier distribution with parameter α if the cumulative distribution function (CDF) and probability distribution function (pdf) of Y are respectively given by

$$F(y;\alpha) = 1 - exp\{1 + \alpha y - e^{\alpha y}\}, \qquad y > 0; \alpha > 0 \tag{1}$$

and

$$f(x;\alpha) = \alpha(e^{\alpha x} - 1)\, exp\{1 + \alpha x - e^{\alpha x}\}, \; y > 0; \alpha > 0 \tag{2}$$

The main drawback of the Teissier distribution in the analysis of real data is that the density and hazard rate functions can only be right skewed and increasing respectively for all parameter values. To introduce flexibility into the Teissier distribution's pdf and hazard rate functions, Sharma et al. (2020) presented a two-parameter extension of the Teissier distribution called the Exponentiated Teissier (ET) distribution, which is flexible enough to model both right and left skewed data, and bathtub and increasing hazard rates. Krishna et al. (2022) presented an interval-bound form of the Teissier called the unit Teissier (UT) distribution, which is though flexible, but not flexible enough to increasing time to event occurrence. Some other distributions introduced to increase the flexibility of include Poonia and Azad (2022) introduced the Alpha Power Exponentiated Teissier (APET) Distribution, Eghwerido (2022) proposed the Marshall – Olkin Teissier Generated Family Alsadat et al. (2023) presented the inverse unit Teissier (IUT) distribution, Eghwerido et al. (2023) introduced the Teissier generated (T-G) family of distributions, Halidu et al. (2025) introduced the Lomax Unit Teissier distribution. To the best of our knowledge, none of these extensions of the Teissier distribution is flexible enough to model time to event data with irregular density function shape density function. Hence, the OTKw distribution.

Alzaatreh et al. (2013) pioneered a novel method for proposing new distributions and class distributions called the "Transformed-Transformer" (T-X) method and it has received great attention in recent years for developing extensions of existing distributions for more flexibility.

Kumaraswamy (1980) introduced a two-parameter probability distribution to model data sets on a unit interval (0, 1).

In this new distribution, we formulate the Odd Teissier Kumaraswamy (OTKw) distribution using the odd ratio operator on the T-X framework, as given by Alzaatreh et al. (2013), with the Teissier distribution as the generator and kumaraswamy distribution as the baseline distribution. We establish that the OTKw distribution exhibit key mathematical and practical properties, which makes it applicable for data analysis. The CDF and pdf for a random variable Y from the Kumaraswamy distribution are respectively, given as

$$G(y) = 1 - \left(1 - y^{\beta}\right)^{\lambda}, \quad 0 < y < 1 \qquad (3)$$

and

$$g(y) = \beta\lambda y^{\beta-1}\left(1 - y^{\beta}\right)^{\lambda-1}, \quad 0 < y < 1 \qquad (4)$$

where $\beta > 0$ and $\lambda > 0$ are two shape parameters of the Kumaraswamy distribution.

In this article, our main objective is to introduce a new flexible extension of the Teissier distribution with support (0, 1) called Odd Teissier Kumaraswamy (OTKw) distribution, study the properties, and show its application to some real data sets. The rest of the paper is organized as follows. In Section 2, present the OTKw distribution with the shape analysis of the main functions (the pdf and hrf), the mathematical properties which include the quantile function, moments, moment generating function, probability-weighted moments, order statistics and Renyi entropy are derived in Section 3. Section 4 presented the maximum likelihood estimates (MLEs) of the unknown parameters of the OTKw distribution, as well as a simulation study to examine the performance of the MLEs. In Section 5, the competency of the proposed OTKw distribution in comparison to some existing distribution models. Finally, Section 6, present the conclusion of the study.

MATERIALS AND METHODS

The Odd Teissier Kumaraswamy (OTKw) Distribution

Let $r(t)$ be the density function and $R(t)$ be the CDF of a random variable $T \in [a, b]$ for which $-\infty \le a < b \le \infty$ and $Z[G(y;\phi)]$ be a function that defines a generator about the CDF of a random variable *Y*, which satisfies the following conditions

i. $Z[G(y;\phi)] \in [a, b]$
ii. $Z[G(y;\phi)]$ is a differentiable and monotonically non-decreasing function
iii. $\lim\limits_{y\to-\infty} Z[G(y;\phi)] = a$ and $\lim\limits_{y\to\infty} Z[G(y;\phi)] = b$

Then Alzaatreh et al. (2013) defined the CDF of the T-X family of distributions as the monotone increasing and continuous function of the cumulative distribution function $G(y;\phi)$ of any random variable Y.

$$F(y) = \int_{a}^{Z[G(y;\phi)]} r(t)\,dt = R\{Z[G(y;\phi)]\} \qquad (5)$$

Suppose a continuous random variable T, is from the Teissier distribution and we define the composite function of any baseline random variable Y, by the odd generator as $Z[G(y;\phi)] = \frac{G(y;\phi)}{\bar{G}(y;\phi)}$, where $G(y;\phi)$ is the CDF of the baseline distribution with a vector of parameters ϕ and $\bar{G}(y;\phi) = 1 - G(y;\phi)$ as the survival function of the baseline distribution. Then using equation (5), we define the CDF of the proposed Odd Teissier Kumaraswamy (OTKw) distribution by

$$F(y) = 1 - exp\left\{1 + \alpha\left[\left(1 - y^{\beta}\right)^{-\lambda} - 1\right] - e^{\alpha\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]}\right\}, 0 < y < 1; \alpha, \beta, \lambda > 0 \qquad (6)$$

and the corresponding pdf of the proposed distribution is obtained as

$$f(y) = \frac{\alpha\beta\lambda y^{a-1}}{\left(1-y^{\beta}\right)^{\lambda+1}}\left(e^{\alpha\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]} - 1\right) exp\left\{1 + \alpha\left[\left(1 - y^{\beta}\right)^{-\lambda} - 1\right] - e^{\alpha\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]}\right\} \qquad (7)$$

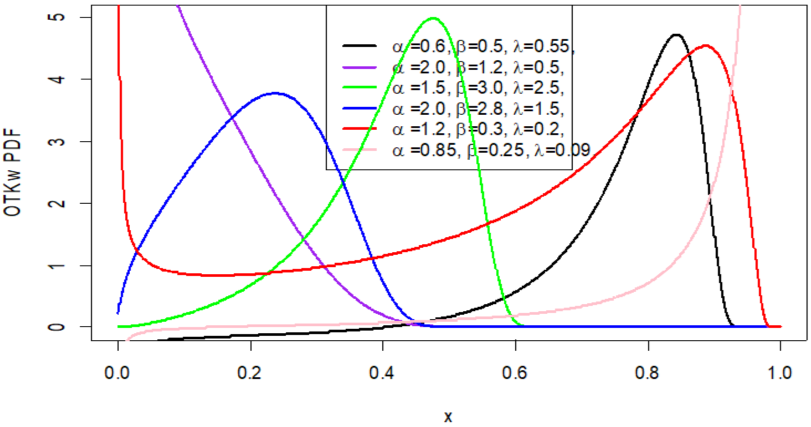

Figure 1: The Plots of the Density Functions of the OT-Kumaraswamy (OTKw) Distribution for Different Parameter Values

Figure 1 presents the plots of the density function of OTKw for some selected parameter values which show that the density function can be increasing, decreasing, left-skewed, right-skewed, symmetric, and irregular (polynomial), which makes it uniquely different from other Teissier based distribution.

The Survival and Hazard Rate Functions of the OTKw Distribution

The survival and hazard rate functions of the Odd Teissier Kumaraswamy (OTKw) distribution are respectively given as

$$S(y) = exp\left\{1 + \alpha\left[\left(1 - y^{\beta}\right)^{-\lambda} - 1\right] - e^{\alpha\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]}\right\} \qquad (8)$$

and

$$h(y) = \frac{\alpha\beta\lambda y^{a-1}}{\left(1-y^{\beta}\right)^{\lambda+1}}\left(e^{\alpha\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]} - 1\right) \qquad (9)$$

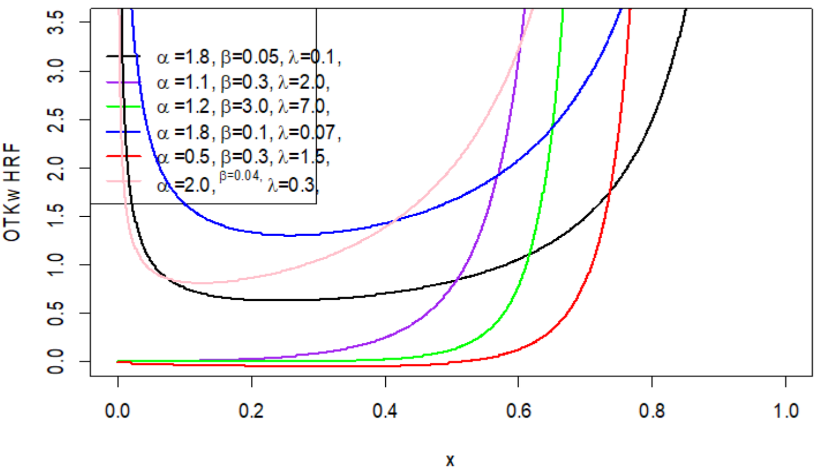

Figure 2: The Plots of the Hazard Rate Functions of the OT-Kumaraswamy (OTKw) Distribution for Different Parameter Values

Figure 2 presents the plots of the hazard function of OTKw for some selected parameter values which show that the hazard rate function exhibits increasing and bathtub shape.

Series Expansion of the Density Functions of OTKw Distribution

We present the series expansion of the probability density function (PDF) of OTKw distribution using the following conditions as defined in Prudnikov et al (1986) and Barreto-Souza et al (2013);

$(1-Z)^n = \sum_{i=0}^{\infty}\binom{n}{i}(-1)^i Z^i$; $(1-Z)^{-n} = \sum_{j=0}^{\infty}\frac{\Gamma(n+j)}{j!\Gamma(n)}Z^j$ and $e^Z = \sum_{k=0}^{\infty}\frac{1}{k!}Z^k$.

Then the series expansion of the density function of the OTKw in equation (7) can be expressed as

$$f(y) = \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\sum_{a=0}^{\infty}\frac{(-1)}{a!}\left(1-e^{\alpha\left[\left(1-y^\beta\right)^{-\lambda}-1\right]}\right)^{a+1} e^{\alpha\left[\left(1-y^\beta\right)^{-\lambda}-1\right]}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\binom{a+1}{b}\frac{(-1)^{b+1}}{a!}e^{\alpha(b+1)\left[\left(1-y^\beta\right)^{-\lambda}-1\right]}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\binom{a+1}{b}\frac{(-1)^{b+c+1}(b+1)^c\alpha^c}{a!\,c!}\left[\left(1-y^\beta\right)^{-\lambda}-1\right]^c$$

$$= \beta\lambda\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{a+1}{b}\binom{c}{d}\frac{(-1)^{b+c+d+1}(b+1)^c\alpha^{c+1}}{a!c!}\left(1-y^\beta\right)^{-\lambda(d+1)-1}y^{\beta-1} \tag{10}$$

Properties of the Odd Teissier Kumaraswamy (OTKw) Distribution

Quantile Functions of OTKw Distribution

The quantile function of a random variable Y is simply the inverse of its CDF. It is denoted as $F_Y^{-1}(q)$ and defined as $q = Pr(Y \le y_q) = F_Y(y_q) \Rightarrow \quad y_q = F_Y^{-1}(q)$, for all$0 < q < 1$. Then the quantile function of the OTKw distribution is obtained as

$$q = F_Y(y_q) = 1 - e^{1-e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]}}e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]}$$

$$\frac{q-1}{e} = -e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]}e^{-e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]}}$$

Letting $H(y_q) = -e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]}$, the equation above is transformed into the form

$$\frac{q-1}{e} = H(y_q)e^{H(y_q)}$$

From Lambert W function defined as$A(x)e^{A(x)} = x$, with the solution of the negative branch given as$A(x) = W_{-1}(x)$, see Corless (1996) for details, then we have

$$H(y_q) = W_{-1}\left(\frac{q-1}{e}\right)$$

$$e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]} = -W_{-1}\left(\frac{q-1}{e}\right), \qquad \text{(Since } H(y_q) = -e^{\alpha\left[\left(1-y_q^\beta\right)^{-\lambda}-1\right]}\text{)}$$

Then, by algebraic simplification, the solution of the above equation becomes

$$y_q = \left[1-\left(1+\frac{1}{\alpha}log\left[-W_{-1}\left(\frac{q-1}{e}\right)\right]\right)^{-\frac{1}{\lambda}}\right]^{\frac{1}{\beta}} \tag{11}$$

For $0 < q < 1$, equation (11) can be used to generate random samples from the OTKw distribution.

Moments and Moment Generating Function of OTKw Distribution

The moment of a continuous random variable Y is defined by$\mu_s^{'} = E(Y^k) = \int_{-\infty}^{\infty} y^k g(y)dy$, such that k is a non-negative integer. Then using the series expansion of the pdf in equation (11), we have the moment of the OTKw as

$$\mu_k^{'} = E(X^k) = \int_0^1 y^k f(y)\, dy \ = \beta\lambda A_1(a,b,c,d,h) \int_0^1 y^{\beta(h+1)-1}\, dy$$

$$= \beta\lambda A_1(a,b,c,d,h) \bullet \frac{y^{\beta(h+1)+k}}{\beta(h+1)+k}\bigg|_0^1 = \frac{\beta\lambda A_1(a,b,c,d,h)}{(h+1)+k} \qquad (12)$$

where

$$A_1(a,b,c,d,h) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty} \binom{a+1}{b}\binom{c}{d} \frac{(-1)^{b+c+d+h+1}(b+1)^c \alpha^{c+1}}{a!c!} \bullet \frac{\Gamma(\lambda d+\lambda+h+1)}{h!\Gamma(\lambda d+\lambda+1)}.$$

Then, the first four raw moments of the OTKw are presented as follows:

$$\mu_1^{'} == A_1(a,b,c,d,h) \bullet \frac{\beta\lambda}{\beta(h+1)+1}; \mu_2^{'} == A_1(a,b,c,d,h) \bullet \frac{\beta\lambda}{\beta(h+1)+2}$$

$$\mu_3^{'} == A_1(a,b,c,d,h) \bullet \frac{\beta\lambda}{\beta(h+1)+3}; \mu_4^{'} == A_1(a,b,c,d,h) \bullet \frac{\beta\lambda}{\beta(h+1)+4}$$

Then, the central moment of the OTKw distribution is as follows

$$\mu_k = E(Y-\mu)^k = E\left\{\sum_{i=0}^{k}\binom{k}{i}(-1)^i X^k \mu^i\right\} = \sum_{i=0}^{k}\binom{k}{i}(-1)^i E\left(X^{k-i}\right)\mu^i$$

$$= \sum_{i=0}^{k}\binom{k}{i}(-1)^i \mu_{k-i}\mu^i$$

It follows that the mean of the random variable Y from the OTKw distribution is the first moment given as

$$\mu = \mu_1^{'} == A_1(a,b,c,d,h) \bullet \frac{\beta\lambda}{\beta(h+1)+1}$$

Also, the variance, measures of skewness, and kurtosis of the OTKw distribution are obtained using the first four kth moments as

Variance $(\mu_2) = \mu - \mu^2$; Skewness $(S_k) = \frac{\mu_3 - \mu_2\mu + 2\mu^3}{(\mu_2-\mu^2)^{\frac{3}{2}}}$ and

Kurtosis $(K_s) = \frac{\mu_4 - 4\mu_3\mu + 6\mu_2\mu^2 - 3\mu^4}{(\mu_2-\mu^2)^2}$

Moment Generating Function (MGF) of MOTLK Distribution

The moment-generating function of the OTKw distribution is obtained as

$$M_Y(z) = E(e^{zy}) = \int_0^1 e^{zy} f(y)dy = \beta\lambda A_1(a,b,c,d,h)\int_0^1 e^{zy} y^{\beta(h+1)-1}\, dy$$

Using the series expansion$e^{zy} = \sum_{i=0}^{\infty}\frac{1}{i!}z^i y^i$, we have

$$M_Y(z) = \beta\lambda A_2(a,b,c,d,h,i)\int_0^1 y^{\beta(h+1)+i-1}\, dy$$

$$= \beta\lambda A_2(a,b,c,d,h,i) \bullet \frac{y^{\beta(h+1)+i}}{\beta(h+1)+i}\bigg|_0^1$$

$$= A_2(a,b,c,d,h,i) \bullet \frac{\beta\lambda}{\beta(h+1)+i} \qquad (13)$$

where $A_2(a,b,c,d,h,i) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\binom{a+1}{b}\binom{c}{d}\frac{(-1)^{b+c+d+h+1}(b+1)^c\alpha^{c+1}z^i}{a!\,c!} \bullet \frac{\Gamma(\lambda d+\lambda+h+1)}{h!i!\Gamma(\lambda d+\lambda+1)}$

Probability Weighted Moments (PWM)

The probability-weighted moment (PWM) of a random variable Y with probability density function $f(y)$and cumulative distribution function $F(y)$ denoted as $\rho_{s,k}$, is formally defined by

$$\rho_{s,k} = \int_{-\infty}^{\infty} y^k f(y) F^k(y) dy$$

But from the CDF and pdf of the OTKw distribution in equations (6) and (7) respectively, we have

$$f(y)[F(y)]^k = \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\left(e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}-1\right)e^{\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)}e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\left[1-e^{\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)}e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right]^k$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\sum_{a=0}^{\infty}\binom{k}{a}(-1)^a\left(e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}-1\right)e^{(a+1)\left(1-\alpha\left[(1-y^\beta)^{-\lambda}-1\right]\right)}e^{\alpha(a+1)\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\binom{k}{a}\frac{(-1)^a(a+1)^b}{b!}\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)^{(b+1)} e^{\alpha(a+1)\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\binom{k}{a}\binom{b+1}{c}\frac{(-1)^{a+c+1}(a+1)^b}{b!}e^{\alpha(a+c+1)\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$= \beta\lambda\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{k}{a}\binom{b+1}{c}\frac{(-1)^{a+c+1}(a+1)^b(a+c+1)^d\alpha^{d+1}}{b!d!} \times \left[(1-y^\beta)^{-\lambda}-1\right]^d (1-y^\beta)^{-\lambda-1} y^{\beta-1}$$

$$= \beta\lambda \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty} \binom{k}{a}\binom{b+1}{c}\binom{d}{h} \frac{(-1)^{a+c+d+h+1}(a+1)^b}{b!} \times \frac{(a+c+1)^d \alpha^{d+1}}{d!} \left(1-y^\beta\right)^{-(d\lambda+\lambda+1)} y^{\beta-1}$$

$$= \beta\lambda \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty} \binom{k}{a}\binom{b+1}{c}\binom{d}{h} \frac{(-1)^{a+c+d+h+1}(a+1)^b}{b!} \times \frac{(a+c+1)^d \alpha^{d+1}}{d!} \frac{\Gamma(\lambda d+\lambda+i+1)}{i!\,\Gamma(\lambda d+\lambda+1)} y^{\beta(i+1)-1}$$

$$= \beta\lambda A_2(a,b,c,d,h,i) y^{\beta(i+1)-1} \tag{14}$$

where $A_3(a,b,c,d,h,i) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty} \binom{k}{a}\binom{b+1}{c}\binom{d}{h} \frac{(-1)^{a+c+d+h+1}(a+1)^b}{b!} \times \frac{(a+c+1)^d\alpha^{d+1}}{d!} \frac{\Gamma(\lambda d+\lambda+i+1)}{i!\Gamma(\lambda d+\lambda+1)}$

Hence,

$$\rho_{s,k} = \beta\,\lambda\, A_3\left(a,b,c,d,h,i\right) \int_{-\infty}^{\infty} y^{\beta(i+1)+k-1} dy = \beta\,\lambda\, A_3\left(a,b,c,d,h,i\right) \left. \frac{y^{\beta(i+1)+k}}{\beta\left(i+1\right)+k} \right|_0^1 = A_3(a,b,c,d,h,i)\frac{\beta\lambda}{\beta(i+1)+k} \tag{15}$$

Distribution of Order Statistics of OTKw Distribution

Given that $Y_1, Y_2, Y_3, \cdots, Y_n$ is a random sample from the OTKw distribution with the corresponding order statistics $Y_{1:n}, Y_{2:n}, Y_{3:n}, \cdots, Y_{n:n}$, then the probability density function of the p^{th} order statistics $Y = Y_{p:n}$, $p = 1,2,3,\dots,n$ denoted by $f_{P:n,}(y)$, for $1 \le y \le n$ is defined as

$$f_{p:n,}(y) = \frac{n!}{(p-1)!(n-p)!} f(y) F^{p-1}(y)[1-F(y)]^{n-p}$$

$$= \frac{n!}{(p-1)!(n-p)!} \sum_{a=0}^{n-p} \binom{n-p}{a} (-1)^a f(y)[F(y)]^{p+a-1} \tag{16}$$

where $f(y)$ and $F(y)$ are respectively the pdf and the CDF of the random variable Y. Using the expansion techniques from Series Expansion of the Density Functions of OTKw Distribution, we have

$$f(y)[F(y)]^{k+a-1} = \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}} \left(e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}-1\right) e^{\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)} e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]} \left[1 - e^{\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)} e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right]^{k+a-1}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}} \sum_{b=0}^{\infty} \binom{k+a-1}{b} (-1)^a \left(e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}-1\right) e^{(b+1)\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)} e^{\alpha(b+1)\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}} \sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \binom{k+a-1}{b} \frac{(-1)^{b+1}(a+1)^c}{c!} \left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)^{(c+1)} e^{\alpha(b+1)\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$= \frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}} \sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty} \binom{k+a-1}{b}\binom{c+1}{d} \frac{(-1)^{b+d+h+1}(a+1)^c}{c!} \frac{(b+d+1)^h\alpha^h}{h!} \left[\left(1-y^\beta\right)^{-\lambda}-1\right]^h$$

$$= \beta\lambda \sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty} \binom{k+a-1}{b}\binom{c+1}{d}\binom{h}{i} \frac{(-1)^{b+d+h+i+1}(a+1)^c}{c!} \times \frac{(b+d+1)^h\alpha^{h+1}}{h!} y^{\beta-1}\left(1-y^\beta\right)^{-(\lambda h+\lambda+1)}$$

$$= \beta\lambda \sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{k+a-1}{b}\binom{c+1}{d}\binom{h}{i} \frac{(-1)^{b+d+h+i+j+1}(a+1)^c}{c!} \times \frac{(b+d+1)^h\alpha^{h+1}}{h!} \frac{\Gamma(\lambda h+\lambda+j+1)}{j!\,\Gamma(\lambda h+\lambda+1)} y^{\beta(j+1)-1}$$

Hence, the density function of the order statistics for OTKw distribution is obtained as follows

$$f_{p:n,}(y) = \frac{n!\beta\lambda}{(p-1)!(n-p)!} A_4(a,b,c,d,h,i,j) y^{\beta(j+1)-1} \tag{17}$$

where

$$A_4(a,b,c,d,h,i,j) = \sum_{a=0}^{n-p}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{n-p}{a}\binom{k+a-1}{b}\binom{c+1}{d}\binom{h}{i} \times \frac{(-1)^{a+b+d+h+i+j+1}(a+1)^c}{c!} \frac{(b+d+1)^h\alpha^{h+1}}{h!} \frac{\Gamma(\lambda h+\lambda+j+1)}{j!\,\Gamma(\lambda h+\lambda+1)}$$

Renyi Entropy of OTKw Distribution

For any random variable Y from the OTKw with pdf $f(y)$, the Renyi entropy is defined as

$$E_R(\kappa) = \frac{1}{1-\kappa} log\, \psi(\kappa), \qquad \kappa > 0 \ \& \ \kappa \neq 1$$

where $\psi(\kappa) = \int_0^\infty f^\kappa(y)\, dy$

But

$$f^\kappa(y) = \left(\frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\right)^\kappa \left(e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}-1\right)^\kappa e^{\kappa\left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)} e^{\kappa\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$= \sum_{a=}^{\infty} \frac{(-1)^{\kappa+a}\kappa^a}{a!} \left(\frac{\alpha\beta\lambda y^{\beta-1}}{(1-y^\beta)^{\lambda+1}}\right)^\kappa \left(1-e^{\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}\right)^{\kappa+a} e^{\kappa\alpha\left[(1-y^\beta)^{-\lambda}-1\right]}$$

$$=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\binom{\kappa+a}{b}\frac{(-1)^{\kappa+a+b}\kappa^{a}}{a!}\frac{\alpha^{\kappa}\beta^{\kappa}\lambda^{\kappa}y^{\kappa\beta-\kappa}}{(1-y^{\beta})^{\kappa\lambda+\kappa}}e^{\alpha(\kappa+1)\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]}$$

$$=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{a=0}^{\infty}\binom{\kappa+a}{b}\frac{(-1)^{\kappa+a+b+c}\kappa^{a}(\kappa+b)^{c}}{a!}\frac{\alpha^{\kappa+c}\beta^{\kappa}\lambda^{\kappa}}{c!}\frac{y^{\kappa\beta-\kappa}}{(1-y^{\beta})^{\kappa\lambda+\kappa}}\left[\left(1-y^{\beta}\right)^{-\lambda}-1\right]^{c}$$

$$=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{\kappa+a}{b}\binom{c}{d}\frac{(-1)^{\kappa+a+b+c+d}\kappa^{a}(\kappa+b)^{c}}{a!}\frac{\alpha^{\kappa+c}\beta^{\kappa}\lambda^{\kappa}}{c!}\frac{y^{\kappa\beta-\kappa}}{(1-y^{\beta})^{\kappa\lambda+\kappa}}\left(1-y^{\beta}\right)^{-(\lambda d+\kappa\lambda+\kappa)}$$

$$=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\binom{\kappa+a}{b}\binom{c}{d}\frac{(-1)^{\kappa+a+b+c+d}\kappa^{a}(\kappa+b)^{c}}{a!}\frac{\alpha^{\kappa+c}\beta^{\kappa}\lambda^{\kappa}}{c!}\frac{\Gamma(\lambda d+\kappa\lambda+\kappa)}{h!\,\Gamma(\lambda d+\kappa\lambda+\kappa)}y^{\beta h+\beta\kappa-\kappa}=A_5(a,b,c,d,h,)y^{\beta h+\beta\kappa-\kappa}$$

where $A_5(a,b,c,d,h,)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\binom{\kappa+a}{b}\binom{c}{d}\frac{(-1)^{\kappa+a+b+c+d}\kappa^{a}(\kappa+b)^{c}}{a!}\frac{\alpha^{\kappa+c}\beta^{\kappa}\lambda^{\kappa}}{c!}\frac{\Gamma(\lambda d+\kappa\lambda+\kappa)}{h!\Gamma(\lambda d+\kappa\lambda+\kappa)}$

Hence, the Renyi entropy of the OTKw distribution is obtained as

$$E_R(\kappa)=\frac{1}{1-\kappa}log\,\psi(\kappa)=\frac{1}{1-\kappa}log\int_0^1 A_5(a,b,c,d,h,)y^{\beta h+\beta\kappa-\kappa}\,dy \tag{18}$$

RESULTS AND DISCUSSION

Statistical Estimation and Inference

In this section, we estimate the parameters of the Odd Teissier Kumaraswamy (OTKw) distribution through the maximum likelihood estimation method and a simulation study to investigate the performance of the maximum likelihood estimates (MLEs of the unknown parameter of the OTKw distribution).

Maximum Likelihood Estimation (MLE) of Odd Teissier Kumaraswamt (OTKw) Distribution

Given a random sample $y_{1,}y_2,y_3,\cdots,y_n$ of size n from the OT-Kumaraswamy (OTKW) distribution with vector of parameters$\Phi=(\alpha,\beta,\lambda)^{T}$. Then the maximum likelihood function denoted by $L(y;\alpha,\beta,\lambda)$ is given as

$L(y;\alpha,\phi)=\prod_{i=1}^{n}f(y;\alpha,\phi)$

$$=\prod_{i=1}^{n}\left\{\frac{\alpha\beta\lambda y_i{}^{\beta-1}}{(1-y_i{}^{\beta})^{\lambda+1}}\left(e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}-1\right)e^{\left(1-e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}\right)}e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}\right\}$$

and the log-likelihood function denoted by$\ell_n=LogL(y;\alpha,\phi)$ is given as

$$\ell_n=LogL(y;\alpha,\phi)=n\,log(\alpha\beta\lambda)+(\beta-1)\sum_{i=1}^{n}log\,y_i-(\lambda+1)\sum_{i=1}^{n}log\left(1-y_i{}^{\beta}\right)+\alpha\sum_{i=1}^{n}\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]$$

$$+\sum_{i=1}^{n}\left(1-e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}\right)+\sum_{i=1}^{n\sum\left(e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}{}_{-1}\right)}log \tag{19}$$

The corresponding partial derivatives of Equation (19) with respect to the parametersα, β and $\lambda$are

$$\frac{\partial\ell_n}{\partial\alpha}=\frac{n}{\alpha}+n\,log(\alpha\beta\lambda)+\sum_{i=1}^{n}\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]+\sum_{i=1}^{n}\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}+\sum_{i=1}^{n}\frac{\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}}{\left(e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}-1\right)} \tag{20}$$

$$\frac{\partial\ell_n}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}log\,y_i-(\lambda+1)\sum_{i=1}^{n}\frac{y_i{}^{\beta}\,log\,y_i}{\left(1-y_i{}^{\beta}\right)}+\alpha\lambda\sum_{i=1}^{n}\frac{y_i{}^{\beta}\,log\,y_i}{\left(1-y_i{}^{\beta}\right)^{\lambda|+1}}-\alpha\lambda\sum_{i=1}^{n}\frac{y_i{}^{\beta}\,log\,y_i e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}}{\left(1-y_i{}^{\beta}\right)^{\lambda|+1}}+\alpha\lambda\sum_{i=1}^{n}\frac{y_i{}^{\beta}\,log\,y_i e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}}{\left(1-y_i{}^{\beta}\right)^{\lambda|+1}\left(e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}{}_{-1}\right)} \tag{21}$$

$$\frac{\partial\ell_n}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}log\left(1-y_i{}^{\beta}\right)-\alpha\sum_{i=1}^{n}\frac{log\left(1-y_i{}^{\beta}\right)}{\left(1-y_i{}^{\beta}\right)^{\lambda}}+\alpha\sum_{i=1}^{n}\frac{log\left(1-y_i{}^{\beta}\right)e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}}{\left(1-y_i{}^{\beta}\right)^{\lambda}}-\alpha\sum_{i=1}^{n}\frac{log\left(1-y_i{}^{\beta}\right)e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}}{\left(1-y_i{}^{\beta}\right)^{\lambda}\left(e^{\alpha\left[\left(1-y_i{}^{\beta}\right)^{-\lambda}-1\right]}{}_{-1}\right)} \tag{22}$$

The MLEs $\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)$ for (α,β,λ) are derived by equating the score function denoted by $U(\Phi)$ to zero, (that is $U(\Phi)=0$) and solve the system of equations $U(\Phi)=\left(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\lambda}\right)^{T}=0$ simultaneously. These systems of equations are non-linear and are solved using iterative scheme called Newton-Raphson method with the help of R statistical package to obtain the MLEs of the unknown parameters.

Simulation Study

In this section, we carry out a simulation study to investigate the performance and accuracy of

Maximum likelihood estimates for the parameters of the proposed OTKw distribution model for various combinations of 6 sample sizes considering four sets of parameter values. The quantile function was used to generate random data from the OTKw distribution. In each simulation, 10,000 samples of size n = 25, 50, 100, 200, 300 and 500 were generated for some sets of parameter values,

$I:\alpha=0.2,\beta=0.5,\lambda=0.75;II:\alpha=0.5,\beta=1.5,\lambda=1.2;III\rightleftarrows:\alpha=2.5;\beta=0.25;\lambda=4.2$ and$IV\rightleftarrows:\alpha=1.2,\beta=2.5;\lambda=2.3$.

The R software was used to evaluate the following quantities in this simulation study:

Mean $=\frac{1}{N}\sum_{i=1}^{N}\hat{\varphi}$

Bias $= \frac{1}{N}\sum_{i=1}^{N}(\hat{\varphi} - \varphi)$

Root Mean Square Error (RMSE) $= \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\varphi} - \varphi)^2}$

Where $N = 10{,}000$, $\hat{\varphi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ is an estimate of $\varphi = (\alpha, \beta, \lambda)$

From the simulation results presented in Tables 1 and 2, it is observed that as the sample size increases, the estimated mean converges progressively toward the true parameter value. In addition, both the bias and the Root Mean Square Error (RMSE) decrease steadily and tend toward zero as the sample size becomes larger. Statistically, this indicates that the maximum likelihood estimators are consistent, since the estimators become more accurate with increasing sample size. The reduction in bias further suggests that the estimators are asymptotically unbiased, while the decreasing RMSE demonstrates improved precision and stability of the estimates. These findings provide strong empirical evidence that the maximum likelihood estimation method is an efficient and reliable estimation technique for the parameters of the Odd Teissier Kumaraswamy (OTKw) distribution model.

Table 1: The Mean, Biases and RMSE of MLEs for Set I and Set II Values of the Parameters of the OTKw Distribution

N		$SetI \Rightarrow (\alpha = 0.2; \beta = 0.5; \lambda = 0.75)$			$SetII \Rightarrow (\alpha = 0.5; \beta = 1.5; \lambda = 1.2)$		
		Mean	**Bias**	**RMSE**	**Mean**	**Bias**	**RMSE**
25	α	0.2110	0.0110	0.1025	0.5112	0.0112	0.2359
	β	0.5551	0.0551	0.2279	1.5538	0.0538	0.3767
	λ	0.7953	0.0453	0.1726	1.3141	0.1141	0.3841
50	α	0.2185	0.0185	0.0910	0.5127	0.0127	0.1769
	β	0.5574	0.0574	0.2069	1.5381	0.0381	0.2645
	λ	0.7691	0.0191	0.1191	1.2571	0.0571	0.2599
100	α	0.2186	0.0186	0.0775	0.5074	0.0074	0.1316
	β	0.5493	0.0493	0.1776	1.5185	0.0185	0.1924
	λ	0.7566	0.0066	0.0856	1.2284	0.0284	0.1830
200	α	0.2139	0.0139	0.0648	0.5021	0.0021	0.0964
	β	0.5360	0.0360	0.1476	1.5139	0.0139	0.1465
	λ	0.7525	0.0025	0.0612	1.2187	0.0187	0.1305
300	α	0.2117	0.0117	0.0562	0.5002	0.0002	0.0834
	β	0.5320	0.0320	0.1286	1.5108	0.0108	0.1226
	λ	0.7524	0.0024	0.0508	1.2161	0.0161	0.1088
500	α	0.2097	0.0097	0.0457	0.4990	-0.0010	0.0728
	β	0.5257	0.0257	0.1013	1.5073	0.0073	0.1045
	λ	0.7509	0.0009	0.0404	1.2124	0.0124	0.0880

Table 2: The Mean, Biases and RMSE of MLEs for Set III and Set IV Values of the Parameters of the OTKw Distribution

N		$Set \rightleftarrows III \rightleftarrows\Rightarrow (\alpha = 2.5; \beta = 0.25; \lambda = 4.2)$			$Set \rightleftarrows IV \Rightarrow (\alpha = 1.2; \beta = 2.5; \lambda = 0.6)$		
		Mean	**Bias**	**RMSE**	**Mean**	**Bias**	**RMSE**
25	α	2.1577	0.1577	0.7228	1.2228	0.0228	0.4302
	β	0.2611	0.0111	0.0467	2.5949	0.0949	0.5214
	λ	4.7892	0.5892	1.7849	2.6359	0.3359	0.9962
50	α	2.1426	0.1426	0.5374	1.2438	0.0438	0.3275
	β	0.2577	0.0077	0.0357	2.5765	0.0765	0.3951
	λ	4.4804	0.2804	1.1556	2.4628	0.1628	0.6456
100	α	2.1077	0.1077	0.4661	1.2485	0.0485	0.2737
	β	0.2548	0.0048	0.0278	2.5533	0.0533	0.2890
	λ	4.3420	0.1420	0.8156	2.3768	0.0768	0.4556
200	α	2.0804	0.0804	0.3738	1.2328	0.0328	0.2213
	β	0.2532	0.0032	0.0215	2.5327	0.0327	0.2184
	λ	4.2671	0.0671	0.5586	2.3390	0.0390	0.3226
300	α	2.0752	0.0752	0.3379	1.2336	0.0336	0.1970
	β	0.2529	0.0029	0.0178	2.5340	0.0340	0.1897
	λ	4.2464	0.0464	0.4635	2.3291	0.0291	0.2619
500	α	2.0729	0.0729	0.3021	1.2289	0.0289	0.1809
	β	0.2528	0.0028	0.0149	2.5259	0.0259	0.1589
	λ	4.2276	0.0276	0.3690	2.3169	0.0169	0.2080

Real Data Application

Here, we consider three real data sets to illustrate that the OTKw distribution is a good lifetime model in comparison to four existing models, namely OLxTL, WTL, MOKw and ETL. The maximum likelihood estimates with their corresponding SEs (in parentheses) of the model's parameters are obtained for each data set. The model comparison was performed using the *log-likelihood* (log*L*), the *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC), the *Cramer–Von Mises* (CVM) and the *Anderson–Darling* (AD). The model with the smallest values of log*L*, AIC, CVM and AD criteria has the best fit for the data. However, since the values of log*L*, AIC, BIC, CVM and AD are sometimes influenced by the number of parameters in the distribution, we also examine the Kolmogorov-Smirnov (KS) statistic and the p-value which are sufficient criteria for comparing the performance of models. In this case, the model with the lowest KS value and highest p-value has the best fit. Furthermore, the fitted densities and fitted CDF.

Data Set 1

The first data set consist of 48 rock samples from a petroleum reservoir reported in Cordeiro and Brito (2012) and the data are:

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Table 3: The *MLEs* and *S.Es* (in Parentheses) for Data Set 1

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
OTKw	39.8623	0.8715		0.0777
	(62.1627)	(0.1286)		(0.1160)
OLxTL	2.4487	0.0411	4.3199	
	(2.7296)	(0.0848)	(0.7778)	
WTL	1.0822	2.1991		4.1126
	(1.1746)	(1.6549)		(7.3569)
MOEKw	0.0213	4.8119	46.3997	
	(0.03311)	(0.6195)	(60.87849)	
ETL	3.0033			11.16189
	(0.3927)			(3.8628)

Table 4: Summary Statistics for the Data Set 1

Model	*log L*	*AIC*	*BIC*	*CVM*	*AD*	*KS*	*p-Value*
OTKw	48.5402	-109.4084	-103.7948	0.0461	0.3262	0.0910	0.8207
OLxTL	51.41682	-106.2988	-100.6852	0.0695	0.5432	0.1071	0.6400
WTL	56.14938	-96.83364	-91.22004	0.2553	1.5485	0.1608	0.1669
MOEKw	57.70418	-91.0804	-85.4668	0.3530	2.1087	0.1722	0.1157
ETL	52.34036	-100.6807	-96.93832	0.2345	1.3972	0.1652	0.1451

Data Set 2

The second data set represents 20 observations of the maximum flood level (in millions of cubic feet per second) for Susquehanna River at Harrisburg, Pennsylvania and is reported in Dumonceaux and Antle (1973). The observations are:

0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.

Table 5: The *MLEs* and *S.Es* (in Parentheses) for Data Set 2

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
OTKw	0.0512	0.0291		0.8015
	(0.0679)	(0.0355)		(0.1500)
OLxTL	0.7549	0.0086	9.6856	
	(0.3478)	(0.0137)	(2.2633)	
WTL	21.7264	0.2452		6.4331
	(5.9419)	(0.3987)		(2.1284)
MOEKw	0.0152	6.4540	5.4005	
	(0.0345)	(1.2643)	(11.1785)	
ETL	3.8079			2.0885
	(1.1249)			(1.0506)

Table 6: Summary Statistics for the Data Set 2

Model	*log L*	*AIC*	*BIC*	*CVM*	*AD*	*KS*	*p-Value*
OTKw	6.9655	-26.5354	-23.5482	0.3654	1.9602	0.2555	0.9269
OLxTL	11.4210	-25.8470	-22.8598	0.2991	1.5145	0.2743	0.8897
WTL	14.5925	-23.1851	-20.1979	0.0933	0.5641	0.1851	0.4992
MOEKw	16.2677	-10.9311	-14.9439	0.0388	0.2384	0.1220	0.09855
ETL	15.9235	-18.8420	-16.8505	0.0410	0.3131	0.1296	0.1466

Data Set 3

This data set is represent the total milk production from the first birth of 107 cows of the SINDI race. The data can be found in Cordeiro and Brito (2012) and analyzed by Muhammad et al. (2023) with the observed values as:

0.4365, 0.4260, 0.5140, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747

Table 7: The *MLEs* and *S.Es* (In Parentheses) for Data Set 3

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
OTKw	0.1218 (0.0692)	0.0275 (0.0181)		0.5537 (0.0536)
OLxTL	2.9521 (1.0826)	6.1070 (3.9298)	1.5329 (0.3981)	
WTL	0.4965 (0.3431)	2.4073 (0.6621)		0.1407 (0.1781)
MOEKw	11.5331 (14.8568)	1.0159 (0.5299)	3.8609 (0.5481)	
ETL	1.0211 (0.3448)			0.2008 (0.0761)

Table 8: Summary Statistics for the Data Set 3

Model	*log L*	*AIC*	*BIC*	*CVM*	*AD*	*KS*	*p-Value*
OTKw	16.5393	-52.9157	-44.8973	0.0270	0.1959	0.0455	0.9796
OLxTL	28.1316	-50.2632	-42.2447	0.0855	0.5057	0.0745	0.5912
WTL	28.1553	-50.3107	-42.2922	0.0692	0.4406	0.0659	0.7409
MOEKw	29.4578	-27.0787	-19.0603	0.4026	2.5280	0.1087	0.1589
ETL	19.2977	-34.5955	-29.2498	0.4474	2.2313	0.1182	0.1004

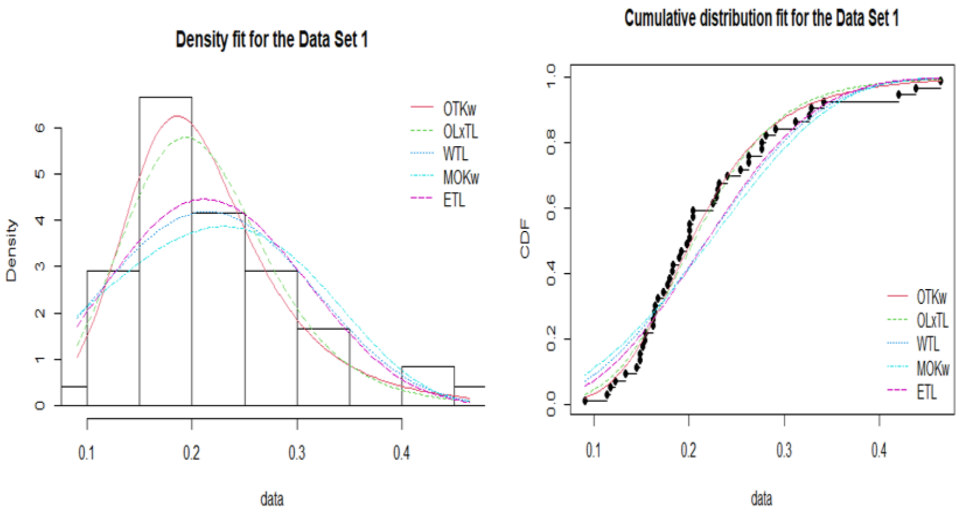

Figure 3: The Empirical Plots of the Fitted Densities and Cumulative Distribution Functions for the Data Set 1

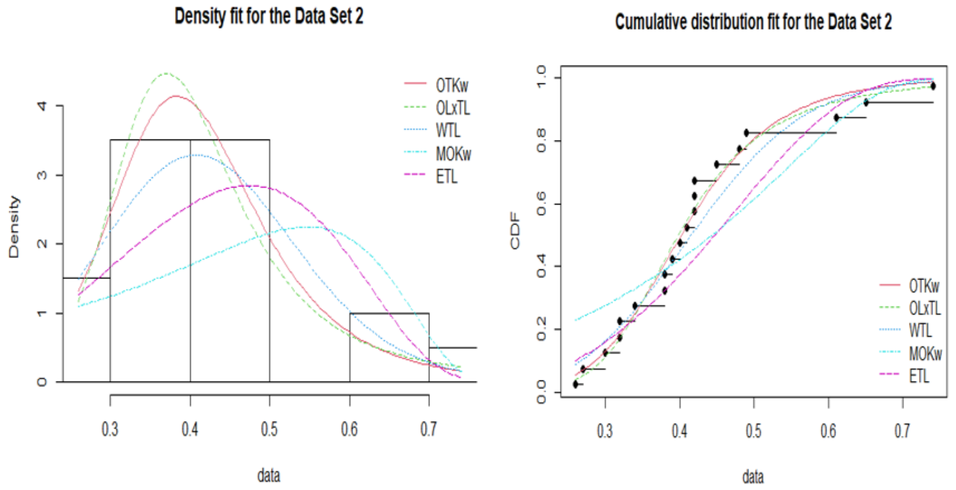

Figure 4: The Empirical Plots of the Fitted Densities and Cumulative Distribution Functions for the Data Set 2

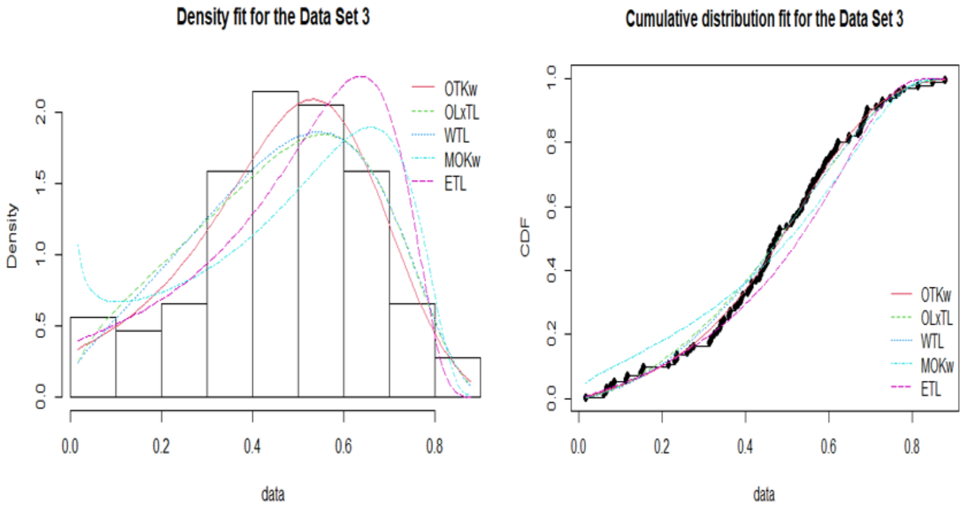

Figure 5: The Empirical Plots of The Fitted Densities and Cumulative Distribution Functions for the Data Set 3

Tables 3, 5 and 7, the Maximum Likelihood estimates (MLE) of the models unknown parameters and their corresponding standard errors indicate that the models were successfully fitted to the data. It shows that parameter estimates with reasonably stable standard errors, suggesting reasonable estimation stability and parameter reliability for the three data sets 1, 2 and 3 respectively. The results in Tables 4, 6 and 8 indicates that the OTKw model has the best fit in all the data sets. It produced the least goodness-of-fit statistics values, namely the log L, AIC, BIC, CVM, AD, KS; with the largest p-value for the three data sets. These results suggest that the OTKw distribution performs better in fitting these data sets, and strongly in agreement with the empirical distribution. Comparatively, the OLxTL model also performed reasonably well but was slightly inferior to the OTKw model across all goodness-of-fit measures. The WTL, MOEKw, and ETL models recorded larger discrepancy statistics and smaller p-values, indicating weaker fitting performance and less adequacy in capturing the characteristics of the data.

Figures 3, 4, and 5 show the empirical density and cumulative distribution functions (CDFs) of the OTKw model in comparison with selected densities and CDFs for the three Data Sets. These figures showed that the three data sets are better fitted by the OTKw model. The empirical density and cumulative distribution functions (CDFs) of the OTKw model compared with some densities and CDFs are presented in Figures 3, 4 and 5 for Data Set 1, 2 and 3 respectively. These plots indicated that the OTKw model provides a better fit for the three data sets.

Overall, the empirical analysis clearly indicates that the OTKw distribution provides the most adequate and statistically reliable fit for the three data sets, among the competing models considered. The superiority of its goodness-of-fit measures and strong consistency with the observed data demonstrate its usefulness and flexibility for modeling lifetime phenomena.

CONCLUSION

In this paper, we presented a novel distribution on a unit interval, called "the the Odd Teissier Kumaraswamy (OTKw) distribution". This is to present a unit interval distribution from the Teissier distribution that is flexible enough to handle left skewed data sets. Some properties of the OTKw distribution are studied. The maximum likelihood estimation was used to estimate the unknown parameters of the model. Simulation study was performed to evaluate the efficiency of MLEs estimates for different sample sizes. An application of the OTKw to three data shows the superior performance of the proposed OTKw distribution over OLxTL, WTL, MOKw and ETL distributions. Hence, we hope that the proposed model will be better alternative model for a wider range in statistical research.

Due to the theoretical and practical flexibility of the proposed Odd Teissier Kumaraswamy (OTKw) distribution, different directions for future research arise from this study. First, many practical experiments or survey yields samples with some incomplete (censored) information. Hence, further studies can be focused on the use of censored data in demonstrating the flexibility of the proposed OTKw distribution. Also, most practical scenarios may be influenced by more than one independent variables. Thus, we suggest that for further study, a bivariate and multivariate distributions may be developed using the proposed OTKw distribution to address scenarios with more than one independent variables. Lastly, further study evaluate the Bayesian estimates for parameters of the proposed OTKw distribution and to investigate their practical usefulness through applications to real datasets from a variety of fields. Such extensions would provide a broader understanding of the flexibility, usefulness, and applicability of the proposed OTKw distribution.

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