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Generalized Hyers-Ulam-Rassias Stability of Perturbed Third-Order Nonlinear Ordinary Differential Equations

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ABSTRACT

This paper establishes generalized Hyers-Ulam-Rassias (H-U-R) stability results for a class of perturbed third-order nonlinear ordinary differential equations (ODEs). The paper considered family of third-order nonlinear differential equations. Under a set of specific criteria on the nonlinearities and using integral transform techniques, the problems are converted into equivalent integral equations. By applying appropriate nonlinear generalizations of the Gronwall-Bellman inequality (Bihari-type inequalities), we derive explicit bounds for the deviation of any approximate solution from an exact solution, thereby proving H-U-R stability. Explicit formulas for the Hyers-Ulam-Rassias constants are provided. The theoretical results are supported by a representative numerical example.

Keywords: Hyers-Ulam-Rassias stability, Nonlinear differential equations, Third order equation, Gronwall-Bellman-Bihari inequality

INTRODUCTION

Stability, a fundamental mathematical concept, is crucial in various disciplines, such as engineering, physics, economics, and biology. It examines the resilience of systems to perturbations, ensuring that small deviations do not lead to catastrophic failures. Stability criteria dictate the reliability of solutions in dynamical systems, control theory, numerical analysis, structural engineering, ecological modeling, financial markets, and machine learning.

The stability problem for functional equations, originally posed by Ulam (1964) and first solved in the context of additive mappings by Hyers (1941), has grown into a significant field of study. Rassias (1978 and 2000) provided a powerful generalization, leading to the concept now known as Hyers-Ulam-Rassias (H-U-R) stability. This framework was later extended from functional equations to differential equations, beginning with the work of Obloza (1993) and Alsina and Ger(1998). Since then, H-U-R stability has been extensively investigated for first- and second-order linear and nonlinear differential equations by the following authors: Aliifiary(2014), Fakunle(2023,), Fakunle (2023), Jung (2004), Jung (2006) and Li and Shen (2010).

For third-order equations, research has primarily focused on linear cases by Abdollapour (2012). The study of H-U-R stability for *nonlinear* third-order equations, particularly those with complex perturbative structures, remains less developed and presents considerable technical challenges. In addition, Aramuraye (2025) talked about application of Elzaki transform to the analytical solution of stiff linear systems of ordinary differential equations.

In this article, we advance this line of research by establishing H-U-R stability for two important classes of perturbed third-order nonlinear ODEs:

$$y''' + (r(t)(y'))'^{\alpha} + p(t)(y')^{\alpha} + q(t)f(y) = P(t, y, y', y'', y'''), \qquad (1)$$

$$y''' + (r(t)\psi(y)y')' + p(t)y' + q(t)f(y) = P(t, y, y', y'', y'''), \qquad (2)$$

Subject to the Initial Conditions

$$y(t_0) = y'(t_0) = y''(t_0) = 0, \quad \forall t > 0. \qquad (3)$$

A special case of (1.1) with $\alpha = 1$ is also treated in detail:

$$y''' + (r(t)y')' + p(t)y' + q(t)f(y) = P(t, y, y', y'', y'''). \qquad (4)$$

Our approach involves: (i) formulating precise H-U-R stability definitions for these problems, (ii) transforming the differential inequalities into integral inequalities via triple integration, (iii) imposing a specific structural criterion on the perturbation term P; and (iv) employing sophisticated Bihari-type integral inequalities to derive quantitative bounds that guarantee stability. The main contributions are Theorems 1, 2, and 3, which provide explicit H-U-R constants.

The remainder of this paper is organized as follows: Section 2 contains preliminary definitions and necessary inequalities. Section 3 discusses the basic existence and uniqueness theory for the IVPs. The core criteria and assumptions are presented in Section 4. Section 5 presents and proves the main H-U-R stability theorems. Section 6 provides a concrete example, and Section 7 concludes the study.

Preliminaries

Let $\mathbf{I} = (0, \infty)$ and $\mathbf{R}_+ = [0, \infty)$. We consider functions in $C^3(\mathbf{I}, \mathbf{R}_+)$, the space of three continuously differentiable functions from $\mathbf{I}$ to $\mathbf{R}_+$.

Definition 1: H-U-R Stability for equation (1) is said to have *generalized Hyers-Ulam-Rassias stability* if there exists a constant $C_\varphi > 0$ with the following property: For every $\varphi \in C(\mathbf{I}, \mathbf{R})$, a positive non-decreasing function, and for every *approximate solution* $y \in C^3(\mathbf{I}, \mathbf{R}_+)$ satisfying the inequality

$$|y''' + (r(t)(y')^{\alpha})' + p(t)(y')^{\alpha} + q(t)f(y) - P(t, y, y', y'', y''')| \le \varphi(t), \qquad (5)$$

there exists an *exact solution* $y_0 \in C^3(\mathbf{I}, \mathbf{R}_+)$ of (1.1)

$$|y(t) - y_0(t)| \le C_\varphi\, \varphi(t), \quad \forall t \in \mathbf{I}. \qquad (6)$$

The constant C_φ is called a *Hyers-Ulam-Rassias constant*.

Analogous definitions hold for Eqs. (1) and (4).

The Cauchy formula for repeated integration and a key nonlinear integral inequality is recalled.

Lemma 1 (Cauchy Formula): Let $g \in C(\mathbf{I}, \mathbf{R}_+)$. For any $t_0, t \in \mathbf{I}$ and $n \in \mathbf{N}$,

$$\int_{t_0}^{t}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{n-1}} g(s_n)\, ds_n \cdots ds_1 = \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1} g(s)\, ds.$$

Lemma 2 (Bihari-Type Inequality): Let $u, f \in C([t_0, T], \mathbf{R}_+)$, let $K \geq 0$, $M > 0$, and let $\omega: \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous, non-decreasing functions such that $\omega(u) > 0$ for $u > 0$. If

$$u(t) \leq K + M \int_{t_0}^{t} f(s)\, \omega(u(s))\, ds, \quad t_0 \leq t \leq T,$$

then for all t in a suitable subinterval $[t_0, T'] \subseteq [t_0, T]$,

$$u(t) \leq \Omega^{-1}\left(\Omega(K) + M \int_{t_0}^{t} f(s)\, ds\right),$$

where $\Omega(v) = \int_{v_0}^{v} \frac{dz}{\omega(z)}$, $v_0 > 0$, and Ω^{-1} is its inverse.

Corollaries for inequalities with multiple nonlinear integral terms, as used in Fakunle and Arawomo (2023, and 2023), and A will be directly applied in the proofs.

Existence and Uniquness of the Solution

Before analyzing the stability, we ensure that the problems are well-posed. We present a representative theorem for equation (1) similar results are obtained for equations (2) and equation (4)

Theorem 1: [Local Existence and Uniqueness] Consider Equation (1) with initial conditions (1.3). Assume:

1. Coefficient functions $r, p, q \in C^1(\mathbf{I}, \mathbf{R}_+)$.
2. $f \in C^1(\mathbf{R}_+, \mathbf{R}_+)$ with $f(0) = 0$.
3. The perturbation $P(t, u, v, w, z)$ is continuous in t and satisfies a uniform Lipschitz condition in (u, v, w, z) on compact sets.
4. The term P has the specific implicit form that allows it to be solved for y''', e.g., P is linear in y''' with coefficient bounded away from 1, or satisfies the conditions of the Implicit Function Theorem locally near $(t_0, 0,0,0,0)$.

Then, there exists an interval $[t_0, t_0 + \delta] \subset \mathbf{I}$ such that the initial value problem (1), (3) has a unique solution $y \in C^3([t_0, t_0 + \delta], \mathbf{R}_+)$.

Proof Sketch. Equation can be written as $y''' = G(t, y, y', y'')$ by solving the implicit structure in assumption (4) under the given criteria. Defining the vector $\mathbf{z} = (y, y', y'')^\top$, the problem is transformed into the first-order system $\mathbf{z}' = \mathbf{F}(t, \mathbf{z})$ with $\mathbf{z}(t_0) = \mathbf{0}$. The Lipschitz conditions on the components of $\mathbf{F}$, derived from the assumptions on r, p, q, f, and P, guarantee a unique local solution according to the standard Picard-Lindelöf theorem.

Assumption and Stability Criteria

To prove H-U-R stability, we require the following specific criteria for the functions and perturbation term: These conditions are in addition to the basic regularity conditions in Theorem 1.

Structure of P*:* There exists a positive function $\phi \in C(\mathbf{I}, \mathbf{R}_+)$ and continuous functions $\varpi, h: \mathbf{R}_+ \to \mathbf{R}_+$ such that the perturbation term has the multiplicative form

$$P(t, y, y', y'', y''') = \phi(t)\, \varpi(y)\, h(y')\, (y'')^n\, (y''')^4, \tag{7}$$

where $n \in \mathbf{N}$.

1. ***Boundedness of approximate solutions:*** The approximate solution y and its derivatives satisfy the global bounds:

$$|y'(t)| \leq \lambda, \quad |y''(t)| \leq \psi, \quad |y'''(t)| \leq \tau, \quad \forall t \in \mathbf{I},$$

for some positive constants λ, ψ, τ.

2. ***Integral Bounds on Coefficients****:* The coefficient functions have bounded integrals on $\mathbf{I}$:

$$\sup_{t \in \mathbf{I}} \int_{t_0}^{t} r(s) ds \leq k_1 < \infty, \quad \sup_{t \in \mathbf{I}} \int_{t_0}^{t} p(s) ds \leq k_2 < \infty,$$

$$\sup_{t \in \mathbf{I}} \int_{t_0}^{t} q(s)\, ds \leq k_4 < \infty, \quad \sup_{t \in \mathbf{I}} \int_{t_0}^{t} \phi(s)\, ds \leq k_3 < \infty.$$

3. ***Admissibility of φ:*** The function φ in Definition 1 satisfies the following:

$$\int_{t_0}^{t} \varphi(s)\, ds \leq \varrho\, \varphi(t), \quad \forall t \in \mathbf{I},$$

for some constant $\varrho > 0$.

These criteria, particularly (C_1), allow us to control the nonlinear perturbation after integration and effectively apply the Bihari-type inequalities.

Main Stability Theorems

We now present the results of core stability. The proofs follow a unified strategy: the differential inequality is integrated thrice, the Cauchy formula and the mean value theorem are applied for integrals, the bounds from Criteria (C_1)-(C_4), and finally employ a nonlinear Gronwall-Bihari inequality to isolate $|y(t)|$.

Theorem 2: (H-U-R Stability for Equation (1)) Assume that the conditions for existence (Theorem 1) and the stability Criteria (C_1)-(C_4) hold. Equation (1) then has generalized Hyers-Ulam-Rassias stability. Moreover, a Hyers-Ulam-Rassias constant is given by

$$\begin{aligned} C_\varphi = \varrho \quad & (1 + \lambda^\alpha k_1 + \lambda^\alpha k_2) \times \\ & \Omega^{-1}(\Omega(1) + h(\lambda)\tau^n \psi^4 k_3\, \varpi(F^{-1}(F(1) + k_4))) \times \\ & F^{-1}(F(1) + k_4), \end{aligned} \tag{8}$$

where $F(u) = \int_1^u \frac{dz}{\beta(z)}$, $\Omega(u) = \int_1^u \frac{dz}{\varpi(z)}$, with $\beta(z) = f(z)$ from the integrated inequality, and F^{-1}, Ω^{-1} denote the inverse functions.

Proof. Let $y \in C^3(\mathbf{I}, \mathbf{R}_+)$ be an approximate solution that satisfies inequality (5). Integrating (5) from t_0 to t three times and using the initial conditions (3) and Lemma 2 yields the following:

$$\begin{aligned} & y(t) + \int_{t_0}^{t} (r(s)(y')^\alpha) ds \\ & + \tfrac{1}{2}\int_{t_0}^{t} (t-s)^2 p(s)(y')^\alpha ds + \tfrac{1}{2}\int_{t_0}^{t} (t-s)^2 q(s) f(y) ds \\ & - \tfrac{1}{2}\int_{t_0}^{t} (t-s)^2 P(s, y, y', y'', y''') ds \leq \tfrac{1}{2}\int_{t_0}^{t} (t-s)^2 \varphi(s) ds. \end{aligned}$$

Applying the mean value theorem for integrals to the second, third, and fifth terms and using Criterion (C_1) to substitute for P, we obtain an inequality of the form:

$$\begin{aligned} |y(t)| \leq A \int_{t_0}^{t} \varphi(s) ds + \int_{t_0}^{t} q(s) f(|y(s)|) ds + \\ B \int_{t_0}^{t} \phi(s) \varpi(|y(s)|) ds, \end{aligned} \tag{9}$$

where $A = 1 + \lambda^\alpha k_1 + \lambda^\alpha k_2$ and $B = h(\lambda)\psi^n \tau^4$ are constants derived from Criteria $(C_2)$and (C_3).Using Criterion (C_4) $(\int_{t_0}^{t} \varphi(s) ds \leq \varrho\varphi(t))$ and defining $\rho(t) = A\varrho\varphi(t)$, (5.2) becomess

$$\begin{aligned} |y(t)| \leq \rho(t) + \int_{t_0}^{t} q(s) f(|y(s)|) ds + \\ B \int_{t_0}^{t} \phi(s) \varpi(|y(s)|) ds. \end{aligned}$$

This integral inequality matches the generalized Bihari inequality in Corollary 2 of [4]. Applying that corollary (noting the order of application due to the functions f and ϖ) leads directly to the bound $|y(t)| \leq C_\varphi \varphi(t)$, with C_φ given by (5.1). According to Theorem 1, an exact solution y_0 exists. Setting y_0 as the unique solution with initial data (0,0,0) validates Definition 1 with the constant C_φ.

Theorem 3: (H-U-R Stability for Equation (2)) Assume that the conditions for existence and Criteria (C_1)-(C_4) hold, with the term λ^α in Criterion (C_2) and subsequent calculations replaced by λ for the derivative in the $(r\psi y')'$ term. Then, equation (1.2) is H-U-R stable. A Hyers-Ulam-Rassias constant is

$$C_\varphi = \varrho \quad (1+\lambda k_2)\Upsilon^{-1}[\Upsilon(1) + h(\lambda)\psi^n\tau^4 k_3\,\varpi(\Theta)] \times \Omega^{-1}(\Omega(1) + \lambda k_4 f(\Theta)) \times \Theta,$$

where $\Theta = F^{-1}(F(1) + \lambda k_1)$, and $\Upsilon(u) = \int_1^u \frac{dz}{\varpi(z)}$.$(F, \Omega$ as in Theorem 1).

The proof follows the same pattern as Theorem 1, but the structure of the integral inequality slightly differs, requiring a nested application of the Bihari-type inequalities as in Fakunle and Arawomo(2023,2023).

***Theorem 4*:** [H-U-R Stability for Equation (4)] Under the assumptions of Theorem 1 with $\alpha = 1$, equation (4) is H-U-R stable. A Hyers-Ulam-Rassias constant is

$$C_\varphi = \varrho(1+\lambda k_1 + \lambda k_2)\,\Omega^{-1}\big(\Omega(1) + h(\lambda)\psi^n\tau^4 k_3\,\varpi(F^{-1}(F(1) + \lambda k_4))\big)\,F^{-1}(F(1) + \lambda k_4).$$

This follows directly from Theorem 1 setting $\alpha = 1$

Numerical Example

To illustrate the applicability of our theoretical results, consider a specific instance of Equation (1.3) on $\mathbf{I} = (0, \infty)$ with $t_0 = 1$:

$$y''' + (e^{-t}y')' + \frac{1}{1+t^2}y' + e^{-2t}\frac{y}{1+y} = \underbrace{\frac{e^{-3t}}{10}\cdot\frac{y}{1+y}\cdot(1+(y')^2)\cdot(y'')^2\cdot(y''')^4}_{P(t,y,y\prime,y\prime\prime,y\prime\prime\prime)}, \qquad (10)$$

with $y(1) = y'(1) = y''(1) = 0$.

Verification of the Criteria

(C_1): Clearly, P has the required form with $\phi(t) = e^{-3t}/10$, $\varpi(y) = y/(1+y)$, $h(v) = 1 + v^2$, $n = 2$.

(C_2): Suppose for the approximate solution we have estimated bounds: $|y'| \le \lambda = 0.5$, $|y''| \le \psi = 0.2$, $|y'''| \le \tau = 0.1$.

(C_3): We compute the integral bounds:

$$k_1 = \sup_{t>1}\int_1^t e^{-s}ds = e^{-1} \approx 0.368,$$

$$k_2 = \sup_{t>1}\int_1^t \frac{1}{1+s^2}ds = \frac{\pi}{2} - \arctan(1) \approx 0.643,$$

$$k_3 = \sup_{t>1}\int_1^t \frac{e^{-3s}}{10}ds = \frac{e^{-3}}{30} \approx 0.00166,$$

$$k_4 = \sup_{t>1}\int_1^t e^{-2s}ds = \frac{e^{-2}}{2} \approx 0.0677.$$

(C_4): Let $\varphi(t) = e^{-t}$, a common choice. Then $\int_1^t e^{-s}ds = e^{-1} - e^{-t} \le e^{-1} = \varrho\varphi(t)$ with $\varrho = e^{-1}/e^{-t}|_{t\text{large}} \to \infty$? This needs adjustmentent. Choose $\varphi(t) = 1/t^2$. Then $\int_1^t s^{-2}ds = 1 - 1/t \le 1 = \varrho\varphi(t)$ for $\varrho = t^2$ which is not constant. To satisfy (C4), one must choose an admissible φ, e.g., $\varphi(t) = e^{-t}$ does NOT satisfy a constant ϱ condition on $(0, \infty)$. This demonstrates the importance ofion. For a finite interval $[1, T]$, we can find a constant ϱ_T.

Assuming we work on a finite interval $[1, T]$ and choose an appropriate φ, we can plug the constants $\lambda, \psi, \tau, k_1, k_2, k_3, k_4, \varrho$ into the formula from Theorem 4. The functions $f(z) = z/(1+z)$, $\varpi(z) = z/(1+z)$, $h(v) = 1 + v^2$ are all non-decreasing for $z, v \ge 0$.We can compute (or estimate numerically) the values of F^{-1} and Ω^{-1} to obtain a numerical value for C_φ, demonstrating that a stability constant exists for this problem.

CONCLUSION

This study successfully established generalized Hyers-Ulam-Rassias stability for two significant classes of third-order nonlinear differential equations with complex perturbations. By imposing a structured form on the perturbation term and employing nonlinear Bihari-type integral inequalities, explicit, computable formulas for the Hyers-Ulam-Rassias constants. The results significantly extend the existing stability theory for higher-order nonlinear ODEs. The presented method is systematic and can potentially be adapted to other classes of equations or BVPs. Future work may focus on relaxing the structural criterion (C_1) or applying these results to specific mathematical models.

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Conflict of Interest

The author declares that there is no conflict of interest regarding this paper's publication..

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