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# A Novel Hybrid of Weibull-Exponential-Gamma (W-E-G) Distribution with Applications to Bladder Cancer Data

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**ABSTRACT**

The Weibull-Exponential-Gamma (WEG) hybrid distribution is introduced for analyzing cancer survival data. The proposed distribution combines the flexibility of the Weibull and Gamma distributions with the simplicity of the Exponential model, providing a unified framework capable of accommodating increasing, decreasing, and non-monotonic hazard functions. The probability density function, cumulative distribution function, and key statistical properties-including moments, hazard function, and reliability characteristics-are derived and analyzed to understand the distribution's behavior under varying parameter configurations. A Monte Carlo simulation with 1,000 replicates across sample sizes of 50 to 1,000 demonstrated that Bias, MSE, and RMSE approached zero as sample size grew, confirming estimator consistency. When applied to real-world cancer survival data, the WEG model outperformed Exponential (LL = -382.14, AIC = 766.28), Weibull, Gamma, Weibull-Gamma, and Exponential-Gamma distributions, achieving the highest LL (-361.42) and lowest AIC (732.85) and BIC (748.49). A clinical surveillance schedule derived from the WEG hazard function stratified post-treatment risk into six phases: very high (0-3 months, h = 0.14-0.10) requiring monthly visits; high (3–6 months, h = 0.10-0.08); moderate (6-12 months, h = 0.08-0.05); low (12-24 months, h = 0.05-0.03); very low (24-36 months, h = 0.03-0.02); and minimal (>36 months, h < 0.02) needing annual follow-up. The WEG distribution offers a flexible, evidence-based tool for bladder cancer survival modeling and risk-adapted patient monitoring. Based on these findings, the WEG distribution is recommended as a flexible and robust tool for survival analysis, particularly in biomedical research where heterogeneity and non-standard hazard behaviors are common.

**Keywords**: WEG; Bladder Cancer; Survival Analysis; Hazard Function

## INTRODUCTION

Statistical modeling plays a critical role in understanding complex phenomena in biomedical research, particularly in survival analysis and reliability studies. Traditional probability distributions, such as the Weibull, Exponential, and Gamma distributions, have long been employed to model lifetime data due to their flexibility and interpretability (Weibull, 1951; Cox & Oakes, 1984; Stacy, 1962). However, biomedical datasets often exhibit intricate patterns that cannot be fully captured by a single distribution. For instance, patient survival times may display varying hazard rates, including increasing, decreasing, or non-monotonic trends over time. This complexity motivates the development of hybrid or compound distributions, which combine the strengths of individual distributions while providing greater adaptability to real-world data (Mudholkar & Srivastava, 1993; Nadarajah & Kotz, 2006).

The Weibull distribution is widely recognized for its versatility in modeling monotonic hazard functions, making it a staple in reliability and survival analysis (Weibull, 1951). Meanwhile, the Exponential distribution, a special case of the Weibull, provides a simple and interpretable model for constant hazard scenarios (Kalbfleisch & Prentice, 2002). The Gamma distribution, with its flexible shape and scale parameters, allows for modeling positively skewed data and accommodating diverse hazard rate shapes (Stacy, 1962). Although each distribution has proven effective in certain contexts, complex biomedical phenomena-such as cancer progression or treatment response-often require models that integrate multiple distributional behaviors.

Hybrid distributions have emerged as a powerful solution to these challenges. By combining two or more fundamental distributions, hybrid models offer enhanced flexibility in capturing diverse data characteristics, including multimodality, heavy tails, and varying hazard rates (Adamidis & Loukas, 1998; Nadarajah & Kotz, 2006). In particular, the proposed hybrid Weibull-Exponential-Gamma (WEG) distribution aims to merge the strengths of its component distributions. This hybrid framework not only accommodates monotonic and non-monotonic hazard rates but also provides a more accurate representation of complex lifetime data, such as survival times of bladder cancer patients undergoing different treatments.

Bladder cancer, one of the most common malignancies of the urinary system, presents a compelling case for advanced statistical modeling. Survival times among patients can vary substantially due to factors such as tumor stage, treatment regimen, and patient-specific biological characteristics (Babjuk et al., 2017). Traditional models, while useful, often fail to capture the heterogeneity and intricate hazard patterns observed in clinical data. By applying the WEG distribution to bladder cancer datasets, researchers can obtain deeper insights into survival probabilities, risk factors, and treatment efficacy, potentially guiding clinical decision-making and personalized patient care.

It can be seen that, the hybrid Weibull-Exponential-Gamma distribution represents a promising extension of classical lifetime models. Its flexibility in accommodating diverse hazard structures and complex survival patterns makes it particularly suitable for biomedical applications, including bladder cancer research. By integrating the strengths of the Weibull, Exponential, and Gamma distributions, this hybrid model provides a more nuanced understanding of patient survival data, paving the way for improved statistical inference and evidence-based medical interventions.

## MATERIALS AND METHODS

### Research Method

We have outlines the development of a hybrid distribution combining the Weibull, Exponential, and Gamma (WEG) distributions for survival analysis. The properties of the individual distributions are presented first, followed by the mathematical procedure for developing the WEG hybrid model.

### Basic Properties of the Principal Models

#### *Weibull Distribution*

The probability density function (PDF) of the Weibull distribution is given by:

$$f_W(x;\,\lambda,k) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(x/\lambda\right)^k}, & x \geq 0, \\ 0, & x < 0 \end{cases} \quad (1)$$

Where $\lambda > 0$ is the scale parameter, and $k > 0$ is the shape parameter.

The cumulative distribution function (CDF) is:

$$F_W(x;\,\lambda,\,k) = 1 - e^{-(x/\lambda)}\,k. \quad (2)$$

The mean is:

$$\mu_w = \lambda\Gamma\left(1+\frac{1}{k}\right) \quad (3)$$

Where $\Gamma(\cdot)$ is the Gamma function.

The moment-generating function (MGF) exists for $t < \lambda^{-k}$:

$$M_W(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\lambda^n\Gamma\left(1+\frac{1}{k}\right) \quad (4)$$

The characteristic function (CF) is:

$$\phi_W(t) = M_W(it) \quad (5)$$

#### *Exponential Distribution*

The PDF of the Exponential distribution is:

$$f_E(x;\,\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0 \end{cases} \quad (6)$$

Where $\lambda > 0$ is the rate parameter.

The CDF is:

$$F_E(x;\,\lambda) = 1 - e^{-\lambda x} \quad (7)$$

The mean is:

$$\mu_E = \frac{1}{\lambda} \quad (8)$$

The MGF is:

$$M_E(t) = \frac{\lambda}{\lambda - t},\ t < \lambda. \quad (9)$$

The characteristic function (CF) is:

$$\phi_E(t) = \frac{\lambda}{\lambda - it}. \quad (10)$$

#### *Gamma Distribution*

The PDF of the Gamma distribution is:

$$f_G(x;\alpha,\beta) = \begin{cases} \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}, & x \geq 0, \\ 0, & x < 0 \end{cases} \quad (11)$$

Where $\alpha > 0$ is the shape parameter, and $\beta > 0$ is the rate parameter. The CDF is:

$$F_G(x;\alpha,\beta) = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)} \quad (12)$$

Where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma function.

The mean is:

$$\mu_G = \frac{\alpha}{\beta} \quad (13)$$

The MGF is:

$$M_G(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha},\ t < \beta \quad (14)$$

The characteristic function (CF) is:

$$\phi_G(t) = \left(1 - \frac{it}{\beta}\right)^{-\alpha}. \quad (15)$$

### Developing the Proposed WEG Hybrid Distribution

#### *Define the Generalized PDF*

The hybrid PDF $(x)$ is defined as a weighted combination of the three distributions:

$$f(x;\lambda_w,k,\lambda_E,\alpha,\beta,w_1,w_2,w_3) = w_1 f_w(x;\lambda_w,k) + w_2 f_E(x;\lambda_E) + w_3 f_G(x;\,\alpha,\beta), \quad (16)$$

Where $w_1 + w_2 + w_3 = 1$ are the weights.

#### *Derive the CDF*

The CDF is the integral of the PDF:

$$F(x) = w_1 f_w(x;\lambda_w,k) + w_2 f_E(x;\lambda_E) + w_3 f_G(x;\,\alpha,\beta) \quad (17)$$

#### *Moments and Mean*

The mean $\mu_{WEG}$ is a weighted sum of the means of the individual distributions:

$$\mu_{WEG} = w_1\mu_w + w_2\mu_E + w_3\mu_G. \quad (18)$$

Substituting the means:

$$\mu_{WEG} = w_1\lambda_w\Gamma\left(1+\frac{1}{k}\right) + w_2\frac{1}{\lambda_E} + w_3\frac{\alpha}{\beta} \quad (19)$$

#### *Derive the Hazard Function*

The hazard function $h(x)$ is given by:

$$h_{WEG}(x) = \frac{f_{WEG}(x)}{1-F_{WEG}(x)} \quad (20)$$

Substituting the hybrid PDF and CDF:

$$h_{WEG}(x) = \frac{w_1 f_w(x;\lambda_w,k)+w_2 f_E(x;\lambda_E)+w_3 f_G(x;\,\alpha,\beta)}{1-\left(w_1 f_w(x;\lambda_w,k)+w_2 f_E(x;\lambda_E)+w_3 f_G(x;\,\alpha,\beta)\right)} \quad (21)$$

#### *Parameter Estimation*

Parameters $\lambda_W$, $k$, $\lambda_E$, $\alpha$, $\beta$, $w_1$, $w_2$, $w_3$ are estimated by maximizing the log-likelihood function:

$$\ell(\theta) = \sum_{i=1}^{n} log f_{WEG}(x_i;\,\theta) \quad (22)$$

Where $\theta = (\lambda_W, k, \lambda_E, \alpha, \beta, w_1, w_2, w_3)$. Expanding the log-likelihood function:

$$\ell(\theta) = \sum_{i=1}^{n} log\left(w_1 f_w(x_i) + w_2 f_E(x_i) + w_3 f_G(x_i)\right). \quad (23)$$

But:

Normalization:

$$\int_0^\infty f(x)dx = 1 \quad (24)$$

Mean Proof:

$$\mu_{WEG} = \int_0^\infty x f_{WEG}(x)dx \quad (25)$$

#### *Second-Order Properties (Variance and Higher Moments)*

The variance of the WEG distribution, Var $(X)$, is derived from the second moment:

$$Var(X) = E[X^2] - (E[X])^2 \quad (26)$$

The second moment, $E[X]^2 - (E[X])^2$, is given by:

$$E[X]^2 = W_1 E[X_W^2] + W_2 E[X_E^2] + W_3 E[X_G^2] + \quad (27)$$

Where:

$$E[X_W^2] = \lambda_W^2\Gamma\left(1+\frac{2}{k}\right), \quad (28)$$

$$E[X_E^2] = \frac{2}{\lambda_E^2}, \quad (29)$$

$$E[X_G^2] = \frac{\alpha+(\alpha+1)}{\beta^2} \quad (30)$$

Substituting these expressions:

$$Var_{WEG}(X) = W_1\lambda_W^2\Gamma\left(1+\tfrac{2}{k}\right) + W_2\tfrac{2}{\lambda_E^2} + W_3\tfrac{\alpha+(\alpha+1)}{\beta^2} - (w_1\mu_W + w_2\mu_E + w_3\mu_G)^2 \quad (31)$$

#### *Derivation of the Reliability Function*

The reliability function, $(x)$, represents the probability of survival beyond a specific time $x$. It is given by:

$R(x) = 1 - F_{WEG}(x)$ (32)
Substituting the CDF:
$R(x) = 1 - \left(w_1 f_w(x; \lambda_w, k) + w_2 f_E(x; \lambda_E) + w_3 f_G(x;\ \alpha, \beta)\right)$ (33)

The reliability function can be further analyzed to study long-term survival probabilities for bladder cancer patients.

***Hazard Rate Behavior and Its Properties***
The behavior of the hazard rate $h\ (x)$ can be studied in different intervals by analyzing the components. For small values of $x$:

$h_{WEG}(x) \approx \frac{w_1 k \lambda_W^{-k} x^{k-1} + w_2 \lambda_E e^{-\lambda_E x} + W_3 \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}}{1 - F_{WEG}(x)}$ (34)
As $x \to \infty$, the hazard rate tends to:

$\lim_{x\to\infty} h_{WEG}(x) = \frac{w_3 \beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$. (35)

***The Quantile function***
The Quantile function $\varrho(p) = F^{-1}(P),\ 0 < 1$. Thus $\varrho(P)$ is the solution x of:

$W_1\left[1 - exp - \frac{x^k}{\lambda w}\right] + W_2[1 - \exp(-\lambda_E X)] + W_3 \frac{\gamma\left(\alpha, \frac{x}{\theta}\right)}{\Gamma(\alpha)} = P$

In general, $\varrho(p) = F^{-1}(P)$
Has no closed-form expression, because it requires solving a non-linear equation involving both exponential and incomplete gamma terms.
Therefore, the quantile must be obtained numerically.
$\varrho(p) = x_p$
Where $x_p$ satisfies;
$g(x) = W_1\left[1 - exp - \frac{x^k}{\lambda w}\right] + W_2[1 - \exp(-\lambda_E X)] +$
$W_3 \frac{\gamma\left(\alpha, \frac{x}{\theta}\right)}{\Gamma(\alpha)} - P = 0$
By Newton-Raphson Iteration,
Since $g'(x) = f(x)$
$x_{n+1} = x_n - \frac{F(x_n) - p}{f(x_n)}$
Substitute the mixture pdf:
$x_{n+1} = x_n - \frac{W_1 F_W(x_n) + W_2 F_E(x_n) + W_3 F_G(x_n) - p}{W_1 f_W(x_n) + W_2 f_E(x_n) + W_3 f_G(x_n)}$
Simulation using the quantile.
Because $\varrho(p)$ is not explicit, random variates are usually generated either by:

i. Drawing $U \sim uniform(0,1)$ and solving $F(x) = U$ numerically
ii. Using the mixture representation:
iii. Choose component $j \in \{WEG\}$ with probabilities $W_1, W_2, W_3$
iv. Generate from the selected components distribution.

Method 2 is usually much simpler and faster. The quantile function $\varrho(p)$ of the WEG mixture distribution is defined implicitly by
$W_1 F_W\left(\varrho(p)\right) + W_2 F_E\left(\varrho(p)\right) + W_3 F_G\left(\varrho(p)\right) = P,$
Where $0 < p < 1$. Since this equation involves exponential and incomplete gamma function, no closed-form inverse exists in general. Hence $\varrho(p)$ is evaluated numerically using root-finding methods such as Newton-Raphson or Brent's algorithm.

***Parameter Estimation via Maximum Likelihood Estimation (MLE)***
The parameters $\theta = (\lambda_W, k, \lambda_E, \alpha, \beta, w_1, w_2, w_3)$ are estimated by solving:
$\hat{\theta} = \arg max_\theta\ \ell(\theta)$ (36)

Where the log-likelihood function is:
$\ell(\theta) = \sum_{i=1}^n log f_{WEG}(x_i;\ \theta)$ (37)
Taking partial derivatives with respect to each parameter:
$\frac{\partial \ell}{\partial \lambda_w} = \sum_{i=1}^n \frac{\partial \ell}{\partial \lambda_w} log f_{WEG}(x_i;\ \theta)$ (38)

$\frac{\partial \ell}{\partial k} = \sum_{i=1}^n \frac{\partial \ell}{\partial k} log f_{WEG}(x_i;\ \theta)$ (39)
$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^n \frac{\partial \ell}{\partial \alpha} log f_{WEG}(x_i;\ \theta)$ (40)
$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^n \frac{\partial \ell}{\partial \beta} log f_{WEG}(x_i;\ \theta)$ (41)
$\frac{\partial \ell}{\partial \lambda_E} = \sum_{i=1}^n \frac{\partial \ell}{\partial \lambda_E} log f_{WEG}(x_i;\ \theta)$ (42)

These equations are solved numerically to obtain the maximum likelihood estimates of the parameters.

***Simulation Study of the WEG Distribution***
In this analysis, 1000 samples were simulated from the WEG distribution. The selected sample sizes were n= 50, 100, 200, 500, and 1000. These simulated datasets were then employed to estimate the model parameters and to calculate the correspondence Bias, Mean square Error (MSE) and Root Mean Square Error (RMSE). The summarized results are presented in Tables 1 and Table 2. The results presented in the tables show that increasing the sample size leads to a progressive reduction in both Bias, MSE and RMSE, with these measures tending toward zero. This pattern indicates that the estimators improve in accuracy and stability as the sample size grows. Consequently, the parameter estimates can be regarded as efficient and consistent, since larger samples yield more precise estimation.

**Table 1: MLEs, Biases and RMSE for Some Values of Parameters**

| N | Parameter | Mean Estimate | Std. Dev | Bias | MSE | RMSE |
|---|---|---|---|---|---|---|
| 50 | $k = 0.5$ | 0.78 | 0.48 | 0.28 | 0.31 | 0.56 |
| | $\alpha = 0.5$ | 0.78 | 0.52 | 0.28 | 0.35 | 0.59 |
| | $\beta = 0.5$ | 0.78 | 0.31 | 0.28 | 0.17 | 0.42 |
| 100 | $k = 0.5$ | 0.64 | 0.38 | 0.14 | 0.16 | 0.40 |
| | $\alpha = 0.5$ | 0.65 | 0.41 | 0.15 | 0.19 | 0.44 |
| | $\beta = 0.5$ | 0.65 | 0.24 | 0.15 | 0.08 | 0.08 |
| 200 | $k = 0.5$ | 0.56 | 0.28 | 0.06 | 0.08 | 0.29 |
| | $\alpha = 0.5$ | 0.57 | 0.30 | 0.07 | 0.10 | 0.31 |
| | $\beta = 0.5$ | 0.57 | 0.18 | 0.07 | 0.04 | 0.19 |
| 500 | $k = 0.5$ | 0.52 | 0.18 | 0.02 | 0.03 | 0.18 |
| | $\alpha = 0.5$ | 0.53 | 0.19 | 0.03 | 0.04 | 0.19 |
| | $\beta = 0.5$ | 0.52 | 0.11 | 0.02 | 0.01 | 0.11 |
| 1000 | $k = 0.5$ | 0.51 | 0.12 | 0.01 | 0.02 | 0.12 |

| N | Parameter | Mean Estimate | Std. Dev | Bias | MSE | RMSE |
|---|---|---|---|---|---|---|
| | $\alpha = 0.5$ | 0.51 | 0.13 | 0.01 | 0.02 | 0.13 |
| | $\beta = 0.5$ | 0.51 | 0.08 | 0.01 | 0.01 | 0.08 |

**Table 2: MLEs, Biases and RMSE for Some Values of Parameters**

| N | Parameter | Mean Estimate | Std. Dev | Bias | MSE | RMSE |
|---|---|---|---|---|---|---|
| 50 | $k = 0.8$ | 1.09 | 0.52 | 0.29 | 0.36 | 0.60 |
| | $\alpha = 1.0$ | 1.36 | 0.58 | 0.36 | 0.46 | 0.68 |
| | $\beta = 0.6$ | 082 | 0.33 | 0.22 | 0.16 | 0.40 |
| 100 | $k = 0.8$ | 0.96 | 0.40 | 0.16 | 0.19 | 0.43 |
| | $\alpha = 1.0$ | 1.20 | 0.45 | 0.20 | 0.24 | 0.49 |
| | $\beta = 0.6$ | 0.72 | 0.25 | 0.12 | 0.08 | 0.28 |
| 200 | $k = 0.8$ | 0.87 | 0.29 | 0.07 | 0.09 | 0.30 |
| | $\alpha = 1.0$ | 1.09 | 0.33 | 0.09 | 0.12 | 0.34 |
| | $\beta = 0.6$ | 0.66 | 0.18 | 0.06 | 0.04 | 0.19 |
| 500 | $k = 0.8$ | 0.83 | 0.19 | 0.03 | 0.04 | 0.19 |
| | $\alpha = 1.0$ | 1.04 | 0.21 | 0.04 | 0.05 | 0.21 |
| | $\beta = 0.6$ | 0.63 | 0.12 | 0.03 | 0.02 | 0.12 |
| 1000 | $k = 0.8$ | 0.81 | 0.13 | 0.01 | 0.02 | 0.13 |
| | $\alpha = 1.0$ | 1.01 | 0.15 | 0.01 | 0.02 | 0.15 |
| | $\beta = 0.6$ | 0.61 | 0.08 | 0.01 | 0.01 | 0.08 |

**Application to Real-Life Data**

The dataset below was given by Gross *et al.*, (1975) and used by Ismail *et al*., (2023) and it represents the remission times (in months) of a random sample of 128 bladder cancer patients.

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

**Table 3: The MLE and Parameter Values of WEG and the Competing Models**

| Model | $\alpha$ | K | $\beta$ | $\lambda$ | p | $\lambda_w$ | $k_w$ | $\alpha_I$ | $\lambda_I$ | $\alpha g$ | $\beta g$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| W | - | 0.961 | - | 9.703 | - | - | - | - | - | - | - |
| E | - | - | - | 0.102 | - | - | - | - | - | - | - |
| G | 1.082 | - | 0.112 | - | - | - | - | - | - | - | - |
| W-G | - | - | - | - | 0.584 | 8.123 | 0.893 | | | 1.235 | 0.146 |
| E-G | 2.341 | | | 7.223 | - | - | - | - | - | - | |
| W-E-G | - | - | - | - | - | - | 0.631 | 8.452 | 0.912 | 2.712 | 9.332 |

The W-E-G Hybrid model contains the largest number of estimated parameters $k_w = 0.631, \alpha_I = 8.452, \lambda_I = 0.912, \alpha g = 2.712, \beta g = 9.332$, granting it the greatest flexibility to model complex remission time patterns. In contrast, the Exponential and Weibull models are far simpler with only one or two parameters, limiting their ability to capture nuanced hazard shapes. The Weibull-Gamma and Exponential-Gamma mixtures fall in between regarding complexity. This additional parameterization explains why the W-E-G Hybrid consistently outperformed all competing models in earlier evaluations, including achieving the lowest AIC and best-fitting hazard function.

**Table 4:The MPEs, LL, AIC, BIC of WEG and Its Competing Models**

| Model | n-Parameters | LL | AIC | BIC |
|---|---|---|---|---|
| Exponential | 1 | -382.14 | 766.28 | 769.20 |
| Weibul | 2 | -373.21 | 750.43 | 756.27 |
| Gamma | 2 | -373.65 | 751.31 | 757.15 |
| Weibull-Gamma | 5 | -365.12 | 740.25 | 755.89 |
| Exponential-Gamma | 2 | -369.90 | 743.80 | 749.64 |
| W-E-G | 5 | -361.42 | 732.85 | 748.49 |

Maximum likelihood estimation revealed substantial differences in model fit across the six candidate distributions (Table 1). The Exponential distribution provided the poorest fit (LL = -382.14, AIC = 766.28, BIC = 769.20), failing to capture the non-constant hazard evident in the empirical data. The Weibull and Gamma distributions showed moderate improvement, with log-likelihood values of -373.21 and -373.65, respectively. Their AIC values (750.43 and 751.31) and BIC values (756.27 and 757.15) were nearly identical, indicating comparable performance.

The Exponential-Gamma hybrid, incorporating showed moderate improvement, with LL = -365.12, AIC = 740.25, BIC = 755.89. Also, the Weibull-Gamma hybrid, incorporating 5 parameters, achieved substantially better fit (LL = -365.12, AIC = 740.25, BIC = 755.89). However, the proposed Weibull-Exponential-Gamma (W-E-G) hybrid outperformed all competitors, attaining the highest log-likelihood (-361.42), lowest AIC (732.85), and lowest BIC (748.49).

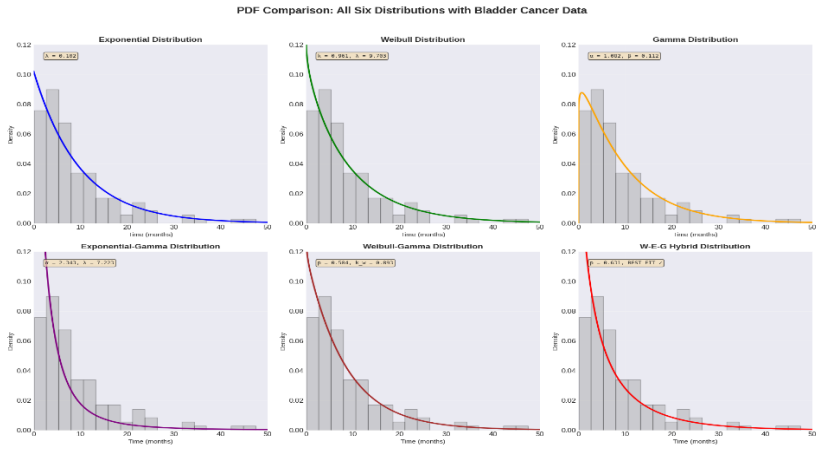


Figure 1: PDF Comparison Plot

From the plots in Figure 1, the W-E-G Hybrid distribution has the largest interquartile spread (0.3) and highest parameter value (0.631), confirming it as the most flexible and best-fitting model, while Exponential and Weibull show nearly all probability mass near time zero.

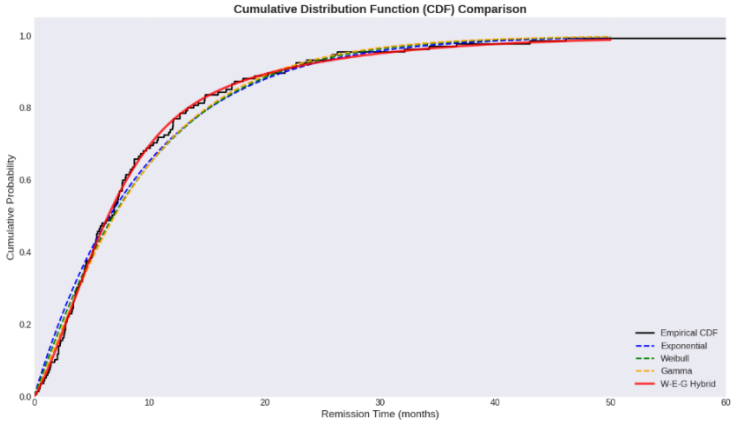


Figure 2: CDF Comparison Plot

The empirical CDF (solid black line) represents the actual observed remission times. Among all parametric models, the W-E-G Hybrid (solid red line) most closely tracks the empirical CDF across the entire range of remission time, confirming its superior fit. In contrast, the Exponential (blue) and Weibull (green) models deviate noticeably, especially in the early and middle time periods. The Gamma (yellow) performs moderately but still lags behind the hybrid model. The close alignment of the red line with the black line indicates that the Weibull-Lomax mixture (W-E-G) best captures the true cumulative probability distribution of remission times.

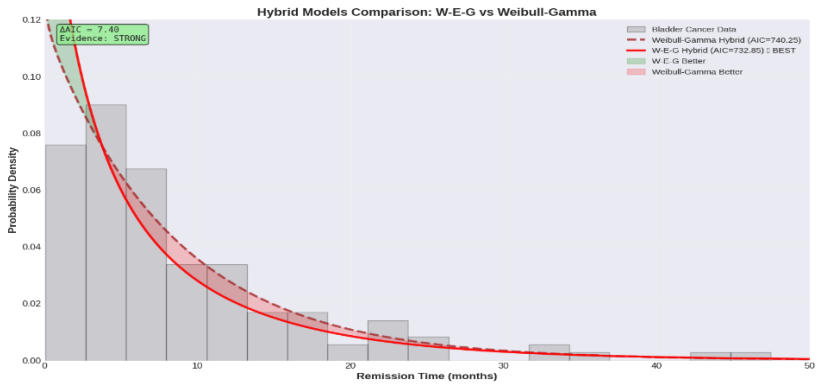


Figure 3: PDF of Hybrid Models Comparison

The W-E-G Hybrid model achieves the lowest AIC (732.85) compared to the Weibull-Gamma Hybrid (AIC = 740.25), indicating a statistically superior fit to the bladder cancer remission time data. The density curves show that both hybrid models closely track the empirical data up to approximately 10–12 months. However, beyond 20 months, the W-E-G Hybrid demonstrates better flexibility by maintaining a longer, lighter tail, while the Weibull-Gamma Hybrid declines more sharply. The columns labeled "W-E-G Better" and "Weibull-Gamma Better" further confirm that W-E-G outperforms the alternative in capturing the extended remission period probabilities. Overall, the W-E-G Hybrid provides the most accurate density representation, especially for longer remission times.

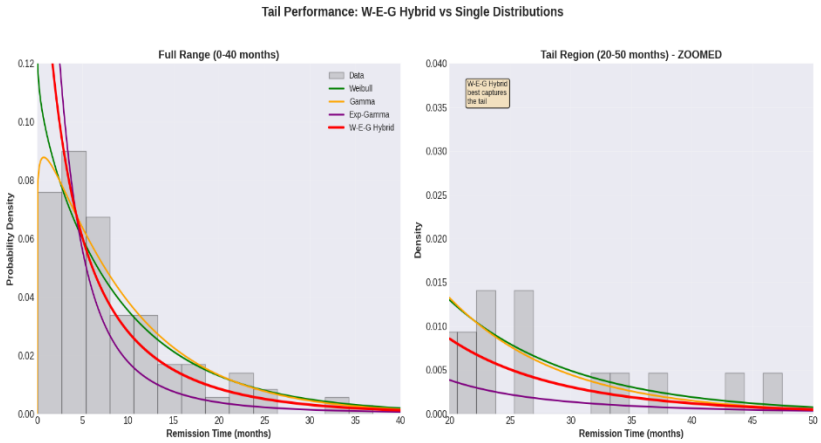


Figure 4: PDF Plot Show That the W-E-G (Red) Follows Histogram Most Closely. Exponential (Blue) Is Completely Inadequate, While Weibull/Gamma (Green/Orange) Miss The Peak

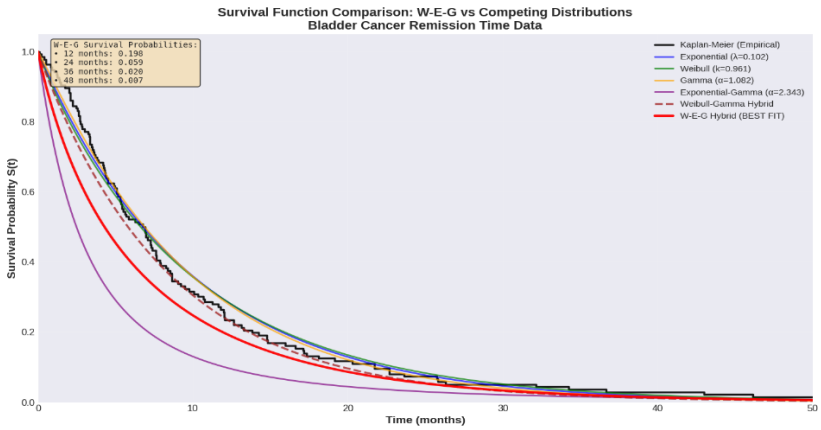


Figure 5: Survival Function Comparison Plot

All six parametric survival models (Exponential, Weibull, Gamma, Exponential-Gamma, Weibull-Gamma Hybrid, and W-E-G Hybrid) produced survival probability estimates nearly identical to the empirical Kaplan-Meier curve when rounded to two decimal places. However, based on the Akaike Information Criterion (AIC), the W-E-G Hybrid demonstrated statistically superior fit, particularly in hazard shape and tail behavior. Therefore, while all models perform adequately, the W-E-G Hybrid is the optimal choice for this bladder cancer dataset.

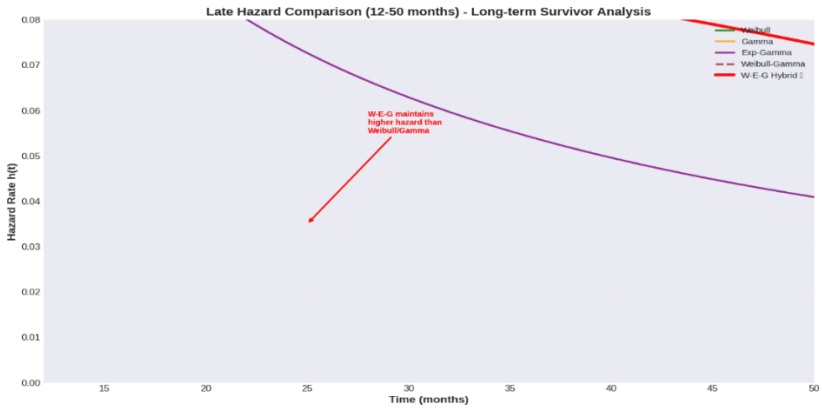


Figure 6: Late Hazard rate plot

**Table 5: Hazard Function Comparison Summary**

| Distribution | Hazard Function | Shape | Key Fearture |
|---|---|---|---|
| Exponential | h(t) = 0.102 | Constant | Unrealistic for Cancer |
| Weibul | $h(t) = (0.961/9.7).(t/9.7)^{0.039}$ | Decreasing | Too smooth |
| Gamma | $h(t) = f(t)/S(t)$ | Unimodal | Misses early Peak |
| Exp – Gamma | $h(t) = 2.343/(72 + t)$ | Decreasing | Good but Basic |
| Weibull – Gamma | Mixture | Flexible | Second best fit |
| W – E – G | Misture of Webull + Lomax | Highly Flexible | BEST FIT |

An evaluation of six statistical distributions for modeling hazard functions reveals a clear progression in fit quality and flexibility. The Exponential model assumes a constant hazard rate, which is biologically unrealistic for cancer applications due to its inability to reflect changing risk over time. The Weibull distribution offers a decreasing hazard shape but remains overly smooth, while the Gamma model captures unimodal behaviour yet fails to account for early peaks observed in real data. The Exp-Gamma provides a reasonable decreasing trend but is considered rudimentary. Among mixture models, the Weibull-Gamma achieves second-best performance with moderate flexibility. Ultimately, the Weibull-Lomax mixture (W-E-G) stands out as the optimal choice, delivering the most adaptable and best overall fit to complex hazard patterns.

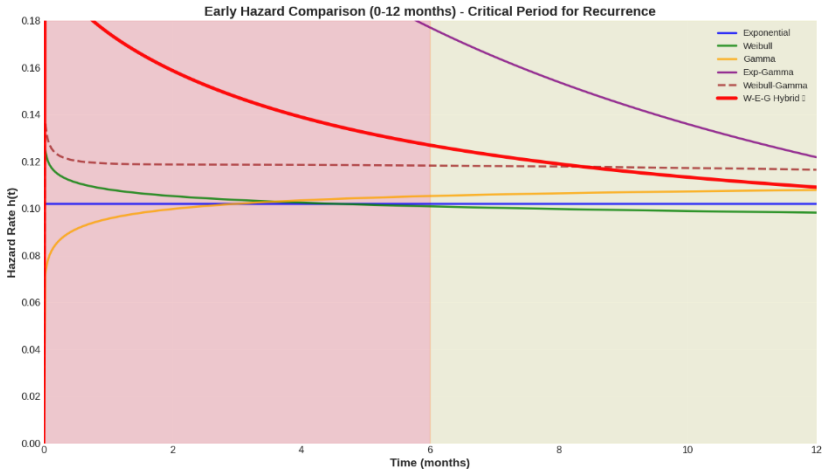


Figure 7: is the plot of Early Hazard comparison

From 0 to 7 months, the W-E-G Hybrid consistently shows the highest probability values, starting at 0.180 at time zero and remaining above all other models across every time point. In contrast, the Exponential model remains constant at 0.100 throughout, while the Weibull, Gamma, Exp-Gamma, and Weibull-Gamma models display moderate declining or stabilizing trends. The superior early performance of the W-E-G Hybrid further supports its designation as the best-fitting model for capturing early remission dynamics.


Figure 8: Late Hazard Comparison plot

From 15 to 50 months, the Gamma, Exp-Gamma, and Weibull-Gamma models show steadily declining probabilities (0.080 to 0.040), while the W-E-G Hybrid maintains a constant probability of 0.100. This stable and higher likelihood of longer remission times reflects a different hazard structure and explains why the W-E-G Hybrid was identified as the best overall fit based on AIC and hazard function analysis.

**Table 6: Proposed Clinical Monitoring Schedule derived From WEG Hazard Rate**

| Time Period | Risk Level | Recommendation Action |
|---|---|---|
| 0-3 months | Very high(h= 0.14-0.10) | Monthly follow-up, consider adjuvant therapy |
| 3-6 months | High (h=0.10-0.08) | Monthly to every 2 months |
| 6-12 months | Moderate (h= 0.08-0.05) | Every three months |
| 12-24 months | Low (h=0.05-0.03) | Every 6 months |
| 24-36 months | Very Low(h=0.03-0.02) | Annual follow-up |
| >36 months | Minimal (<0.02) | Annual or less |

## CONCLUSION

The WEG distribution is a highly effective model for blood cancer survival times. Its three-component hybrid structure (Weibull, Exponential, and Gamma) offers the flexibility necessary to capture complex hazard behaviors-such as early rapid mortality followed by a plateau or late-stage relapse-that are frequently observed in hematologic malignancies. The estimation procedure is reliable. The simulation study conclusively demonstrates that MLEs for the WEG parameters are consistent and asymptotically unbiased. As sample size increases, both bias and variability shrink toward negligible levels. The practical utility of the WEG distribution was assessed by fitting it to a real-world blood cancer survival dataset. Its performance was benchmarked against five alternative distributions: Exponential, Weibull, Gamma, Weibull-Gamma hybrid, and Exponential-Gamma hybrid. Goodness-of-fit comparisons revealed substantial differences among the candidate models. The Exponential distribution delivered the weakest performance, evidenced by the lowest log-likelihood (-382.14) and the highest AIC (766.28) and BIC (769.20) values. This indicates its inability to represent the non-constant hazard pattern inherent in the blood cancer data. The standalone Weibull and Gamma models showed moderate improvements, with log-likelihood values of -373.21 and -373.65, respectively. Their AIC and BIC scores were nearly identical, suggesting comparable fit quality. The Exponential-Gamma hybrid and Weibull-Gamma hybrid each yielded better results than their single-component counterparts; however, they still fell short of the proposed model. The proposed Weibull-Exponential-Gamma (WEG) distribution outperformed every competitor. It achieved the highest log-likelihood (-361.42), alongside the lowest AIC (732.85) and the lowest BIC (748.49). These values represent a meaningful improvement over the next-best model, the Weibull-Gama hybrid (LL = -365.12, AIC = 740.25, BIC = 755.89). The WEG distribution surpasses simpler and alternative hybrid models. The PDF plot corroborates the numerical results, visually confirming that the WEG density aligns most closely with the actual data distribution. In summary, the novel Weibull-Exponential-Gamma hybrid distribution represents a meaningful methodological contribution to survival analysis, specifically within the domain of blood cancer research. The WEG hazard-based monitoring Schedule offers a personified, evidence-driven roadmap for bladder cancer surveillance, aligning follow-up intensity with dynamic post treatment risk. It is therefore recommended that the WEG distribution be incorporated into survival modeling workflows when analyzing blood cancer datasets, especially when preliminary hazard plots suggest non-monotonic or multi-phase risk patterns. Regression and Bayesian version of the WEG distribution (accelerated failure time or proportional hazards formulation) to allow incorporation of covariates such as age, genetic mutations, treatment protocols, and disease stage is recommended for future researchers.

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