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A Continuous and Optimized Three-Step Third-Derivative Hybrid Block Method for Third-Order IVPs Based On Volterra Integral Equations of the Second Kind

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ABSTRACT
This paper presents an optimization of two hybrid points within a three-step method based on Volterra integral equations of the second kind for third-order initial value problems. The scheme is formulated by combining power series expansion with exponentially fitted functions as basic functions to construct a continuous hybrid formulation. This continuous scheme is then discretized into a block method using a three-step framework with appropriately selected hybrid points. The two off-grid points are optimized by equating the leading terms of the truncation error to zero, and the resulting error equations are used to determine the approximate values of the unknown parameters. The theoretical properties of the proposed method are investigated. The order of accuracy and associated error constant are derived, and the method is shown to satisfy the conditions of consistency and zero-stability, thereby establishing convergence. Stability analysis further confirms the robustness of the scheme. Numerical experiments conducted for solving stiff problems demonstrate that the proposed method provides highly accurate approximations with improved computational efficiency compared with existing block and hybrid methods. The results indicate that the method constitutes a reliable and efficient computational tool for solving stiff problems encountered in applied mathematics and scientific computing.

Keywords: Convergence, Hybrid Block Method, Numerical Stability, Optimized Numerical Schemes, Volterra Integral Equations

INTRODUCTION
Mathematical models are frequently formulated as second-kind Volterra integral equations. However, these equations can rarely be solved analytically, particularly when the kernel or forcing function is nonlinear or highly complicated. Consequently, obtaining approximate solutions for the equations discussed by V. Nuriyeva requires the use of efficient and accurate numerical methods. Over the years, numerous numerical techniques have been developed for solving second-kind Volterra integral equations involving integral operators. Among the earliest approaches were the classical quadrature and collocation methods, in which the integral term is approximated through numerical integration or polynomial interpolation procedures. Collocation techniques based on Hermite or Lagrange polynomials have proven successful in transforming integral equations into systems of algebraic equations that are easier to solve without sacrificing computational accuracy. High-order collocation and barycentric interpolation schemes, as reported by E. John et al. (2024) and Ahmed Abdulkareem Hadi (2023), exhibit improved convergence rates and enhanced accuracy, especially when suitable transformation functions and interpolation nodes are employed. Although these methods provide effective approximations, they may encounter limitations related to computational complexity and stiffness in solving certain classes of problems, as observed by F. O. Obarhua (2023). Comparative studies have further demonstrated that hybrid block schemes can achieve higher-order accuracy with lower error constants than traditional single-step and classical multistep methods, as noted by I. Mohammed Dibal and S. H. Yeak (2025). Several researchers have extended hybrid block methods for solving Volterra integral equations. In particular, three-step implicit hybrid block methods employing interpolation and collocation techniques have been developed for second-kind Volterra integral equations, demonstrating improved stability and accuracy over existing numerical methods. Similarly, exponentially fitted hybrid block approaches were investigated by D. Raymond et al. (2023) and T. Y. Kyagya (2020). Nevertheless, the lack of optimization of two hybrid points within three-step methods remains a notable gap in the numerical analysis literature. The proposed method in this study seeks to improve accuracy, stability, and computational efficiency by incorporating third-derivative information into an optimized hybrid block framework.

This study considers a third-order Initial Value Problem (IVP) represented as a hybrid Volterra Integral Equation (VIE) of the form

$$y'''(x) = f'''(x) + \int_{\alpha(x)}^{\beta(x)} K\left(x, s, y(s)y'(s), y''(s)\right)ds, x \in [x_0, X] \quad (1)$$

Where $y'''(x)$ is the unknown function, $f'''(x)$ is the known function, $k(x,s)$ is the kernel or the nucleus of the integral equation that define the relationship between the unknown function and the integral, that is a function of two variable(also known), $y(s)$ is the solution function evaluated at s, $y'(s)$ is the slope of the solution at s and $y''(s)$ is the curvature of the solution at s. ds is the integration variable. It is to be noted that the limit of integration $\alpha(x)$ and $\beta(x)$ may be both variables, constant or mixed.

It follows that solving equation (1) is equivalent to solving the third-order ordinary differential equation (ODE) initial value problem given by

$$y'''(x) = f'''(x) + \varphi\left(x, y(x), y'(x), y''(x)\right), y(x_0) = f(x_0), y'(x_0) = f'(x_0), y''(x_0) = f''(x_0), \quad (2)$$

Hence, both the integral equation (1) and the initial value problem (2) can be solved using a unified numerical approach. To address the resolution of (1) and (2) as equivalence, we have devised a method utilizing a combination of power series and an exponentially fitted function as the approximate solution, with the algorithm structured as follows:

$$\varphi'''(x) = \phi'''(x) + \sum_{j=0}^{s} \mu_i x^j + \sum_{j=t}^{q} \psi_j e^{x^j} \quad (3)$$

Serve as the approximate solution to (1) and (2). where μ_i and $\psi_i$are the coefficients to be determined, $x^j$is the polynomial terms account for smooth components of the solution, $e^{x^j}$is the exponential terms provide exponential fitting for stiff behavior. Contingent upon the following stipulation.

$$\eta(x) = \varphi'''(x) - \phi'''(x) \tag{4}$$

MATERIALS AND METHODS

Derivation of the Methods

To develop this approach, two off-grid points are introduced between the interpolation points, which are u and$v$ respectively. For the three-step scheme, the value of (k) is taken as three; consequently, the collocation approach is by using the combination of power series and exponential function as the approximate solution of the form

$$\varphi'''(x) = \phi'''(x) + \sum_{j=0}^{3} \mu_i x^j + \sum_{j=1}^{5} \psi_j e^{x^j} \tag{5}$$

By taking the first, second, and third derivatives of equation (5), the following expressions are obtained

$$\varphi'(x) - \phi'(x) = \sum_{j=0}^{3} j x^{j-1} \mu_j(x) + \sum_{j=1}^{5} j x^{j-1} \psi_j(x) e^{x^j} \tag{6}$$

$$\varphi''(x) - \phi''(x) = \sum_{j=0}^{3} j(j-1) x^{j-2} \mu_j(x) + \sum_{j=1}^{5} \psi_j j\left[(j-1)x^{j-2} + j x^{2(j-2)}\right] e^{x^j} \tag{7}$$

$$\varphi'''(x) - \phi'''(x) = \sum_{j=0}^{3} j(j-1)(j-2) x^{j-3} \mu_j + \sum_{j=1}^{5} j x^{j-3} \psi_j(x) e^{x^j}\left(j^2 x^{2j} - 3j - 3j x^j + 3j^2 x^j + j^2 + 2\right) \tag{8}$$

Equations (5)–(7) are then interpolated at the selected points, while equation (8) is collocated at all grid points. As a result, equation (8) generates a system of nonlinear equations.

$$MD = V \tag{9}$$

$$\begin{bmatrix}
12 & 12x_n & 9x_n^2 & 7x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}{x_n}^5 & \frac{1}{120}{x_n}^6 & \frac{1}{840}{x_n}^7 & \frac{1}{6720}{x_n}^8 \\
0 & 12 & 18x_n & 21x_n^2 & x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}x_n^6 & \frac{1}{840}{x_n}^7 \\
0 & 0 & 18 & 42x_n & 3x_n^2 & 3x_n^3 & \frac{1}{4}x_n^4 & \frac{1}{20}x_n^5 & \frac{1}{120}{x_n}^6 \\
0 & 0 & 0 & 42 & 6x_n & 3x_n^2 & {x^3}_n & \frac{1}{4}x_n^4 & \frac{1}{20}{x_n}^5 \\
0 & 0 & 0 & 42 & 6(x_n + \frac{4979}{20000}h) & 3(x_n + \frac{4979}{20000}h)^2 & (x_n + \frac{4979}{20000}h)^3 & \frac{1}{4}(x_n + \frac{4979}{20000}h)^4 & \frac{1}{20}(x_n + \frac{4979}{20000}h)^5 \\
0 & 0 & 0 & 42 & 6(x_n + \frac{68919}{100000}h) & 3(x_n + \frac{68919}{100000}h)^2 & (x_n + \frac{68919}{100000}h)^3 & \frac{1}{4}(x_n + \frac{68919}{100000}h)^4 & \frac{1}{20}(x_n + \frac{68919}{100000}h)^5 \\
0 & 0 & 0 & 42 & 6(x_n + h) & 3(x_n + h)^2 & (x_n + h)^3 & \frac{1}{4}(x_n + h)^4 & \frac{1}{20}(x_n + h)^5 \\
0 & 0 & 0 & 42 & 6(x_n + 2h) & 3(x_n + 2h)^2 & (x_n + 2h)^3 & \frac{1}{4}(x_n + 2h)^4 & \frac{1}{20}(x_n + 2h)^5 \\
0 & 0 & 0 & 42 & 6(x_n + 3h) & 3(x_n + 3h)^2 & (x_n + 3h)^3 & \frac{1}{4}(x_n + 3h)^4 & \frac{1}{20}(x_n + 3h)^5
\end{bmatrix}
\begin{bmatrix}
\mu_0 \\ \mu_0' \\ \mu_0'' \\ \psi_0 \\ \psi_{\frac{4979}{20000}} \\ \psi_{\frac{68919}{100000}} \\ \psi_1 \\ \psi_2 \\ \psi_3
\end{bmatrix}
=
\begin{bmatrix}
(\varphi_0 - \phi_0) \\ (\varphi_0' - \phi_0') \\ (\varphi_0'' - \phi_0'') \\ g_n \\ g_{n+\frac{4979}{20000}} \\ g_{n+\frac{68919}{100000}} \\ g_{n+1} \\ g_{n+2} \\ g_{n+3}
\end{bmatrix}$$

Applying the Gaussian elimination technique to equation (9) produces the coefficients of μ_j , ψ_j , $j = 0(1)8$. These coefficients are subsequently substituted into equation (5) to obtain the implicit continuous optimized hybrid Volterra integral equation of the second kind in the following form

$$p\big((\varphi_n + \xi h) - (\phi_n + \xi h)\big) = \mu_0(\varphi_n - \phi_n) + h\mu_1(\varphi'_n - \phi'_n) + h^2\mu_2(\varphi''_n - \phi''_n) + h^3[\psi_0 g_n + \psi_u g_{n+u} + \psi_v g_{n+v} + \psi_1 g_{n+1} + \psi_2 g_{n+2} + \psi_3 g_{n+3}] \tag{10}$$

Differentiating equation (10) successively with respect to the independent variable yields the following expressions

$$hp'\big((\varphi_n + qh) - (\phi_n + qh)\big) = h\mu_0(\varphi'_n - \phi'_n) + h^2\mu_1(\varphi''_n - \phi''_n) + h^3[\psi_0 g_n + \psi_u g_{n+u} + \psi_v g_{n+v} + \psi_1 g_{n+1} + \psi_2 g_{n+2} + \psi_3 g_{n+3}] \tag{11}$$

$$h^2 p''\big((\varphi_n + qh) - (\phi_n + qh)\big) = h^2\mu_0(\varphi''_n - \phi''_n) + h^3\begin{bmatrix}\psi_0 g_n + \psi_u g_{n+u} + \psi_v g_{n+v} + \psi_1 g_{n+1} \\ + \psi_2 g_{n+2} + \psi_3 g_{n+3}\end{bmatrix} \tag{12}$$

Substituting $\xi = 1$ into equation (10) gives an approximate Volterra integral equation for the solution of equation (5) at the specified pointt_{n+1} , yielding

$$p\big((\varphi_n + h) - (\phi_n + h)\big) = (\varphi_n - \phi_n) + h\mu_1(\varphi'_n - \phi'_n) + \frac{1}{2}h^2\mu_2(\varphi''_n - \phi''_n) + h^3\begin{bmatrix}
\frac{1}{10080}\left(\frac{-188u - 188v + 1064uv + 57}{uv}\right)g_n + \frac{1}{1680}\left(\frac{188v - 88v - 57}{u(u-3)(u-1)(u-2)(u-v)}\right)g_{n+u} \\
-\frac{1}{1680}\left(\frac{188u - 57}{v(v-3)(v-1)(v-2)(u-v)}\right)g_{n+v} + \frac{1}{3360}\left(\frac{-106u - 106v + 294uv + 49}{(v-1)(u-1)}\right)g_{n+1} \\
-\frac{1}{3360}\left(\frac{-36u - 36v + 112uv + 15}{(v-2)(u-2)}\right)g_{n+2} + \frac{1}{10080}\left(\frac{-22u - 22v + 70uv + 9}{(v-3)(u-3)}\right)g_{n+3}
\end{bmatrix} \tag{13}$$

Similarly, substituting $\xi = 1$into equation (11) produces an approximate Volterra integral equation for the solution of equation (5) att'_{n+1}, yielding

$$hp'\big((\varphi_n+h)-(\phi_n+h)\big)=h(\varphi'_n-\phi'_n)+h^2\mu_2(\varphi''_n-\phi''_n)+h^3+$$
$$h^3\begin{bmatrix}\frac{1}{2520}\left(\frac{-147u-147v+679uv+53}{uv}\right)g_n+\frac{1}{420}\left(\frac{147v-53}{u(u-3)(u-1)(u-2)(u-v)}\right)g_{n+u}\\-\frac{1}{420}\left(\frac{147u-53}{v(v-3)(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{840}\left(\frac{-119u-119v+266uv+66}{(v-1)(u-1)}\right)g_{n+1}\\-\frac{1}{840}\left(\frac{-35u-35v+91uv+17}{(v-2)(u-2)}\right)g_{n+2}+\frac{1}{2520}\left(\frac{-21u-21v+56uv+10}{(v-3)(u-3)}\right)g_{n+3}\end{bmatrix}\qquad(14)$$

Finally, substituting $\xi=1$ into equation (12) yields an approximate Volterra integral equation for the solution of equation (5) att''_{n+1}, resulting in

$$h^2p^{''}\,{}^{((\varphi_n+h)-(\phi_n+h))}=h^2(\varphi''_n-\phi''_n)$$
$$+h^3\begin{bmatrix}\frac{1}{360}\left(\frac{-38u-38v+135uv+17}{uv}\right)g_n+\frac{1}{60}\left(\frac{38v-17}{u(u-3)(u-1)(u-2)(u-v)}\right)g_{n+u}\\-\frac{1}{60}\left(\frac{38u-17}{v(v-3)(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{120}\left(\frac{-57u-57v+95uv+40}{(v-1)(u-1)}\right)g_{n+1}\\-\frac{1}{120}\left(\frac{-12u-12v+25uv+7}{(v-2)(u-2)}\right)g_{n+2}+\frac{1}{360}\left(\frac{-7u-7v+15uv+4}{(v-3)(u-3)}\right)g_{n+3}\end{bmatrix}\qquad(15)$$

Expanding equations (13)–(15) in Taylor series about the specified pointt_n, we obtain

$$\begin{bmatrix}\sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j-(\varphi_n-\phi_n)-h(\varphi'_n-\phi'_n)-\frac{1}{2}h^2(\varphi''_n-\phi''_n)-\sum_{j=0}^{2}\frac{h^{j+2}}{j!}+h^3+h^3\begin{bmatrix}\frac{1}{10080}\left(\frac{-188u-188v+1064uv+57}{uv}\right)g_n+\frac{1}{1680}\left(\frac{188v-88v-57}{u(u-3)(u-1)(u-2)(u-v)}\right)g_{n+u}\\-\frac{1}{1680}\left(\frac{188u-57}{v(v-3)(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{3360}\left(\frac{-106u-106v+294uv+49}{(v-1)(u-1)}\right)g_{n+1}\\-\frac{1}{3360}\left(\frac{-36u-36v+112uv+15}{(v-2)(u-2)}\right)g_{n+2}+\frac{1}{10080}\left(\frac{-22u-22v+70uv+9}{(v-3)(u-3)}\right)g_{n+3}\end{bmatrix}\\ \sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j-h(\varphi'_n-\phi'_n)-\frac{1}{2}h^2(\varphi''_n-\phi''_n)-\sum_{j=0}^{2}\frac{h^{j+2}}{j!}++h^3\begin{bmatrix}\frac{1}{2520}\left(\frac{-147u-147v+679uv+53}{uv}\right)g_n+\frac{1}{420}\left(\frac{147v-53}{u(u-3)(u-1)(u-2)(u-v)}\right)g_{n+u}\\-\frac{1}{420}\left(\frac{147u-53}{v(v-3)(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{840}\left(\frac{-119u-119v+266uv+66}{(v-1)(u-1)}\right)g_{n+1}\\-\frac{1}{840}\left(\frac{-35u-35v+91uv+17}{(v-2)(u-2)}\right)g_{n+2}+\frac{1}{2520}\left(\frac{-21u-21v+56uv+10}{(v-3)(u-3)}\right)g_{n+3}\end{bmatrix}\\ \sum_{j=0}^{2}\frac{(h^3)}{j!}(\varphi_n-\phi_n)^j-h^2(\varphi''_n-\phi''_n)-\sum_{j=0}^{2}\frac{h^{j+2}}{j!}++h^3+h^3\begin{bmatrix}\frac{1}{360}\left(\frac{-38u-38v+135uv+17}{uv}\right)g_n+\frac{1}{60}\left(\frac{38v-17}{u(u-3)(u-1)(u-2)(u-v)}\right)g_{n+u}\\-\frac{1}{60}\left(\frac{38u-17}{v(v-3)(v-1)(v-2)(u-v)}\right)g_{n+v}+\frac{1}{120}\left(\frac{-57u-57v+95uv+40}{(v-1)(u-1)}\right)g_{n+1}\\-\frac{1}{120}\left(\frac{-12u-12v+25uv+7}{(v-2)(u-2)}\right)g_{n+2}+\frac{1}{360}\left(\frac{-7u-7v+15uv+4}{(v-3)(u-3)}\right)g_{n+3}\end{bmatrix}\end{bmatrix}=0\qquad(16)$$

Algorithm for the Derivation of the Optimized Method

The algorithm for the implementation of the optimize three-step method is given by the following steps; **Step 1**: Expanding (16) using the Taylor series to obtain the corresponding Local truncation error given as

$L[\varphi'(t_{n+1});h]=-\frac{1}{302400}(-171u-171v+564uv+68)+o(h^9)$ (17)

$L[\varphi''(t_{n+1});h]=-\frac{1}{50400}(-106u-106v+294uv+49)+o(h^8)$ (18)

$L[\varphi'''(t_{n+1});h]=-\frac{1}{25200}(-119u-119v+266uv+66)+o(h^7)$ (19)

Step 2: Equate the principal term of the local truncation errors in (18) and (19) to zero, to obtain

$v=\frac{91}{194}+\frac{1}{1358}\sqrt{89355}$ or $\frac{68919}{100000}, u=\frac{91}{194}-\frac{1}{1358}\sqrt{89355}$ or $\frac{4979}{20000}$ (20)

Step 3: Substitute the values of $u=\frac{4979}{20000}$and $v=\frac{68919}{100000}$ as obtain in step 2 into (9).

Step 4: Using Gaussian elimination method on (9) in step 3, gives the coefficients of

$\mu_0,\mu_0^{'},\mu_0^{''},\psi_0,\psi_{\frac{4979}{20000}},\psi_{\frac{68919}{100000}},\psi_1,\psi_2,\psi_3$

Step 5: The values of step 4 are subsequently swapped into (5) to get the implicit continuous optimized hybrid Volterra integral equation of the second kinds of the form

$$p\big((\varphi_n+\xi h)-(\phi_n+\xi h)\big)=\mu_0(\varphi_n-\phi_n)+h\mu'_0(\varphi'_n-\phi'_n)+h^2\mu''_0(\varphi''_n-\phi''_n)$$
$$+h^3\left[\psi_0 g_n+\psi_{\frac{4979}{20000}}g_{n+\frac{4979}{20000}}+\psi_{\frac{68919}{100000}}g_{n+\frac{68919}{100000}}+\psi_1 g_{n+1}+\psi_2 g_{n+2}+\psi_3 g_{n+3}\right],\xi=\left\{\frac{4979}{20000},\frac{68919}{100000},1,2,3\right\}\qquad(21)$$

Calculating the first and second derivatives of (21) and assessing their values at the specified places $t_{n+\psi}=t_n+\psi h,\psi=\left\{0,\frac{4979}{20000},\frac{68919}{100000},1,2,3\right\}$ substituting into (22), we derive the discrete.

$$A^{(0)}B_m^{[1]}=A^{(1)}C_m^{[0]}+\sum_{i=0}D^{[i]}\,G_m^{[i]}+\sum_{j=u,v,1,2,3}D^{[j]}\,G_m^{[j]}\qquad(22)$$

Where $G_m{}^{(i)}=\begin{bmatrix}g_{n+\frac{4979}{20000}}\\g_{n+\frac{63919}{100000}}\\g_{n+1}\\g_{n+2}\\g_{n+3}\end{bmatrix}$

$$
A^{(1)} = \begin{bmatrix}
1 & \frac{4979h}{20000} & \frac{24790441h^2}{800000000} \\
1 & \frac{68919h}{100000} & \frac{4749828561h^2}{20000000000} \\
1 & h & \frac{h^2}{2} \\
1 & 2h & 2h^2 \\
1 & 3h & \frac{9h^2}{2} \\
0 & 1 & \frac{4979h}{20000} \\
0 & 1 & \frac{68919h}{100000} \\
0 & 1 & h \\
0 & 1 & 2h \\
0 & 1 & 3h \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} C_m^{[0]}
$$

$$
= \begin{bmatrix}
(\varphi_n - \phi_n) \\
(\varphi'_n - \phi'_n) \\
(\varphi''_n - \phi''_n)
\end{bmatrix}, D^{(i)} = \begin{bmatrix}
\frac{27963028920760100672214116337691h^3}{44461025280000000000000000000000000} \\
\frac{480908536710325931887847736089779683h^3}{27882000000000000000000000000000000000} \\
\frac{201577000000000000000h^3}{1850436953491820099133} \\
\frac{1261264000000000000000h^3}{1850436953491820099133} \\
\frac{1185813000000000000000h^3}{4266666951} \\
\frac{9380887697223718473372715153h^2}{555762816000000000000000000000} \\
\frac{3588316695089378861753800945021h^2}{69706000000000000000000000000000} \\
\frac{63184128979h^2}{864732206520} \\
\frac{4453189396h^2}{108091525815} \\
\frac{9824560791h^2}{32027118760} \\
\frac{125646335438597156480291h}{1323244800000000000000000} \\
\frac{6564392840505104758467257h}{99580000000000000000000000} \\
\frac{601753309h}{8235544824} \\
-\frac{521068897h}{3088329309} \\
\frac{60971977h}{70389272}
\end{bmatrix}, G_m^{[i]} = [g_n]
$$

$$A^{(0)}=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\end{bmatrix},\ B_m^{[1]}=\begin{bmatrix}\left(\varphi_{n+\frac{4979}{20000}}-\phi_{n+\frac{4979}{20000}}\right)\\\left(\varphi_{n+\frac{68919}{100000}}-\phi_{n+\frac{68919}{100000}}\right)\\(\varphi_{n+1}-\phi_{n+1})\\(\varphi_{n+2}-\phi_{n+2})\\(\varphi_{n+3}-\phi_{n+3})\\\left(\varphi'_{n+\frac{4979}{20000}}-\phi'_{n+\frac{4979}{20000}}\right)\\\left(\varphi'_{n+\frac{68919}{100000}}-\phi'_{n+\frac{68919}{100000}}\right)\\(\varphi'_{n+1}-\phi'_{n+1})\\(\varphi'_{n+2}-\phi'_{n+2})\\(\varphi'_{n+3}-\phi'_{n+3})\\\left(\varphi''_{n+\frac{4979}{20000}}-\phi''_{n+\frac{4979}{20000}}\right)\\\left(\varphi''_{n+\frac{68919}{100000}}-\phi''_{n+\frac{68919}{100000}}\right)\\(\varphi''_{n+1}-\phi''_{n+1})\\(\varphi''_{n+2}-\phi''_{n+2})\\(\varphi''_{n+3}-\phi''_{n+3})\end{bmatrix}$$

$$G^{(i)}=\begin{bmatrix}
\text{[illegible]} & -\text{[illegible]} & \text{[illegible]} & -\text{[illegible]} & \text{[illegible]}\\
\text{[illegible]} & -\text{[illegible]} & \text{[illegible]} & -\text{[illegible]} & \text{[illegible]}\\
\frac{2157000000000h}{1849825492099155} & \frac{15294530000000000h}{748352301961978287} & \frac{-57h}{4449510} & -\frac{14599h}{31841300} & \frac{74179h}{1386201268}\\
\frac{12124000000000h}{1849825492099155} & \frac{12953000000000000h}{748352301961978287} & \frac{1945162h}{5467986} & \frac{8098279h}{42011085} & -\frac{386469h}{57184454}\\
\frac{11583000000000h}{6612178066911} & \frac{2512500000000h}{367290426153653} & \frac{5457148h}{19496640} & \frac{1529940h}{31841397} & \frac{1319807h}{25937520}\\
\text{[illegible]} & -\text{[illegible]} & \text{[illegible]} & -\text{[illegible]} & \text{[illegible]}\\
\text{[illegible]} & \text{[illegible]} & -\text{[illegible]} & -\text{[illegible]} & \text{[illegible]}\\
\frac{4818000000000h}{165920842833817} & \frac{102718500000000h}{748352301961978287} & -\frac{176h}{19684940} & \frac{147894h}{12834568} & -\frac{23019h}{40508256}\\
\frac{15578400000000h}{165920842833817} & \frac{12795000000000000h}{748352301961978287} & \frac{5701859h}{40211085} & \frac{4928128h}{42011085} & -\frac{1159518h}{40508256}\\
\frac{3429000000000h}{6612178066911} & \frac{25751500000000h}{27710254812678313} & \frac{4596171h}{3611230} & \frac{16104549h}{12834568} & \frac{3701890h}{52948280}\\
\text{[illegible]} & -\text{[illegible]} & \text{[illegible]} & -\text{[illegible]} & \text{[illegible]}\\
\text{[illegible]} & \text{[illegible]} & -\text{[illegible]} & \text{[illegible]} & -\text{[illegible]}\\
\frac{9152000000000h}{3796525834817} & \frac{15812500000000000h}{357624405713457} & \frac{1302499h}{11740424} & \frac{-126653h}{11740424} & \frac{9119h}{3940256}\\
\frac{2027400000000h}{3796525834817} & \frac{-12495000000000000h}{10710428853197001} & \frac{2179284h}{14051013} & \frac{4640751h}{13771813} & -\frac{894880h}{11428839}\\
\frac{-13200000000h}{9808015528948} & \frac{15832500000000h}{367290426153653} & \frac{-7262019h}{4449510} & \frac{8574183h}{5458544} & \frac{9989170h}{3940256}
\end{bmatrix}$$

$$\left(\varphi_{n+\frac{4979}{20000}}-\phi_{n+\frac{4979}{20000}}\right)=\left(\varphi_n-\phi_n\right)+\frac{4979h}{20000}\left(\varphi'_n-\phi'_n\right)+\frac{24790441h^2}{800000000}\left(\varphi''_n-\phi''_n\right)+\frac{27963028920760100672214116337691h^3}{44461025280000000000000000000000}g_n$$
$$+\frac{22724278631326319872923756389871h^3}{190283936571161255424000000000000}g_{n+\frac{4979}{20000}}-\frac{15283792274483048362648574906821917h^3}{12285043020627414184901143756800000000}g_{n+_{n+\frac{68919}{100000}}}+\frac{19423470562238024962685294861062791h^3}{11155026025600000000000000000000000}g_{n+1}$$
$$-\frac{27935605732760460878852405317588371h^3}{3290533264076800000000000000000000000}g_{n+2}+\frac{78755135338540325085182620222807h^3}{117175059772416000000000000000000000}g_{n+3}$$

$$\left(\varphi_{n+_{n+\frac{68919}{100000}}}-\phi_{n+_{n+\frac{68919}{100000}}}\right)=\left(\varphi_n-\phi_n\right)+\frac{68919h}{100000}\left(\varphi'_n-\phi'_n\right)+\frac{4749828561h^2}{20000000000}\left(\varphi''_n-\phi''_n\right)+\frac{48090853671032593188784773608977968h^3}{2788240000000000000000000000000000000}g_n$$
$$+\frac{13326004539225907358417334510154198161h^3}{3524641816174895426920000000000000000000}g_{n+\frac{4979}{20000}}-\frac{2745085501480774331714535714004983h^3}{15473881570096243951360000000000000}g_{n+\frac{68919}{100000}}$$
$$+\frac{38218368140767783411311626118391614244h^3}{2904594584000000000000000000000000000000}g_{n+1}-\frac{29176775207072094771893700426643835456h^3}{3672470160800000000000000000000000000000}g_{n+2}$$
$$+\frac{154651742764954258080154185212765746441h^3}{237333743752000000000000000000000000000000}g_{n+3}$$

$$\left(\varphi_{n+1}-\phi_{n+1}\right)=\left(\varphi_n-\phi_n\right)+h(\varphi'_n-\phi'_n)+\frac{1}{2}h^2\left(\varphi''_n-\phi''_n\right)+\frac{20157700000000000000000h^3}{1850436953491820099133}g_n+\frac{20157700000000000000000h^3}{1850436953491820099133}g_{n+\frac{4979}{20000}}$$
$$+\frac{159334375000000000000000h^3}{7498195203019661978089077}g_{n+\frac{68919}{100000}}-\frac{677h^3}{4149935120}g_{n+1}-\frac{18467969h^3}{321341139070}g_{n+2}+\frac{74217907h^3}{12816022162608}g_{n+3}$$

$$\left(\varphi_{n+2}-\phi_{n+2}\right)=\left(\varphi_n-\phi_n\right)+2h(\varphi'_n-\phi'_n)+2h^2\left(\varphi''_n-\phi''_n\right)+\frac{1261264000000000000000000h^3}{1850436953491820099133}g_n+\frac{1261264000000000000000000h^3}{1850436953491820099133}g_{n+\frac{4979}{20000}}$$
$$+\frac{126995000000000000000000000h^3}{7498195203019661978089077}g_{n+\frac{68919}{100000}}+\frac{1894630522h^3}{5446789845}g_{n+1}+\frac{8299081279h^3}{428011708605}g_{n+2}-\frac{386864598h^3}{572143846545}g_{n+3}$$

$$\left(\varphi_{n+3}-\phi_{n+3}\right)=\left(\varphi_n-\phi_n\right)+3h(\varphi'_n-\phi'_n)+\frac{9}{2}h^2\left(\varphi''_n-\phi''_n\right)+\frac{1185813000000000000000000h^3}{4266666951}g_n+\frac{1185813000000000000000000h^3}{616812317830606699711}g_{n+\frac{4979}{20000}}$$
$$+\frac{253125000000000000000000h^3}{396729904921675236935 93}g_{n+\frac{68919}{100000}}+\frac{54557115489h^3}{29049545840}g_{n+1}+\frac{15939099000h^3}{32134113907}g_{n+2}+\frac{17319581721h^3}{2373337437520}g_{n+3}$$

$$\left(\varphi'_{n+\frac{4979}{20000}}-\phi'_{n+\frac{4979}{20000}}\right)=(\varphi'_n-\phi'_n)+\frac{4979h}{20000}h\left(\varphi''_n-\phi''_n\right)+\frac{938088769722371847337271515 3h^2}{5557628160000000000000000000000}g_n+\frac{93914239527014040276250787 9h^2}{5351735716063910308800000000}g_{n+\frac{4979}{20000}}$$
$$-\frac{21411844562744505885799726354 1h^2}{38390759439460669327816074240000}g_{n+\frac{68919}{100000}}+\frac{2807690615432466197861057233213h^2}{12549403802880000000000000000000}g_{n+1}-\frac{4447287972696777976210710 71h^2}{4113166580096000000000000000000}g_{n+2}$$
$$+\frac{8752740362874043476726691932 13h^2}{10252817730086400000000000000000}g_{n+3}$$

$$\left(\varphi'_{n+\frac{68919}{100000}}-\phi'_{n+\frac{68919}{100000}}\right)=(\varphi'_n-\phi'_n)+\frac{68919h}{100000}h\left(\varphi''_n-\phi''_n\right)+\frac{358831669508937886175380094502 1h^2}{69706000000000000000000000000000}g_n$$
$$+\frac{25590839460665052037751408357710443h^2}{154203079457651674927750000000000000}g_{n+\frac{4979}{20000}}+\frac{2108184819329898289056009413 7h^2}{9670863481310152469600000000000}g_{n+\frac{68919}{100000}}-\frac{12365984891105287876954749910674 33h^2}{72623864600000000000000000000000000}g_{n+1}$$
$$-\frac{30342076679810790979583321504931 03h^2}{642682278140000000000000000000000000}g_{n+2}+\frac{34763243919369157469913324673256 7h^2}{593334359380000000000000000000000000}g_{n+3}$$

$$\left(\varphi'_{n+1}-\phi'_{n+1}\right)=(\varphi'_n-\phi'_n)+h\left(\varphi''_n-\phi''_n\right)+\frac{63184128979h^2}{864732206520}g_n+\frac{4831093000000000000000000h^2}{1665393258142638089219 7}g_{n+\frac{4979}{20000}}+\frac{10252718750000000000000000h^2}{7498195203019661978089077}g_{n+\frac{68919}{100000}}$$
$$-\frac{15767h^2}{196084434420}g_{n+1}+\frac{147786403h^2}{1285364556280}g_{n+2}-\frac{23201093h^2}{4005006925815}g_{n+3}$$

$$\left(\varphi'_{n+2}-\phi'_{n+2}\right)=(\varphi'_n-\phi'_n)+2h\left(\varphi''_n-\phi''_n\right)+\frac{4453189396h^2}{108091525815}g_n+\frac{16756784000000000000000000h^2}{16653932581426380892197}g_{n+\frac{4979}{20000}}$$
$$+\frac{122795000000000000000000000h^2}{7498195203019661978089077}g_{n+\frac{68919}{100000}}+\frac{50701385954h^2}{49021108605}g_{n+1}+\frac{40928012186h^2}{482011708605}g_{n+2}-\frac{11195932186h^2}{4005006925815}g_{n+3}$$

$$\left(\varphi'_{n+3}-\phi'_{n+3}\right)=(\varphi'_n-\phi'_n)+3h\left(\varphi''_n-\phi''_n\right)+\frac{9824560791h^2}{32027118760}g_n+\frac{5472990000000000000000000h^2}{616812317830606699711}g_{n+\frac{4979}{20000}}+\frac{235753125000000000000000000h^2}{277710933445172665855151}g_{n+\frac{68919}{100000}}$$
$$+\frac{4654691721h^2}{3631193230}g_{n+1}+\frac{1431004155489h^2}{1285364556280}g_{n+2}+\frac{36701093907h^2}{593334359380}g_{n+3}$$

$$\left(\varphi''_{n+\frac{4979}{20000}}-\phi''_{n+\frac{4979}{20000}}\right)=$$
$$(\varphi''_n-\phi''_n)+\frac{1256463354385971564802 91h}{132324480000000000000000000}g_n+\frac{1380261090691341362862 41h}{7645336737234157584000000}g_{n+\frac{4979}{20000}}-\frac{4588021707936241259624 29h}{10753714128700467598828032}g_{n+\frac{63919}{100000}}$$
$$+\frac{1511813872841607469880933 33h}{8963859859200000000000000000000}g_{n+1}-\frac{7085452219440672832229333 3h}{8813928385920000000000000000000}g_{n+2}$$
$$+\frac{100850981465298949142687h}{1595520966400000000000000000}g_{n+3}$$

$$\left(\varphi''_{n+\frac{63919}{100000}}-\phi''_{n+\frac{63919}{100000}}\right)$$
$$=(\varphi''_n-\phi''_n)+\frac{656439284050510475846725 7h}{99580000000000000000000000000}g_n$$
$$+\frac{18179957655167531805687542879 7h}{440580227021861928365000000000}g_{n+\frac{4979}{20000}}$$
$$+\frac{136535327929704754147079h}{55262077036058014112 0000}g_{n+\frac{63919}{100000}}-\frac{38837082286296276049222841061h}{1037483780000000000000000000000}g_{n+1}$$
$$+\frac{98315001962820628039182969549h}{918117540200000000000000000000}g_{n+2}$$
$$-\frac{620308055869515281504004106 1h}{8476205134000000000000000000000}g_{n+3}$$

$$(\varphi''_{n+1}-\phi''_{n+1})=(\varphi''_n-\phi''_n)+\frac{601753309h}{8235544824}g_n+\frac{9189220000000000000000000h}{2379133225918054413171}g_{n+\frac{4979}{20000}}$$
$$+\frac{157801250000000000000000000h}{357056914429507713242337}g_{n+\frac{63919}{100000}}+\frac{1130214319h}{11204824824}g_{n+1}$$
$$-\frac{12666505h}{110174104824}g_{n+2}+\frac{98189h}{33904820536}g_{n+3}$$

$$(\varphi''_{n+2}-\phi''_{n+2})=(\varphi''_n-\phi''_n)-\frac{521068897h}{3088329309}g_n+\frac{27027040000000000000000000h}{2379133225918054413171}g_{n+\frac{4979}{20000}}$$
$$-\frac{1248950000000000000000000000h}{1071170743288523139727011}g_{n+\frac{63919}{100000}}+\frac{2617982804h}{1400603103}g_{n+1}$$

$$+\frac{4640075701h}{13771763103}g_{n+2}-\frac{849488000h}{114428769309}g_{n+3}$$

$$(\varphi''_{n+3}-\phi''_{n+3})=(\varphi''_{n}-\phi''_{n})+\frac{60971977h}{70389272}g_{n}-\frac{1782000000000000000h}{968308191256839403}g_{n+\frac{4979}{20000}}+\frac{15638125000000000000000000h}{3967299049216752369359 3}g_{n+\frac{63919}{100000}}-\frac{782620299h}{414993512}g_{n+1}+\frac{8537240187h}{5246385944}g_{n+2}+\frac{9864819701h}{33904820536}g_{n+3} \quad (23)$$

Analysis of Basic Properties of the Methods

Order of the Block

As stated by Chollom et al. (2007), let the linear difference operator ℓbe defined using the newly developed technique in equation (24).

$$L[\varphi(x);h]=\sum_{j=0}^{k}(\varphi-\phi)\,(x+jh)-h^{3}\left(\sum_{j=0}^{3}\mu_{i}x^{j}+\sum_{j=1}^{5}\psi_{j}e^{x^{j}}\right) \quad (24)$$

Where $\left(\varphi(x)-\phi(x)\right)$ represents the exact solution satisfying equation (24). This solution can be expressed in terms of a Taylor series expansion about the pointx_n, leading to the following expression

$$\ell\left[\left(\varphi(x)-\phi(x)\right):h\right]=\overline{c_o}\left(\varphi(x)-\phi(x)\right)+\overline{c_1}h\left(\varphi'(x)-\phi'(x)\right)+\overline{c_2}h^{2}\left(\varphi''(x)-\phi''(x)\right)+\overline{c_3}h^{3}\left(\varphi'''(x)-\phi'''(x)\right)+\cdots+\overline{c_{p+3}}h^{p+3}\left(\varphi^{(p+3)}(x)-\phi^{(p+3)}(x)\right)+\cdots \quad (25)$$

Similarly,

The newly formulated Three-Step Hybrid Optimized Volterra Integral Equation of the second kind (23) is of a specific order.p If,

$$\ell\left[\left(\varphi(x)-\phi(x)\right):h\right]=o\left(h^{p+3}\right),\overline{c_o}=\overline{c_1}=\overline{c_2}=\overline{c_3}=\cdots=\overline{c_{p+2}}=0,\overline{c_{p+3}}\neq 0$$

Therefore, the principal local truncation error x_n+k is then defined as

$$\overline{c_{p+3}}h^{p+3}(\varphi-\phi)^{p+3}(x_n)$$

Where=

$$C_0=\sum_{j=0}^{k}\mu_j$$

$$C_1=\sum_{j=1}^{k}j\mu_j-\sum_{j=0}^{k}\psi_j$$

$$C_2=\frac{1}{2}\sum_{j=1}^{k}j^2\mu_j-\sum_{j=0}^{k}j\psi_j$$

$$C_q=\frac{1}{q!}\sum_{j=1}^{k}j^{q}\mu_j-\frac{1}{(q-1)!}\sum_{j=0}^{k}j^{q-1}\psi_j-\frac{1}{(q-2)!}\sum_{j=0}^{k}j^{q-2}\varpi_j\,,q=3,4,5,\ldots \quad (26)$$

Expanding (24) in Taylor series about the point x_n and comparing the coefficient of h in (25)

$$\bar{C}_9=\frac{1}{9!}\left(\left(\frac{4979}{20000}\right)^{9}\overline{\mu}_{\frac{4979}{20000}}+\left(\frac{68919}{100000}\right)^{9}\overline{\mu}_{\frac{68919}{100000}}+(1)^{9}\overline{\mu}_1+(2)^{9}\overline{\mu}_2+(3)^{9}\overline{\mu}_3\right)+\frac{1}{8!}\left(\left(\frac{4979}{20000}\right)^{8}\overline{\mu}'_{\frac{4979}{20000}}+\left(\frac{68919}{100000}\right)^{8}\overline{\mu}'_{\frac{68919}{100000}}+(1)^{8}\overline{\mu}'_1+(2)^{8}\overline{\mu}'_2+(3)^{8}\overline{\mu}'_3\right)+\frac{1}{7!}\left(\left(\frac{16613}{100000}\right)^{7}\overline{\mu}''_{\frac{16613}{100000}}+\left(\frac{68919}{10000}\right)^{7}\overline{\mu}''_{\frac{68919}{100000}}+(1)^{7}\overline{\mu}''_1+(2)^{7}\overline{\mu}''_2+(3)^{7}\overline{\mu}''_3\right)-\frac{1}{6!}\left(\overline{\psi}_{\frac{16613}{100000}}\left(\frac{16613}{100000}\right)^{6}+\overline{\psi}_{\frac{68919}{100000}}\left(\frac{68919}{100000}\right)^{6}+\overline{\psi}_1(1)^{6}+\overline{\psi}_2(2)^{6}+\overline{\psi}_3(3)^{6}\right)$$

$$
=\begin{pmatrix}
\frac{1}{9!}\left(\left(\frac{16613}{100000}\right)^{9}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{2272\,427863132\,631987292\,375683987}{1902839\,365711612\,554240000\,000000000}\left(\frac{4979}{20000}\right)^{6}\\-\frac{5283792\,274483048\,362648574\,906821917}{12\,285043020\,627414184\,901143756\,800000000}\left(\frac{68919}{10000}\right)^{6}+\frac{1942347\,056223802\,496268529\,486106279}{11\,155025602\,560000000\,000000000\,000000000}(1)^{6}\\-\frac{2793560\,573276046\,087885240\,531758837}{329\,053326407\,680000000\,000000000\,000000000}(2)^{6}+\frac{787551\,353385403\,250851826\,202228073}{1171\,750597724\,160000000\,000000000\,000000000}(2)^{6}\end{array}\right)\\
\frac{1}{9!}\left(\left(\frac{68919}{100000}\right)^{9}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{44280\,083007812\,500000000}{3072384\,565501992\,884264331}\left(\frac{16613}{100000}\right)^{6}-\frac{465752807}{3423\,967727040}\left(\frac{1}{2}\right)^{6}\\+\frac{9018554687\,500000000}{19975\,785048636\,257042559}\left(\frac{85609}{100000}\right)^{6}-\frac{614124493}{2481\,276913920}(1)^{6}+\frac{961136219}{676660\,636657152}(2)^{6}\end{array}\right)\\
\frac{1}{9!}\left(\left(\frac{85609}{100000}\right)^{9}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{62353\,024134447\,881991329\,803930667\,218406237}{983163\,060960637\,722964585\,920000000\,000000000}\left(\frac{16613}{100000}\right)^{6}\\+\frac{2593\,861576647\,465694679\,662445314\,814826883}{106998\,991470000\,000000000\,000000000\,000000000}\left(\frac{1}{2}\right)^{6}-\frac{4811\,902111686\,529441832\,085269543}{4432345\,925004728\,640000000\,000000000}\left(\frac{85609}{100000}\right)^{6}\\+\frac{220\,651222503\,334243049\,383434220\,660462727}{336006\,248760000\,000000000\,000000000\,000000000}(1)^{6}-\frac{2915542540\,868217377\,891488313\,404610797}{1448331\,842160000\,000000000\,000000000\,000000000}(2)^{6}\end{array}\right)\\
\frac{1}{9!}((1)^{9})-\frac{1}{6!}\left(\begin{array}{l}\frac{285286\,562500000\,000000000}{3072384\,565501992\,884264331}\left(\frac{16613}{100000}\right)^{6}+\frac{3533283781}{74\,899294029}\left(\frac{1}{2}\right)^{6}-\frac{1663\,437500000\,000000000}{605932\,146475299\,796957623}\left(\frac{85609}{100000}\right)^{6}\\+\frac{51847927}{168\,003124380}(1)^{6}-\frac{266643781}{105728\,224477680}(2)^{6}\end{array}\right)\\
\frac{1}{9!}((2)^{9})-\frac{1}{6!}\left(\begin{array}{l}\frac{568270\,000000000\,000000000}{3072384\,565501992\,884264331}\left(\frac{16613}{100000}\right)^{6}+\frac{335\,643000992}{374\,496470145}\left(\frac{1}{2}\right)^{6}-\frac{20570\,000000000\,000000000}{24901\,321088026\,019053053}\left(\frac{85609}{100000}\right)^{6}\\+\frac{110\,934224876}{126\,002343285}(1)^{6}+\frac{659972003}{90\,520740135}(2)^{6}\end{array}\right)\\
\frac{1}{8!}\left(\left(\frac{16613}{100000}\right)^{8}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{277\,252189626\,557557832\,341723493}{36987\,715229061\,492617400\,000000000}\left(\frac{16613}{100000}\right)^{6}\\-\frac{3810702\,614766049\,485272861\,956280899}{1872482350\,725000000\,000000000\,000000000}\left(\frac{1}{2}\right)^{6}+\frac{58223\,492843632\,669794406\,945208687}{40395476\,431686653\,130508200\,000000000}\left(\frac{85609}{10000}\right)^{6}\\-\frac{197111\,323543531\,568388395\,688781211}{280005207\,300000000\,000000000\,000000000}(1)^{6}+\frac{858364\,028261116\,106810154\,684530899}{264\,320561194\,200000000\,000000000\,000000000}(2)^{6}\end{array}\right)\\
\frac{1}{8!}\left(\left(\frac{1}{2}\right)^{8}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{267160\,683593750\,000000000}{3072384\,565501992\,884264331}\left(\frac{16613}{100000}\right)^{6}+\frac{96\,020320393}{5991\,943522320}\left(\frac{1}{2}\right)^{6}-\frac{164\,042968750\,000000000}{46610\,165113484\,599765971}\left(\frac{85609}{100000}\right)^{6}\\+\frac{144152411}{103\,386538080}(1)^{6}-\frac{336081281}{105728\,224477680}(2)^{6}\end{array}\right)\\
\frac{1}{8!}\left(\left(\frac{85609}{100000}\right)^{8}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{114109027\,002447532\,137868571\,929942331}{614476913\,100398576\,852866200\,000000000}\left(\frac{16613}{100000}\right)^{6}\\+\frac{49216556\,922388110\,969171841\,772710589}{374496470\,145000000\,000000000\,000000000}\left(\frac{1}{2}\right)^{6}+\frac{13\,960459207\,460977409\,299086491}{1415\,580479798\,385209400\,000000000}\left(\frac{85609}{100000}\right)^{6}\\-\frac{70521\,820283337\,505855313\,094533137}{168003124\,380000000\,000000000\,000000000}(1)^{6}-\frac{366001\,413184339\,291691791\,603780589}{52\,864112238\,840000000\,000000000\,000000000}(2)^{6}\end{array}\right)\\
\frac{1}{8!}((1)^{8})-\frac{1}{6!}\left(\begin{array}{l}\frac{686578\,750000000\,000000000}{3072384\,565501992\,884264331}\left(\frac{16613}{100000}\right)^{6}+\frac{70\,043500496}{374\,496470145}\left(\frac{1}{2}\right)^{6}\\+\frac{27621\,250000000\,000000000}{605932\,146475299\,796957623}\left(\frac{85609}{100000}\right)^{6}-\frac{503652073}{168\,003124380}(1)^{6}-\frac{44443781}{52864\,112238840}(2)^{6}\end{array}\right)\\
\frac{1}{8!}((2)^{8})-\frac{1}{6!}\left(\begin{array}{l}-\frac{42710\,000000000\,000000000}{65369\,884372382\,827324773}\left(\frac{16613}{100000}\right)^{6}+\frac{961\,422200992}{374\,496470145}\left(\frac{1}{2}\right)^{5}\\-\frac{3730\,000000000\,000000000}{991\,705640712\,438292893}\left(\frac{85609}{100000}\right)^{6}+\frac{10\,630263452}{3230829315}(1)^{5}+\frac{290\,489824876}{6608\,014029855}(2)^{5}\end{array}\right)
\end{pmatrix}
$$

$$
\begin{bmatrix}
\frac{1}{7!}\left(\left(\frac{16613}{100000}\right)^{7}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{12345020\,936540690\,809421371}{105679186\,368747121\,764000000}\left(\frac{16613}{100000}\right)^{6}\\-\frac{62\,527587257\,221387913\,182249093}{2674\,974786750\,000000000\,000000000}\left(\frac{1}{2}\right)^{6}+\frac{16\,903585262\,389151669\,740815391}{1038\,740822529\,085366213\,068000000}\left(\frac{85609}{100000}\right)^{6}\\-\frac{28\,524419861\,904165804\,500549093}{3600\,066951000\,000000000\,000000000}(1)^{6}+\frac{13\,683473806\,604835462\,137149093}{377600\,801706000\,000000000\,000000000}(2)^{6}\end{array}\right)\\
\frac{1}{7!}\left(\left(\frac{1}{2}\right)^{7}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{130362\,382812500\,000000000}{438912\,080785998\,983466333}\left(\frac{16613}{100000}\right)^{6}+\frac{16\,422037441}{85\,599193176}\left(\frac{1}{2}\right)^{6}-\frac{12480\,898437500\,000000000}{259685\,205632271\,341553267}\left(\frac{85609}{100000}\right)^{6}\\+\frac{2422183781}{115\,202142432}(1)^{6}-\frac{888873049}{12083\,225654592}(2)^{6}\end{array}\right)\\
\frac{1}{7!}\left(\left(\frac{85609}{100000}\right)^{7}\right)-\frac{1}{6!}\left(\begin{array}{l}\frac{457\,681207794\,956857510\,458227159}{1755\,648323143\,995933865\,332000000}\left(\frac{16613}{100000}\right)^{6}+\frac{1044\,777909434\,769206124\,661240571}{2674\,974786750\,000000000\,000000000}\left(\frac{1}{2}\right)^{6}\\+\frac{86283\,943305368\,099269781}{449390\,628507423\,876000000}\left(\frac{85609}{100000}\right)^{6}-\frac{4883954927\,461740010\,405728127}{133\,335813000\,000000000\,000000000}(1)^{6}\\+\frac{20\,196017535\,620332567\,186459429}{377600\,801706000\,000000000\,000000000}(2)^{6}\end{array}\right)\\
\frac{1}{7!}((1)^{7})-\frac{1}{6!}\left(\begin{array}{l}\frac{116005\,625000000\,000000000}{438912\,080785998\,983466333}\left(\frac{16613}{100000}\right)^{6}+\frac{584118014}{1528557021}\left(\frac{1}{2}\right)^{6}+\frac{385\,625000000\,000000000}{1373\,995796996\,144664303}\left(\frac{85609}{100000}\right)^{6}\\+\frac{6585271}{266671626}(1)^{6}+\frac{5555000}{188\,800400853}(2)^{6}\end{array}\right)\\
\frac{1}{7!}((2)^{7})-\frac{1}{6!}\left(\begin{array}{l}-\frac{1287820\,000000000\,000000000}{438912\,080785998\,983466333}\left(\frac{16613}{100000}\right)^{6}+\frac{62\,577920000}{10\,699899147}\left(\frac{1}{2}\right)^{6}+\frac{2667740\,000000000\,000000000}{259685\,205632271\,341553267}\left(\frac{85609}{100000}\right)^{6}\\+\frac{28\,444569268}{3600066951}(1)^{6}+\frac{43\,111226951}{188\,800400853}(2)^{6}\end{array}\right)
\end{bmatrix}
$$

Hence, the new methods (23) has order $c_{p+2} = 8, p = (6,6,6,6,6,6,6,6,6,6,6,6,6,6,6)^T$, and error constant

$$c_9 = \left[-\frac{1313436\,585571426\,493117707\,741574257\,064045183}{453600\,000000000\,000000000\,000000000\,000000000\,000000000}, -\frac{63\,484686679}{2322432000\,000000000}, \frac{2977732\,717769931\,557989089\,665200539\,071569357}{226800\,000000000\,000000000\,000000000\,000000000\,000000000}, \frac{866529637}{36288000\,000000000}, -\frac{31\,466800853}{1134000\,000000000}, \frac{332\,800771432\,895109382\,607591565\,846889181}{6048000000\,000000000\,000000000\,000000000\,000000000}, \frac{465752807}{27648000\,000000000}, \frac{30\,071775335\,194163225\,963618112\,342388151}{288000000\,000000000\,000000000\,000000000\,000000000}, \frac{51847927}{1008000\,000000000}, -\frac{48\,177956219}{378000\,000000000}, -\frac{4605990\,626266000\,708268693\,052389209}{7560\,000000000\,000000000\,000000000\,000000000}, \frac{96\,020320393}{96768000\,000000000}, -\frac{1195140\,952291526\,815032685\,495408019}{2520\,000000000\,000000000\,000000000\,000000000}, -\frac{503652073}{2016000\,000000000}, -\frac{2472\,622456219}{189000\,000000000}\right]$$

Hence our method is of order six (6).

Consistency of the Method

Definition 1. The optimize hybrid block method (23) is said to be consistent if it has order p≥1 .Thus, our method whose order is 6 satisfied this condition.

Zero Stability

The proposed novel three-step optimized hybrid method based on Volterra integral equation of the second kind in equation (23) for (k = 3), incorporating two off-grid collocation points defined in equation (24), to yield

$$\rho(z) = det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right] = 0 \qquad (27)$$

Gives

$$\rho(z) = \left| z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&1&0&0&0&0&\frac{4979}{20000}&0&0&0&0&\frac{24790441}{800000000}\\ 0&0&0&0&1&0&0&0&0&\frac{68919}{100000}&0&0&0&0&\frac{4749828561}{20000000000}\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&\frac{1}{2}\\ 0&0&0&0&1&0&0&0&0&2&0&0&0&0&2\\ 0&0&0&0&1&0&0&0&0&3&0&0&0&0&\frac{9}{2}\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&\frac{4979}{20000}\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&\frac{68919}{100000}\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&2\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&3\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \right|$$

Calculating its determinant and resolving the characteristic equation produces

$= (z-1)^3 z^{12}$

$\rho(z) = (z-1)^3 z^{12} = 0, z = 0,0,0,0,0,0,0,0,0,0,0,0,1,1,1$

since

$|z_i| = |0,0,0,0,0,0,0,0,0,0,0,0,1,1,1| \leq 1$

The method is zero-stable.

Convergence

Theorem 1 (Henrici, 1962). Zero stability and consistency are necessary requirements for the convergence of a linear multistep algorithm. For a numerical method to be convergent, it must satisfy both consistency and zero-stability. Therefore, the three-step third derivative method with two optimize off grid hybrid points based on the Volterra integral equation of the second kind is convergent since it is both consistent and zero-stable

Region of Absolute Stability

To plot the region of absolute stability of New Two-Step Hybrid Optimized Volterra Integral Equation of the second kind (23), the methods were formulated as general linear method expressed as

$$(\varphi-\phi)_m^{(i)} = \left[(\varphi-\phi)_{n+\frac{4979}{20000}}^{(i)} \quad (\varphi-\phi)_{n+\frac{68919}{100000}}^{(i)} \quad (\varphi-\phi)_{n+1}^{(i)} \quad (\varphi-\phi)_{n+2}^{(i)} \quad (\varphi-\phi)_{n+3}^{(i)}\right]^T$$

$$(\varphi-\phi)_n^{(i)} = \left[(\varphi-\phi)_{n-\frac{4979}{20000}}^{(i)} \quad (\varphi-\phi)_{n-\frac{68919}{100000}}^{(i)} \quad (\varphi-\phi)_{n-1}^{(i)} \quad (\varphi-\phi)_{n-2}^{(i)} \quad (\varphi-\phi)_{n}^{(i)}\right]^T$$

$$G(\varphi_m-\phi_m) = \left[g_{n+\frac{4979}{20000}}^{(i)} \quad g_{n+\frac{68919}{100000}}^{(i)} \quad g_{n+1}^{(i)} \quad g_{n+2}^{(i)} \quad g_{n+3}^{(i)}\right]^T$$

$$g(\varphi_n-\phi_n) = \left[g_{n-\frac{4979}{20000}}^{(i)} \quad g_{n-\frac{68919}{100000}}^{(i)} \quad g_{n-1}^{(i)} \quad g_{n-2}^{(i)} \quad g_{n}^{(i)}\right]^T$$

$A^{(0)} = 5 \times 5$ Identity matrix

The stability polynomial of (23) gives

$$h(w) = -h^5\left(\frac{114382567}{80000000000}w^5 + \frac{4238102567}{80000000000}w^4\right) - h^4\left(\frac{22288856079}{80000000000}w^4 - \frac{1670256079}{80000000000}w^5\right) - h^3\left(\frac{17348983103}{120000000000}w^5 + \frac{99823383103}{120000000000}w^4\right)$$

$$+ h^2\left(\frac{33600827701}{60000000000}w^5 - \frac{95456627701}{60000000000}w^4\right) - h\left(\frac{346907}{300000}w^5 + \frac{553093}{300000}w^4\right) + w^5 - w^4$$

Which produce the absolute stability region using Mat Lab.

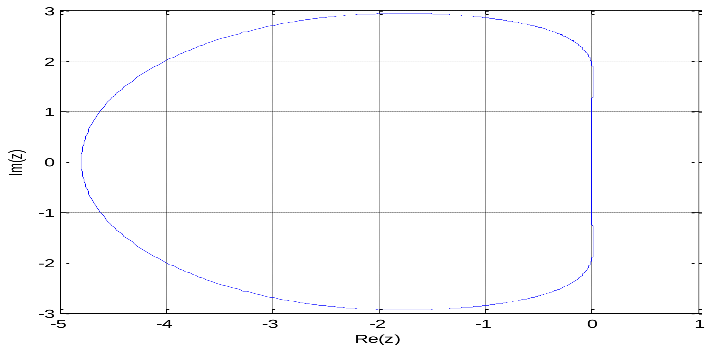

Figure 1: Region of Absolute Stability (RAS) for the Method (23)

RESULTS AND DISCUSSION

Problem1: Consider the Stiff problem

$y''' = x - 4y', y(0) = 0, y'(0) = 0, y''(0) = 1$

$ExactSolution; : y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2, h = \frac{1}{10}$

Source: Kayode and Adebisi. (2025)

Table 1 Comparing the Absolute Errors in the New Method with Error from Kayode and Adebisi (2025) for problem1

X	Error in our method	Error in Kayode and Adebisi (2025)
0.1	3.184300e-16	1.7404575e-15
0.2	2.430400e-16	8.1219170e-15
0.3	2.462618e-15	1.8908564e-14
0.4	3.786418e-15	3.3652581e-14
0.5	1.654370e-15	4.9627030e-14
0.6	6.732670e-15	6.5769060e-14
0.7	4.455711e-15	8.1443290e-14
0.8	3.076450e-15	9.6018590e-14
0.9	8.342530e-14	1.0716268e-13
1.0	5.754320e-14	1.1260826e-13

Problem 2: Consider the Stiff problem

$\gamma''' = -5\gamma'' - 7\gamma' - 3\gamma, \gamma(0) = 1, \gamma'(0) = 0, \gamma''(0) = -1$

$ExactSolution; : \gamma(r) = exp^{-r} + r\, exp^{-r}, h = \frac{1}{10}$

Source: Tumba *et al* (2021)

Table 2 Comparing the Absolute Errors in the New Method with Error from Tumba *et al* (2021) for problem2

X	Error in our method	Tumba *et al* (2021)
0.1	3.41390e-16	1.0434e-14
0.2	1.39926e-15	9.8731e-14
0.3	2.68021e-15	3.1317e-13
0.4	4.00970e-15	6.6668e-13
0.5	5.16657e-15	1.1507e-12
0.6	6.13497e-15	1.7445e-12
0.7	6.82559e-15	2.4220e-12
0.8	7.28686e-15	3.1554e-12
0.9	7.48948e-15	3.9178e-12
1.0	7.50042e-15	4.6852e-12

Table 1 and Figure 2 display the results for a highly stiff third-order linear problem. Despite the stiffness, the numerical solutions produced by all methods closely follow the exact solution across the range of values. Region of Absolute Stability (RAS) for the method (23) confirms finite region of absolute stability or bounded region of absolute stability. The new block method maintains excellent consistency and stability, especially for larger values of the independent variable x, indicating its suitability for stiff problems.

CONCLUSION

The three-step method based on Volterra integral equations of the second kind for third-order initial value problems demonstrates a marked improvement in the numerical treatment of stiff problems. The numerical results indicate that the proposed technique produces approximate solutions that are highly consistent with one another, while some grid points exhibit even closer agreement with the exact solution. This enhanced performance highlights the improved accuracy achieved through the third-order hybrid block formulation.

Moreover, comparisons with established methods presented in Tables 1 and 2 show that the newly developed scheme consistently produces smaller absolute errors, confirming its superior convergence rate and computational efficiency. Overall, the proposed method offers a stable, accurate, and efficient computational framework, establishing it as a reliable numerical tool for solving stiff problems.

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