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A Modified Hestenes-Stiefel Type Method for Finding an Approximate Solution of Nonlinear Monotone Equations

***[1]Aliyu Yusuf and Paul [2]Victor Eshiokhe**

[1]Department of Science, School of Continuing Education, Bayero University, Kano, Nigeria
[1]Numerical Optimization Research Group, Department of Mathematical Sciences, Bayero University, Kano State, Nigeria.
[2]Berger Paint Junction, Suleja, Niger State, Nigeria.

*Corresponding authors' email: ayusuf.sce@buk.edu.ng

ABSTRACT
This research presents a Modified Hestenes-Stiefel (HS) conjugate gradient method for solving systems of monotone equations. A modified Hestenes-Stiefel (MHS) type method designed to efficiently obtain approximate solutions for monotone equations. The paper introduces an innovative three-term derivative-free projection algorithm based on a modified Hestenes-Stiefel (MHS) parameter for solving nonlinear monotone equations. The proposed algorithm is designed to be effective, derivative-free, and exhibits low memory requirements, making it particularly suitable for large-scale problems. A key feature of the algorithm is its ability to generate bounded descent search directions at each iteration, independent of the line search procedures. Under standard assumptions, we establish the global convergence properties of the method. Comprehensive numerical experiments demonstrate the efficiency of the algorithm in handling large-scale nonlinear monotone equations. Preliminary numerical experiments indicate that the propose method is promising and competitive with existing techniques for solving monotone equations. The promising results suggest that the MHS method is a viable and efficient alternative for solving a wide range of monotone equations.

Keywords: Derivative-free Method, Nonlinear Monotone Equations, Projection Method, Global Convergence, Numerical Experiments, Signal Recovery Problems

INTRODUCTION
Constrained systems of nonlinear monotone equations constitute a class of problems that can be formulated as follows:

$$F(x) = 0, x \in G, \quad (1)$$

Where $G \subseteq \mathbb{R}^n$ represents a convex, closed, and nonempty set. The mapping $F : F^n \to F^n$ satisfies the monotonicity condition, that is

$$(F(x) - F(y))^T (x - y) \geq 0, \text{ for all } x, y \in \mathbb{R}^n \quad (2)$$

Problem (1) has attracted considerable research interest due to its wide-ranging applications across multiple domains. These include economic equilibrium models (Dirkse & Ferris, 1995; Wang et al., 2014), financial forecasting systems (Kang et al., 2020), power flow analysis (Chen et al., 2012; Wang & Wang, 2009), and generalized proximal algorithms incorporating Bregman distances (Iusem & Solodov, 1997). Furthermore, monotone variational inequalities can be reformulated as monotone nonlinear equations (Fukushima, 1992; Yusuf et al., 2025a; Yusuf et al., 2025b). This formulation also proves valuable in sparse signal and image reconstruction problems.
Various numerical approaches have been developed for solving (1), including Levenberg-Marquardt techniques, quasi-Newton methods, Newton-type algorithms, and their numerous variants (Dennis & Moré, 1974; Ioannis, 2007; Mohammad & Waziri, 2015). These methods typically exhibit superlinear convergence properties (Donghui & Fukushima, 1999; Guanglu & Chuan, 2005; Mohammad & Waziri, 2015; Zhifeng & Huan, 2020). However, they generally require computing or approximating the Jacobian matrix to solve systems of linear equations. To circumvent this limitation, many researchers have explored derivative-free methods. Notable contributions in this direction include the works presented in (Waziri et al., 2020; Yusuf et al., 2024a; Yusuf et al., 2024b; Yusuf et al., 2025a).
This research focuses on derivative-free three-term methods inspired by the modified Hestenes-Stiefel (HS) conjugate gradient approach. The classical HS method was originally proposed by Hestenes and Stiefel (1952) for unconstrained optimization problems. Yan et al. (Zhen et al., 2010) developed a globally convergent derivative-free method for large-scale nonlinear monotone equations by integrating two modified HS schemes with the projection technique introduced in (Solodov & Svaiter, 1998). Their approach successfully handled non-smooth equations, and numerical results confirmed its computational efficiency. Subsequently, Dai and Zhu (Zhifeng & Huan, 2020) extended the modified HS methodology to large-scale nonlinear monotone equations through the incorporation of a hyperplane projection strategy. Further improvements were reported by Ibrahim et al. (Abdulkarim et al., 2022), who proposed an accelerated derivative-free variant.
Koorapetse (2019) presented a novel three-term conjugate gradient-based projection method for solving large-scale nonlinear monotone equations, establishing both sufficient descent properties and global convergence. Notably, the HS-type, PRP, and DY methods can be viewed as special cases of the three-term conjugate gradient framework proposed in (Abubakar et al., 2021). Recent work by Jie and Zhong (2023) introduced a modified HS method that guarantees local and global convergence under standard Wolfe line search conditions. Their approach generates search directions dk that consistently satisfy sufficient descent properties independent of line search procedures while maintaining the conjugacy condition. Motivated by these developments, we extend this methodology to address nonlinear monotone equations and signal recovery applications.

The remainder of this paper is structured as follows: Section 2 presents the algorithmic framework and theoretical foundations. Section 3 establishes the boundedness properties and global convergence analysis. Sections 4 report the numerical experiments using bench mark test problems of nonlinear monotone equations. Conclusions are presented in Section 5.

Throughout this manuscript, $\|\cdot\|$ denotes the Euclidean norm, with $F_k = F(x_k)$ and $y_k = F_k - F_{k-1}$. The operator H_G [.] represents the orthogonal projection onto the convex set G, defined as $H_G[x] = \text{argmin}\{\| x - z \|: z \in G\}$ for any nonempty, convex, and closed set $G \subset \mathbb{R}^n$.

Algorithmic Framework

This section introduces the conceptual foundations and implementation details of our proposed algorithm. Building upon the ideas presented in (Jie & Zhong, 2023), we define the search direction as follows:

$$d_k = \begin{cases} -F_k, & if \quad k = 0 \\ -F_k + \beta_k^{MHS} d_{k-1} - \gamma_k(y_{k-1}), & if \quad k \geq 1, \end{cases} \quad (3)$$

Where,

$$\ell_k^{MHS} = \frac{F_k^T(F_k - F_{k-1})}{\mu(F_k^T d_{k-1}) + (d_{k-1}^T y_{k-1})} \quad (4)$$

$$\gamma_k = \frac{F_k^T d_{k-1}}{\mu(F_k^T d_{k-1}) + (d_{k-1}^T y_{k-1})} \quad (5)$$

It is worth emphasizing that this method originated from an unconstrained optimization problem (Jie & Zhong, 2023) and requires appropriate modifications to extend the method to constrained nonlinear monotone equations.

Remark 2.1.1 The direction is defined piecewise. For the initial step, it defaults to the steepest descent; for subsequent steps, it blends the current gradient with the previous direction and a correction term.

$$d_k = \begin{cases} -F_k, & if \quad k = 0 \\ -F_k + \beta_k^{MHS} d_{k-1} - \gamma_k(y_{k-1}), & if \quad k \geq 1, \end{cases} \quad (6)$$

MATERIALS AND METHODS

The Modified Parameters

The scalars β_k^{MHS} and γ_k are designed to improve the conjugacy and stability of the algorithm. They share a common denominator, to avoid division by zero and ensure the descent property.

The MHS Update Parameter:

$$\beta_k^{MHS} = \frac{F_k^T(F_k - F_{k-1})}{\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})} \quad (7)$$

The Acceleration Parameter:

$$\gamma_k = \frac{F_k^T d_{k-1}}{\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})} \quad (8)$$

The Denominator: The term $\mu(F_k^T d_{k-1}) + (d_{k-1} y_{k-1})$ is a safeguard. In standard CG, this is often just. $d_{k-1}^T y_{k-1}$. Adding μ the term (where $\mu \geq 0$) helps maintain $d_k^T F_k < 0$.

The γ_k term: This acts as a quasi-Newton correction. It adjusts the direction based on the change in the gradient. $(F_k - F_{k-1})$, which helps the algorithm adapt to the curvature of the function more effectively than standard Hestenes-Stiefel.

We modified the denominator of Hestenes-Stifel as follows:

$$d_{k-1}^T F_{k-1} = d_{k-1}^T(F_{k-1} + t d_{k-1}) = d_{k-1}^T F_{k-1} + t\|d_{k-1}\|^2 = d_{k-1}^T F_{k-1} + \left(1 + max\left\{0, \frac{-d_{k-1}^T F_{k-1}}{\|d_{k-1}\|^2}\right\}\right)\|d_{k-1}\|^2 \geq d_{k-1}^T F_{k-1} + \|d_{k-1}\|^2 - d_{k-1}^T F_{k-1} \geq \|d_{k-1}\| > 0 \quad (9)$$

This establishes the positivity of $d_{k-1}^T P_{k-1}$ whenever $d_{k-1} \neq 0$.

For the case of the second term of the denominator of our d_k that is $(F_k^T d_{k-1})$ since our proposed direction d_k is descent as shown in Remark 1 below:

Remark 1:

We want to show that our proposed direction is always descent.

We start with the definition of d_k for k = o then k ≥ 1

When k = 0

$$F_0^T d_0 = -F_0^T F_0 = \|F_0\|^2$$

$$F_0^T d_0 \leq -\|F_0\|^2 \quad (10)$$

When k ≥ 1

$$d_k = -F_k + \beta_k^{MHS} d_{k-1} - \gamma_k(y_{k-1}) \quad (11)$$

Multiply equation (11) both sides by F_k^T.

$$F_k^T d_k = -\|F_0\|^2 + \beta_k^{MHS} F_k^T d_{k-1} - \gamma_k F_k^T(y_{k-1})$$

$$F_k^T d_k = -\|F_0\|^2 + \beta_k^{MHS} F_k^T d_{k-1} - \gamma_k F_k^T(y_{k-1}) \quad (12)$$

Where;

$$\beta_k^{MHS} = \frac{F_k^T(y_k)}{\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})} \quad (13)$$

And

$$\gamma_k = \frac{F_k^T d_{k-1}}{\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})} \quad (14)$$

Substitute equation (13) and equation (14) into equation (12)

$$F_k^T d_k = -\|F_0\|^2 + \frac{(F_k^T y_k)(F_k^T d_{k-1})}{\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})} - \frac{(F_k^T d_{k-1})(F_k^T y_k)}{\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})} \quad (15)$$

Equation (15) deduce to:

$$F_k^T d_k \leq -\|F_0\|^2 \quad (16)$$

Then the denominator of the third term of d_k is always descent that is either is equal to zero (0) or negative number, by considering the whole denominator of second and third terms of d_k that is $\mu(d_{k-1}^T y_{k-1}) - (F_k^T d_{k-1})$, we can see that, $\mu > 0$, $(d_{k-1}^T y_{k-1}) > 0$ whenever $d_k \neq 0$ and $(F_k^T d_{k-1})$ is either zero (0) or negative number, so in any case of $(F_k^T d_{k-1})$ the whole denominator of d_k is positive.

Algorithm 1: Modified Hestenes-Stiefel Algorithm (MHS)

Input, choose $x_k \in \mathbb{R}^n$, $0 < \theta < 2$, $\rho \in (0,1)$, μ, ε, $q > 0$. Set $k = 0$.

Step 1. If $\|F(x_k)\| < \varepsilon$ Then terminate. Else go to **Step 2**.

Step 2. Compute d_k using equations (3), equation (4) and equation (5).

Step 3. Compute the step-size $v_k = \mu\rho^i$, where $i = 1, 2, 3, \ldots$, is the least positive integer that satisfies the following inequality:

$$-F(x_k + v_k d_k)^T d_k \geq q v_k \|d_k\|^2 \quad (17)$$

Step 4. Let $\gamma_k = x_k + v_k d_k$. If $\gamma_k \in G$ and $\|F(\gamma_k)\| < \varepsilon$, stop. Else, compute H_G by:

$$x_{k+1} = H_G[x_k - \theta\zeta_k F(\gamma_k)], \quad (18)$$

Where, $\zeta_k = \frac{F(\gamma_k)^T(x_k - \gamma_k)}{\|F(\gamma_k)\|^2}$ (19)

Step 5. Finally, set $k = k + 1$ and repeat from **Step 1**.

RESULTS AND DISCUSSION

Convergence Analysis

This section establishes the global convergence properties of the MHS algorithm. We begin by stating the fundamental assumptions underlying our analysis.

Condition 3.1 The feasible set G ⊆ $\mathbf{R}^n$ is closed, convex and nonempty.

Condition 3.2 The mapping F is both monotone and L-Lipschitz continuous on $\mathbf{R}^n$. Specifically,

$$(F(x_1) - F(x_2))^T(x_1 - x_2) \geq 0, \text{ for all } x_1, x_2 \in \mathbb{R}^n, \quad (20)$$

And there exists a constant L > 0 such that

$$\| F(x_1) - F(x_2) \| \leq L \| x_1 - x_2 \|, \forall x_1, x_2 \in \mathbb{R}^n \quad (21)$$

Lemma 3.3 [1] Under Conditions 3.1 and 3.2, the sequences $\{x_k\}$ and $\{\Gamma_k\}$ generated by Algorithm 1 remain bounded. Moreover,

$$\lim_{k\to\infty} v_k \|d_k\| = 0 \quad (22)$$

Remark 3.4 Lemma 3.3 implies the existence of a constant. $\delta > 0$ Such that $\|x_k\| < \delta$ for all k. The continuity of F together with the boundedness of $\{x_k\}$ guarantees that $\{F_k\}$ is also bounded; consequently, there exists B > 0 satisfying $\|F_k\| \leq B$ for all k.

Lemma 3.5 The search direction defined by (3) satisfies the descent condition given by relation (7).

Proof: For $k = 0$, direct substitution yields $F_0^T d_0 \leq -\|F_k\|^2$.

For $k \geq 1$, using equations (3), (4) and (8), we obtain:

$$F_k^T d_k = -\|F_k\|^2 + \frac{F_k^T(y_{k-1} - d_{k-1})}{\lambda_k^*} F_k^T d_{k-1} - \frac{(F_k^T d_{k-1})(F_k^T y_{k-1})}{\lambda_k^*}. = -\|F_k\|^2 - \frac{(F_k^T d_{k-1})^2}{\lambda_k^*} + \frac{(F_k^T d_{k-1})(F_k^T y_{k-1})}{\lambda_k^*} - \frac{(F_k^T d_{k-1})(F_k^T y_{k-1})}{\lambda_k^*}. = -\|F_k\|^2 - \frac{(F_k^T d_{k-1})^2}{\lambda_k^*} \leq -\|F_k\|^2 \quad (23)$$

Thus, the required property holds for all k.■

Lemma 3.6: Assume Conditions 3.1 and 3.2 are satisfied. Let d_k and x_k, ζ_k be defined by equations (3), (18), and (19), respectively. Then:

(i) For every iteration k, there exists a step size $v_k = \mu\rho^i$ satisfying (17) for some non-negative integer i. (ii) The step size v_k satisfies the following bound:

$$v_k > min\left\{\mu, \frac{\rho\|F_k\|^2}{(L+q)\|d_k\|^2}\right\} \quad (24)$$

Proof

(i) We proceed by contradiction. Suppose there exists $k_0 \geq 0$ such that equation (17) fails for all integers $i \geq 0$. Then:

$$-F(x_k + v_k d_k)^T d_k \geq q v_k \|d_{k_0}\|^2, \quad \forall \text{ i} \quad (25)$$

Taking the limit as $i \to \infty$ and exploiting the continuity of F, we obtain:

Thus, equation (24) is well established. ■

-F (x_(k_0))^T d_(k_0)⩽0 (26)

However, equation (16) yields:

-F(x_(k_0))^T d_(k_0)= ‖ F(x_(k_0)) ‖ ^2 > 0,

Which contradicts (16). Hence, a suitable step size must exist. When v_k≠ μ the step size v_k^I=v_k/ ρ fails to satisfy equation (17), implying:

-F (x_k+v_k^I d_k) ^T d_k⩾ 〖qv_k^I〗 _k ‖ d_k ‖ ^2 (27)

From (7), we have:

‖ F_k ‖ ^2=-F_k^T d_k= (F (x_k+v_k^I d_k)-F_k)^T d_k-F(x_k+v_k^I d_k)^T d_k (28)

Substituting equation (27) into equation (28), by using the Cauchy-Schwarz inequality and the Lipschitz continuity of F, we obtain:

$$\|d_k\|^2 < \|(F(x_k + \alpha_k' d_k) - F_k)\|\|d_k\| + t\alpha_k'\|d_k\|^2 < L\alpha_k'\|d_k\|^2 + t\alpha_k'\|d_k\|^2 = (L+t)\alpha_k'\|d_k\|^2. \quad (29)$$

Substituting $v_k' = \frac{v_k}{\rho}$ In equation (29), we have:

$$v_k > \frac{\rho\|F_k\|^2}{(L+t)\|d_k\|^2} \quad (30)$$

Theorem 1

Let $\{x_k\}$ be a sequence defined by equation (18), if Condition 3.1 and Condition 3.2 are satisfied then,

$$\lim_{k\to\infty} inf \|F_k\| = 0 \quad (31)$$

Consequently, $\{x_k\}$ converges to a solution of problem (1).

Proof

By contradiction, suppose that

$$\lim_{k\to\infty} inf \|F_k\| \neq 0 \quad (32)$$

Then there exists a constant $m > 0$ such that for all $k \geq 0$,

$$\|F_k\| \geq m \quad (33)$$

Using Cauchy-Schwarz inequality, (16) and (33) can be deduced as:

$$\|d_k\| \geq \|P_k\| \geq m \quad (34)$$

From Lemma 3.3, both $\{F_k\}$ and $\{x_k\}$ are bounded. We want to establish the boundedness of $\{d_k\}$. For k = 0, we have:

$$\|d_0\| = \|P_0\| \leq m \quad (35)$$

For k ≥ 1, by triangle inequality and the Cauchy-Schwarz inequality, equation (3) gives:

$$\|d_k\| = \|-F_k + \beta_k^{MHS} d_{k-1} - \gamma_k(y_{k-1})\| \quad (36)$$

From equations (4) and (5), we have:

$$\|d_k\| \leq \|F_k\| + \frac{\|F_k\|\|y_{k-1}\|}{|\mu|(\|F_{k-1}\|\|d_{k-1}\| + \|d_{k-1}\|\|y_{k-1}\|)}\|d_{k-1}\| + \frac{\|F_k\|\|d_{k-1}\|}{|\mu|(\|F_{k-1}\|\|d_{k-1}\| + \|d_{k-1}\|\|y_{k-1}\|)}\|y_{k-1}\| \quad (37)$$

$$\|d_k\| \leq \|F_k\| + \frac{\|F_k\|\|y_{k-1}\|}{|\mu|(\|F_{k-1}\| + \|y_{k-1}\|)} + \frac{\|F_k\|\|y_{k-1}\|}{|\mu|(\|F_{k-1}\| + \|y_{k-1}\|)} \quad (38)$$

$$\|d_k\| \leq \|F_k\| + \frac{\|F_k\|\|y_{k-1}\|}{|\mu|(\|F_{k-1}\| + \|y_{k-1}\|)} + \frac{\|F_k\|\|y_{k-1}\|}{|\mu|(\|F_{k-1}\| + \|y_{k-1}\|)} \quad (39)$$

Using the property of norm, relation (39) reduces to:

$$\|d_k\| \leq \|F_k\| + \frac{\|F_{k-1}\| + \|y_{k-1}\|}{|\mu|(\|F_{k-1}\| + \|y_{k-1}\|)} + \frac{\|F_{k-1}\| + \|y_{k-1}\|}{|\mu|(\|F_{k-1}\| + \|y_{k-1}\|)} \quad (40)$$

$$\|d_k\| \leq \|F_k\| + \frac{1}{|\mu|} + \frac{1}{|\mu|} \quad (41)$$

By letting $M = B + \frac{2}{|\mu|}$, we have:

$$\|d_k\| \leq M \quad for\ all\ k \quad (42)$$

Multiplying inequality (30) both side by $\|d_k\|$ gives:

$$v_k\|d_k\| > min\left\{\mu\|F_k\|, \frac{\rho\|F_k\|^2}{(L+q)\|d_k\|^2}\right\} \geq min\left\{\mu m, \frac{\rho m^2}{(L+q)M^2}\right\} > 0 \quad (43)$$

This contradicts equation (22), which asserts that $v_k \|d_k\| \to 0$. Therefore, our initial assumption must be false, and (31) holds. The continuity of F with (31) guarantees that $\{x_k\}$ possesses an accumulation point $\bar{x}_k$ satisfying $F(\bar{x}_k) = 0$, that is x¯is a solution to (1). Since $\bar{x}_k$ is an accumulation point, then Lemma 3.3 implies convergence of $\{\|x_k - \bar{x}_k\|\}$, and consequently $\{x_k\}$ converges to $\bar{x}_k$.

Computational Experiments

This section demonstrates the numerical performance of the MHS algorithm through comparative scheme with established methods. All experiments were conducted on a personal computer equipped with 4GB RAM and a 2.13 GHz processor. The test problems were implemented in MATLAB. The following experimental setup was employed:

i. Five distinct Initial Points (IP):

$x_1 = (0.1, 0.1, \dots, 0.1)^T$,
$x_2 = (0.2, 0.2, \dots, 0.2)^T$,
$x_3 = (0.5, 0.5, \dots, 0.5)^T$,
$x_3 = (0.5, 0.5, \dots, 0.5)^T$
$x_4 = (1.5, 1.5, \dots., 1.5)^T$
$x_5 = (2, 2, \dots, 2)^T$

i. Five problem dimensions: 1000, 5000, 10000, 50000, 100000.
ii. Eight benchmark problems (detailed in Table 1).

Algorithm parameters were set as follows: $q = 0.0001$, $\rho = 0.8$, $\mu = 1$, $\theta = 1.2$. The stopping criterion was $\| F_k \| \leq 10^{-5}$. Performance comparisons were conducted against the MHS method of Zhen and Li (2010), the MCDPM method of Aji et al. (2020), and the PRPFR method proposed by Yuan et al. (2020). Evaluation metrics included number of iterations (NOI), number of function evaluations (NFE), and CPU time in seconds (TIME).

Table 1: Description of Test Problems

No.	Problem Description and Reference
1	Modified exponential function 2 (La Cruz et al., 2006)
2	Logarithmic function (La Cruz et al., 2006)
3	Problem 1 from Zhao and Li (2001)
4	Strictly convex function I (La Cruz et al., 2006)
5	Strictly convex function II (La Cruz et al., 2006)
6	Tridiagonal exponential function (Bing & Lin, 1991)
7	Nonsmooth function (Abubakar et al., 2020)
8	Problem 4 from Ding et al. (2017)

The complete numerical results are presented in tables 2, 3, 4, 5, 6, 7, 8 and 9. For clarity, scientific notation is used to represent the final residual norms.

Table 2: MHS Results of Experiment for Problem 1 Compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	1	7	0.0047	0	29	87	0.0405	9.59e-06	7	22	0.0312	9.24e-06	1	6	0.0325	0
	x_2	1	7	0.0023	0	31	94	0.0451	9.97e-06	7	23	0.0088	7.02e-06	1	6	0.0159	0
	x_3	1	7	0.0082	0	34	102	0.0206	9.84e-06	7	22	0.0054	3.88e-06	17	70	0.0075	7.83e-07
	x_4	1	9	0.0034	0	31	93	0.0273	8.64e-06	21	65	0.0064	6.29e-06	18	74	0.0116	8.83e-07
	x_5	1	10	0.0035	0	1	4	0.0037	0	1	6	0.0029	0	9	38	0.0100	8.36e-07
5,000	x_1	1	7	0.0078	0	28	85	0.0588	9.63e-06	7	23	0.0128	9.97e-06	1	6	0.0076	0
	x_2	1	7	0.0170	0	31	93	0.0360	8.42e-06	8	26	0.0148	2.12e-06	1	6	0.0067	0
	x_3	1	7	0.0094	0	33	100	0.0403	9.92e-06	7	22	0.0133	7.23e-06	17	70	0.0403	7.83e-07
	x_4	1	9	0.0138	0	31	94	0.0385	9.95e-06	22	68	0.0317	6.76e-06	17	69	0.0380	9.69e-07
	x_5	1	10	0.0145	0	1	4	0.0043	0	1	6	0.0048	0	10	41	0.0216	9.79e-07
10,000	x_1	1	7	0.0089	0	28	84	0.0733	9.29e-06	8	26	0.0226	1.92e-06	1	6	0.0084	0
	x_2	1	7	0.0215	0	30	91	0.0876	9.63e-06	8	26	0.0224	2.99e-06	1	6	0.0089	0
	x_3	1	7	0.0292	0	33	99	0.1008	9.54e-06	7	22	0.0174	9.96e-06	17	70	0.0420	7.83e-07
	x_4	1	9	0.0093	0	32	96	0.0839	8.56e-06	22	68	0.0462	9.55e-06	16	66	0.0536	9.62e-07
	x_5	1	10	0.0209	0	1	4	0.0114	0	1	6	0.0064	0	10	42	0.0308	3.62e-07
50,000	x_1	1	7	0.0527	0	28	84	0.2385	7.80e-06	8	26	0.0816	4.28e-06	1	6	0.0210	0
	x_2	1	7	0.0600	0	30	90	0.3050	8.36e-06	8	26	0.0895	6.68e-06	1	6	0.0221	0
	x_3	1	7	0.0287	0	32	97	0.3011	9.79e-06	8	26	0.0735	1.44e-06	17	70	0.1358	7.83e-07
	x_4	1	9	0.1162	0	33	99	0.2583	7.81e-06	23	72	0.2024	8.72e-06	15	62	0.1626	8.85e-07
	x_5	1	10	0.1358	0	1	4	0.0308	0	1	6	0.0217	0	10	42	0.1079	8.11e-07
100,000	x_1	1	7	0.0510	0	27	82	0.3805	9.68e-06	8	26	0.1689	6.06e-06	1	6	0.0362	0
	x_2	1	7	0.0516	0	29	88	0.3797	9.95e-06	8	26	0.1824	9.45e-06	1	6	0.0399	0
	x_3	1	7	0.0450	0	32	96	0.3557	9.54e-06	8	26	0.1676	2.03e-06	17	70	0.2869	7.83e-07
	x_4	1	9	0.1699	0	33	99	0.5366	8.20e-06	24	74	0.4688	7.01e-06	15	61	0.2661	8.80e-07
	x_5	1	10	0.1008	0	1	4	0.0257	0	1	6	0.0440	0	11	45	0.1737	5.96e-07

Table 3: MHS Results of Experiment for Problem 2 Compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	5	17	0.0048	2.69e-06	41	122	0.0515	8.12e-06	10	29	0.0073	6.78e-06	8	23	0.0144	8.24e-07
	x_2	5	17	0.0047	3.77e-06	42	125	0.0179	8.01e-06	10	29	0.0067	8.75e-06	8	23	0.0101	6.96e-07
	x_3	5	16	0.0063	2.52e-06	35	104	0.0449	9.94e-06	10	28	0.0076	4.63e-06	8	24	0.0112	3.73e-07
	x_4	5	16	0.0096	3.74e-06	36	106	0.0174	8.80e-06	10	28	0.0101	5.12e-06	10	30	0.0092	4.77e-07
	x_5	5	16	0.0045	4.40e-06	41	120	0.0286	9.67e-06	12	33	0.0118	4.69e-06	10	30	0.0121	4.77e-07
5,000	x_1	5	17	0.0143	5.33e-06	43	129	0.0696	9.94e-06	11	32	0.0329	4.20e-06	8	24	0.0258	3.29e-07
	x_2	5	17	0.0150	7.48e-06	44	132	0.0900	9.84e-06	11	32	0.0327	5.45e-06	8	24	0.0283	2.91e-07
	x_3	5	16	0.0264	4.77e-06	38	112	0.0978	9.91e-06	10	28	0.0203	9.91e-06	8	24	0.0288	3.52e-07
	x_4	5	16	0.0307	6.82e-06	39	116	0.1324	9.78e-06	10	29	0.0387	9.47e-06	11	32	0.0338	5.29e-07
	x_5	5	16	0.0144	7.11e-06	44	129	0.0883	9.31e-06	12	34	0.0245	7.14e-06	11	32	0.0332	5.29e-07
10,000	x_1	3	9	0.0133	6.46e-06	45	134	0.1661	8.54e-06	11	32	0.0446	5.94e-06	8	24	0.0465	4.58e-07
	x_2	6	19	0.0304	1.78e-06	46	137	0.2143	8.46e-06	11	32	0.0459	7.69e-06	8	24	0.0304	4.07e-07
	x_3	5	16	0.0673	6.60e-06	40	118	0.1703	8.06e-06	10	29	0.0508	9.76e-06	8	24	0.0344	3.49e-07
	x_4	5	16	0.0737	9.38e-06	41	121	0.1242	8.57e-06	11	31	0.0396	5.46e-06	11	32	0.0629	7.47e-07
	x_5	5	16	0.0241	9.62e-06	45	132	0.2928	9.98e-06	13	36	0.0502	4.03e-06	11	32	0.0527	7.47e-07
50,000	x_1	5	17	0.1818	4.25e-07	48	143	0.8830	8.38e-06	11	33	0.1799	9.28e-06	9	26	0.1318	5.06e-07
	x_2	6	19	0.2374	3.93e-06	49	146	0.6727	8.30e-06	12	35	0.1793	4.81e-06	8	24	0.1251	9.03e-07
	x_3	5	17	0.0882	4.28e-07	42	125	0.7231	9.85e-06	11	31	0.1699	8.69e-06	8	24	0.1193	3.47e-07
	x_4	5	17	0.0753	6.06e-07	44	130	0.7131	8.54e-06	11	32	0.1552	8.67e-06	11	33	0.1577	3.50e-07
	x_5	5	17	0.0787	6.12e-07	48	141	0.6030	9.78e-06	13	36	0.1767	8.98e-06	11	33	0.1773	3.50e-07
100,000	x_1	5	17	0.3812	5.99e-07	49	146	1.0319	9.01e-06	12	35	0.3151	5.25e-06	9	26	0.2957	7.14e-07
	x_2	6	19	0.1678	5.55e-06	50	149	1.1322	8.92e-06	12	35	0.3322	6.80e-06	9	26	0.2572	6.38e-07
	x_3	5	17	0.1491	6.03e-07	44	130	1.3780	8.47e-06	11	32	0.2797	8.60e-06	8	24	0.1927	3.47e-07
	x_4	5	17	0.2361	8.53e-07	45	133	0.8626	9.20e-06	12	34	0.3034	4.91e-06	11	33	0.3486	4.94e-07
	x_5	5	17	0.2240	8.60e-07	49	145	1.0232	9.98e-06	13	37	0.3271	8.88e-06	11	33	0.3661	4.94e-07

Table 4: MHS Results of Experiment for Problem 3 Compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	–	–	–	0	47	141	0.0225	8.32e-06	10	31	0.0080	9.32e-06	1	5	0.0040	0
	x_2	–	–	–	0	49	147	0.0188	9.57e-06	11	33	0.0099	7.35e-06	1	5	0.0033	0
	x_3	–	–	–	0	52	157	0.0264	9.73e-06	12	36	0.0099	4.66e-06	1	5	0.0040	0
	x_4	9	31	0.0036	6.08e-06	56	168	0.0202	8.28e-06	11	33	0.0089	7.77e-06	1	5	0.0040	0
	x_5	–	–	–	0	56	168	0.0183	8.28e-06	13	39	0.0081	6.13e-06	1	5	0.0041	0
5,000	x_1	–	–	–	0	50	150	0.0826	8.16e-06	11	33	0.0259	8.34e-06	1	5	0.0056	0
	x_2	–	–	–	0	52	156	0.0805	9.40e-06	12	36	0.0246	4.60e-06	1	5	0.0090	0
	x_3	–	–	–	0	55	166	0.1051	9.55e-06	12	37	0.0232	7.29e-06	1	5	0.0105	0
	x_4	10	33	0.0186	4.30e-06	59	177	0.0910	8.13e-06	12	36	0.0241	4.87e-06	1	5	0.0113	0
	x_5	–	–	–	0	59	177	0.1063	8.13e-06	13	40	0.0259	9.60e-06	1	5	0.0095	0
10,000	x_1	–	–	–	0	51	153	0.1529	8.77e-06	11	34	0.0406	8.25e-06	1	5	0.0125	0
	x_2	–	–	–	0	53	160	0.1379	9.59e-06	12	36	0.0377	6.51e-06	1	5	0.0095	0
	x_3	–	–	–	0	57	171	0.1729	8.21e-06	13	39	0.0408	4.12e-06	1	5	0.0138	0
	x_4	10	33	0.0275	6.08e-06	60	180	0.1544	8.73e-06	12	36	0.0354	6.88e-06	1	5	0.0134	0
	x_5	–	–	–	0	60	180	0.1449	8.73e-06	14	42	0.0462	5.43e-06	1	5	0.0081	0
50,000	x_1	–	–	–	0	54	162	0.5012	8.61e-06	12	36	0.1505	7.38e-06	1	5	0.0385	0
	x_2	–	–	–	0	56	168	0.5242	9.91e-06	13	39	0.1547	4.08e-06	1	5	0.0366	0
	x_3	–	–	–	0	60	180	0.5406	8.06e-06	13	39	0.1577	9.22e-06	1	5	0.0216	0
	x_4	10	34	0.2125	7.70e-06	63	189	0.5762	8.57e-06	13	39	0.1590	4.31e-06	1	5	0.0355	0
	x_5	–	–	–	0	63	189	0.5815	8.57e-06	14	43	0.1855	8.50e-06	1	5	0.0227	0
100,000	x_1	–	–	–	0	55	165	0.9004	9.26e-06	12	37	0.3255	7.31e-06	1	5	0.0731	0
	x_2	–	–	–	0	58	174	1.0207	8.10e-06	13	39	0.2766	5.77e-06	1	5	0.0373	0
	x_3	–	–	–	0	61	183	1.0622	8.66e-06	13	40	0.3014	9.13e-06	1	5	0.0395	0
	x_4	11	36	0.7042	3.44e-06	64	192	1.0626	9.21e-06	13	39	0.2869	6.10e-06	1	5	0.0387	0
	x_5	–	–	–	0	64	192	1.0574	9.21e-06	15	45	0.3550	4.81e-06	1	5	0.0389	0

Table 5: MHS Results of Experiment for Problem 4 Compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	–	–	–	0	47	141	0.0225	8.32e-06	10	31	0.0080	9.32e-06	1	5	0.0040	0
	x_2	–	–	–	0	49	147	0.0188	9.57e-06	11	33	0.0099	7.35e-06	1	5	0.0033	0
	x_3	–	–	–	0	52	157	0.0264	9.73e-06	12	36	0.0099	4.66e-06	1	5	0.0040	0
	x_4	9	31	0.0036	6.08e-06	56	168	0.0202	8.28e-06	11	33	0.0089	7.77e-06	1	5	0.0040	0
	x_5	–	–	–	0	56	168	0.0183	8.28e-06	13	39	0.0081	6.13e-06	1	5	0.0041	0
5,000	x_1	–	–	–	0	50	150	0.0826	8.16e-06	11	33	0.0259	8.34e-06	1	5	0.0056	0
	x_2	–	–	–	0	52	156	0.0805	9.40e-06	12	36	0.0246	4.60e-06	1	5	0.0090	0
	x_3	–	–	–	0	55	166	0.1051	9.55e-06	12	37	0.0232	7.29e-06	1	5	0.0105	0
	x_4	10	33	0.0186	4.30e-06	59	177	0.0910	8.13e-06	12	36	0.0241	4.87e-06	1	5	0.0113	0
	x_5	–	–	–	0	59	177	0.1063	8.13e-06	13	40	0.0259	9.60e-06	1	5	0.0095	0
10,000	x_1	–	–	–	0	51	153	0.1529	8.77e-06	11	34	0.0406	8.25e-06	1	5	0.0125	0
	x_2	–	–	–	0	53	160	0.1379	9.59e-06	12	36	0.0377	6.51e-06	1	5	0.0095	0
	x_3	–	–	–	0	57	171	0.1729	8.21e-06	13	39	0.0408	4.12e-06	1	5	0.0138	0
	x_4	10	33	0.0275	6.08e-06	60	180	0.1544	8.73e-06	12	36	0.0354	6.88e-06	1	5	0.0134	0
	x_5	–	–	–	0	60	180	0.1449	8.73e-06	14	42	0.0462	5.43e-06	1	5	0.0081	0
50,000	x_1	–	–	–	0	54	162	0.5012	8.61e-06	12	36	0.1505	7.38e-06	1	5	0.0385	0
	x_2	–	–	–	0	56	168	0.5242	9.91e-06	13	39	0.1547	4.08e-06	1	5	0.0366	0
	x_3	–	–	–	0	60	180	0.5406	8.06e-06	13	39	0.1577	9.22e-06	1	5	0.0216	0
	x_4	10	34	0.2125	7.70e-06	63	189	0.5762	8.57e-06	13	39	0.1590	4.31e-06	1	5	0.0355	0
	x_5	–	–	–	0	63	189	0.5815	8.57e-06	14	43	0.1855	8.50e-06	1	5	0.0227	0
100,000	x_1	–	–	–	0	55	165	0.9004	9.26e-06	12	37	0.3255	7.31e-06	1	5	0.0731	0
	x_2	–	–	–	0	58	174	1.0207	8.10e-06	13	39	0.2766	5.77e-06	1	5	0.0373	0
	x_3	–	–	–	0	61	183	1.0622	8.66e-06	13	40	0.3014	9.13e-06	1	5	0.0395	0
	x_4	11	36	0.7042	3.44e-06	64	192	1.0626	9.21e-06	13	39	0.2869	6.10e-06	1	5	0.0387	0
	x_5	–	–	–	0	64	192	1.0574	9.21e-06	15	45	0.3550	4.81e-06	1	5	0.0389	0

Table 6: TTHS Results of Experiment for Problem 5 Compared with PRPFR, MHS and MCD

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	1	4	0.0170	0	46	139	0.0104	9.75e-06	10	31	0.0061	7.77e-06	1	6	0.0092	0
	x_2	1	4	0.0082	0	49	147	0.0114	8.44e-06	11	33	0.0042	5.05e-06	1	5	0.0065	0
	x_3	2	8	0.0105	0	51	154	0.0227	9.57e-06	11	33	0.0035	4.15e-06	1	5	0.0106	2.22e-16
	x_4	1	7	0.0088	0	53	159	0.0121	9.18e-06	1	5	0.0032	0	21	85	0.0214	7.15e-07
	x_5	1	9	0.0146	0	52	156	0.0188	9.60e-06	47	143	0.0163	9.29e-06	21	85	0.0184	7.15e-07
5,000	x_1	1	4	0.0042	0	49	148	0.0613	9.57e-06	11	33	0.0167	6.95e-06	1	6	0.0048	0
	x_2	1	4	0.0081	0	52	156	0.0713	8.29e-06	11	34	0.0151	7.90e-06	1	5	0.0037	0
	x_3	2	8	0.0120	0	54	162	0.0742	9.89e-06	11	33	0.0151	9.28e-06	1	5	0.0045	2.22e-16
	x_4	1	7	0.0104	0	56	168	0.0768	9.01e-06	1	5	0.0043	0	22	89	0.0671	7.38e-07
	x_5	1	9	0.0160	0	55	165	0.0553	9.42e-06	50	152	0.0735	8.44e-06	22	89	0.0799	7.38e-07
10,000	x_1	1	4	0.0112	0	51	153	0.0829	8.23e-06	11	33	0.0286	9.83e-06	1	6	0.0069	0
	x_2	1	4	0.0167	0	53	159	0.0985	8.91e-06	12	36	0.0288	4.47e-06	1	5	0.0053	0
	x_3	2	8	0.0149	0	56	168	0.1017	8.08e-06	11	34	0.0283	9.19e-06	1	5	0.0064	2.22e-16
	x_4	1	7	0.0081	0	57	171	0.1008	9.68e-06	1	5	0.0062	0	23	92	0.1029	7.31e-07
	x_5	1	9	0.0079	0	56	169	0.0929	9.62e-06	51	155	0.1158	8.85e-06	23	92	0.0935	7.31e-07
50,000	x_1	1	4	0.0502	0	54	162	0.3860	8.08e-06	12	36	0.0928	6.16e-06	1	6	0.0323	0
	x_2	1	4	0.0375	0	56	168	0.3355	8.75e-06	12	36	0.0962	9.99e-06	1	5	0.0499	0
	x_3	2	8	0.0259	0	58	175	0.4236	9.91e-06	12	36	0.0947	8.22e-06	1	5	0.0157	2.22e-16
	x_4	1	7	0.0229	0	60	180	0.3840	9.50e-06	1	5	0.0158	0	24	96	0.2440	7.52e-07
	x_5	1	9	0.0259	0	59	177	0.3929	9.94e-06	54	164	0.3567	8.04e-06	24	96	0.1974	7.52e-07
100,000	x_1	1	4	0.0775	0	55	165	0.6428	8.68e-06	12	36	0.1633	8.71e-06	1	6	0.0717	0
	x_2	1	4	0.0525	0	57	171	0.7653	9.40e-06	12	37	0.1575	9.89e-06	1	5	0.0248	0
	x_3	2	8	0.0399	0	60	180	0.6388	8.52e-06	12	37	0.1810	8.13e-06	1	5	0.0334	2.22e-16
	x_4	1	7	0.0467	0	61	184	0.6649	9.70e-06	1	5	0.0312	0	24	97	0.5083	6.99e-07
	x_5	1	9	0.0472	0	61	183	0.5916	8.12e-06	55	167	0.6841	8.43e-06	24	97	0.5071	6.99e-07

Table 7: MHS results of experiment for problem 6 compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	21	66	0.0526	8.82e-06	59	177	0.0256	8.09e-06	13	39	0.0073	7.68e-06	28	113	0.0337	6.96e-07
	x_2	19	60	0.0063	6.04e-06	58	175	0.0263	9.72e-06	13	39	0.0096	7.38e-06	1001	4007	0.6696	2.07e+00
	x_3	52	164	0.0572	9.80e-06	58	174	0.0252	9.02e-06	13	39	0.0113	6.50e-06	1001	4007	0.6576	2.21e+00
	x_4	25	82	0.0355	9.84e-06	56	168	0.0409	8.57e-06	12	37	0.0131	8.93e-06	13	39	0.0103	3.53e-07
	x_5	66	208	0.0354	8.95e-06	54	162	0.0209	8.75e-06	12	36	0.0077	7.52e-06	39	156	0.0657	8.55e-07
5,000	x_1	25	77	0.0530	4.23e-06	61	184	0.0981	9.93e-06	14	42	0.0337	4.81e-06	1001	4012	3.5336	1.87e+00
	x_2	–	–	–	0	61	184	0.0926	9.55e-06	14	42	0.0317	4.63e-06	1001	4007	3.5429	5.24e+00
	x_3	71	222	0.0913	8.67e-06	61	183	0.1146	8.86e-06	14	42	0.0314	4.08e-06	1001	4007	3.0632	5.83e+00
	x_4	27	88	0.0494	9.02e-06	59	177	0.0952	8.42e-06	13	39	0.0330	8.00e-06	16	48	0.0354	4.96e-07
	x_5	–	–	–	0	57	171	0.1018	8.60e-06	13	39	0.0255	4.72e-06	41	164	0.0763	9.01e-07
10,000	x_1	24	74	0.0506	6.75e-06	63	189	0.1870	8.54e-06	14	42	0.0401	6.81e-06	1001	4012	6.1108	2.97e+00
	x_2	74	231	0.2061	9.96e-06	63	189	0.1726	8.22e-06	14	42	0.0403	6.55e-06	1001	4007	6.2559	7.52e+00
	x_3	71	223	0.2479	9.67e-06	62	186	0.1689	9.52e-06	14	42	0.0403	5.77e-06	1001	4007	6.3759	8.34e+00
	x_4	28	90	0.2574	9.86e-06	60	180	0.1623	9.05e-06	13	40	0.0538	7.92e-06	33	122	0.1552	8.42e-07
	x_5	75	235	0.1798	8.40e-06	58	174	0.1475	9.24e-06	13	39	0.0733	6.67e-06	42	168	0.2237	9.88e-07
50,000	x_1	25	77	0.3148	5.61e-06	66	198	0.6841	8.38e-06	15	45	0.3357	4.26e-06	1001	4013	23.7433	2.33e+00
	x_2	78	243	0.6318	8.68e-06	66	198	0.6602	8.06e-06	15	45	0.3199	4.10e-06	1001	4007	23.9866	1.70e+01
	x_3	75	234	0.7316	9.44e-06	65	195	0.6375	9.35e-06	14	43	0.2865	9.03e-06	1001	4007	23.8049	1.88e+01
	x_4	31	99	0.3037	6.97e-06	63	189	0.6468	8.89e-06	14	42	0.2920	7.09e-06	1001	4004	23.8772	1.95e+00
	x_5	81	253	0.9695	8.96e-06	61	183	0.7621	9.07e-06	14	42	0.3227	4.18e-06	47	187	0.9724	8.88e-07
100,000	x_1	25	77	0.4084	8.03e-06	67	201	1.4151	9.01e-06	15	45	0.6704	6.03e-06	1001	4013	44.0654	5.52e+00
	x_2	79	247	1.3003	9.74e-06	67	201	1.5302	8.67e-06	15	45	0.3599	5.80e-06	1001	4007	44.5654	2.41e+01
	x_3	77	240	1.2071	8.87e-06	–	199	1.3289	9.54e-06	15	45	0.3651	5.11e-06	1001	4007	45.6114	2.66e+01
	x_4	32	102	0.7202	7.49e-06	64	192	1.2856	9.55e-06	14	43	0.3702	7.01e-06	1001	4004	45.0273	2.97e+00
	x_5	83	259	1.2496	8.49e-06	62	186	1.2076	9.75e-06	14	42	0.6487	5.91e-06	49	196	2.0699	8.31e-07

Table 8: MHS Results of Experiment for Problem 7 Compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	10	32	0.0398	4.17e-06	7	18	0.0237	9.06e-06	7	19	0.0095	8.68e-06	21	85	0.0296	7.81e-07
	x_2	10	32	0.0080	4.95e-06	7	18	0.0046	9.06e-06	7	19	0.0068	8.68e-06	22	88	0.0155	8.33e-07
	x_3	10	32	0.0142	7.87e-06	7	18	0.0046	9.06e-06	12	38	0.0118	5.25e-06	22	88	0.0308	8.64e-07
	x_4	12	38	0.0062	9.46e-06	7	18	0.0059	9.06e-06	108	330	0.0403	9.80e-06	22	88	0.0262	7.12e-07
	x_5	14	44	0.0163	5.98e-06	7	18	0.0074	9.06e-06	181	550	0.0734	9.56e-06	22	88	0.0333	7.12e-07
5,000	x_1	9	29	0.0490	9.58e-06	6	16	0.0101	6.77e-07	7	23	0.0215	1.45e-06	22	89	0.0633	8.05e-07
	x_2	9	29	0.0190	9.62e-06	6	16	0.0075	6.77e-07	18	57	0.0407	5.21e-06	23	92	0.0815	8.58e-07
	x_3	9	29	0.0389	9.31e-06	6	16	0.0113	6.77e-07	87	268	0.1183	9.92e-06	23	92	0.0535	8.91e-07
	x_4	8	26	0.0496	4.38e-06	6	16	0.0110	6.77e-07	682	2056	0.8973	9.92e-06	23	92	0.0835	7.33e-07
	x_5	6	20	0.0142	9.41e-06	6	16	0.0100	6.77e-07	682	2056	1.1662	9.92e-06	23	92	0.0611	7.33e-07
10,000	x_1	7	23	0.0247	5.06e-06	7	19	0.0204	5.87e-06	8	26	0.0251	1.42e-06	23	92	0.1371	7.97e-07
	x_2	7	23	0.0476	5.19e-06	7	19	0.0182	5.87e-06	46	143	0.1257	8.80e-06	23	93	0.1009	7.98e-07
	x_3	7	23	0.0507	5.44e-06	7	19	0.0200	5.87e-06	363	1098	0.8190	9.76e-06	23	93	0.1172	8.28e-07
	x_4	7	23	0.0616	4.69e-06	7	19	0.0153	5.87e-06	605	1825	1.5028	9.95e-06	23	93	0.0771	6.81e-07
	x_5	7	23	0.0246	3.30e-06	7	19	0.0161	5.87e-06	605	1825	1.4498	9.95e-06	23	93	0.0921	6.81e-07
50,000	x_1	5	18	0.1735	4.72e-06	12	35	0.1084	8.08e-06	86	265	0.9701	9.18e-06	24	96	0.5177	8.19e-07
	x_2	5	18	0.2605	4.66e-06	12	35	0.0995	8.08e-06	402	1216	4.6388	9.98e-06	24	97	0.6265	8.20e-07
	x_3	5	18	0.0838	4.46e-06	12	35	0.0998	8.08e-06	402	1216	5.4102	9.98e-06	24	97	0.6308	8.52e-07
	x_4	5	18	0.1788	3.18e-06	12	35	0.0993	8.08e-06	402	1216	5.2769	9.98e-06	24	97	0.5803	7.01e-07
	x_5	5	18	0.2733	2.11e-06	12	35	0.0968	8.08e-06	402	1216	7.1626	9.98e-06	24	97	0.5415	7.01e-07
100,000	x_1	5	18	0.2353	5.15e-06	18	55	0.3769	9.84e-06	188	574	5.2414	9.94e-06	24	97	1.1388	7.61e-07
	x_2	5	18	0.2033	4.96e-06	18	55	0.3269	9.84e-06	315	955	7.0752	9.94e-06	25	100	1.2857	8.12e-07
	x_3	5	18	0.1749	4.39e-06	18	55	0.3791	9.84e-06	315	955	7.0767	9.94e-06	25	100	1.1054	8.43e-07
	x_4	5	18	0.2011	2.46e-06	18	55	0.3458	9.84e-06	315	955	6.7129	9.94e-06	24	97	1.2276	9.91e-07
	x_5	5	18	0.2749	1.47e-06	18	55	0.3773	9.84e-06	315	955	6.8062	9.94e-06	24	97	1.2367	9.91e-07

Table 9: MHS Results of Experiment for Problem 8 Compared with PRPFR, YAN2010 and MCDPM

		MHS				PRPFR				YAN2010				MCDPM			
Dim	IP	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm	NOI	NFE	TIME	Norm
1,000	x_1	7	25	0.0557	3.71e-06	13	40	0.0275	8.78e-06	12	38	0.0051	4.31e-06	17	70	0.0453	8.77e-07
	x_2	7	25	0.0134	3.34e-06	13	39	0.0049	7.19e-06	11	35	0.0090	9.79e-06	11	46	0.0243	5.26e-07
	x_3	5	19	0.0108	6.56e-06	13	39	0.0052	6.85e-06	11	35	0.0067	9.34e-06	11	47	0.0192	3.38e-07
	x_4	7	25	0.0042	4.80e-06	15	45	0.0046	5.54e-06	13	41	0.0115	5.21e-06	12	51	0.0109	7.22e-07
	x_5	8	27	0.0049	3.93e-06	15	45	0.0064	7.95e-06	13	41	0.0134	7.48e-06	11	47	0.0128	8.31e-07
5,000	x_1	7	25	0.0400	8.29e-06	14	42	0.0105	8.52e-06	12	38	0.0224	9.65e-06	11	46	0.0505	5.78e-07
	x_2	7	25	0.0173	7.46e-06	14	42	0.0128	5.16e-06	12	38	0.0318	5.84e-06	15	63	0.0742	5.24e-07
	x_3	6	22	0.0335	2.35e-06	14	42	0.0187	4.92e-06	12	38	0.0247	5.57e-06	12	51	0.0273	3.59e-07
	x_4	8	28	0.0421	1.72e-06	15	46	0.0182	9.15e-06	13	42	0.0219	7.99e-06	11	46	0.0237	6.44e-07
	x_5	8	27	0.0179	8.78e-06	16	48	0.0143	5.71e-06	14	44	0.0157	4.46e-06	11	46	0.0512	5.72e-07
10,000	x_1	8	28	0.0552	1.88e-06	14	43	0.0195	8.91e-06	12	39	0.0363	9.36e-06	12	50	0.0757	5.16e-07
	x_2	8	28	0.0928	1.69e-06	14	42	0.0354	7.30e-06	12	38	0.0446	8.26e-06	14	58	0.0732	7.59e-07
	x_3	6	22	0.0392	3.33e-06	14	42	0.0184	6.96e-06	12	38	0.0500	7.88e-06	13	55	0.0830	4.13e-07
	x_4	8	28	0.0709	2.43e-06	16	48	0.0231	5.62e-06	14	44	0.0273	4.39e-06	12	50	0.0383	7.62e-07
	x_5	9	30	0.0349	1.99e-06	16	48	0.0343	8.07e-06	14	44	0.0238	6.31e-06	12	50	0.0884	5.76e-07
50,000	x_1	8	28	0.2289	4.20e-06	15	45	0.1216	8.66e-06	13	41	0.0970	8.14e-06	12	51	0.2783	6.78e-07
	x_2	8	28	0.1548	3.78e-06	15	45	0.0909	5.24e-06	13	41	0.0979	4.93e-06	19	78	0.3115	9.64e-07
	x_3	6	22	0.2631	7.44e-06	15	45	0.0896	5.00e-06	13	41	0.1132	4.70e-06	16	67	0.3375	2.99e-07
	x_4	8	28	0.2111	5.44e-06	16	49	0.1013	9.30e-06	14	44	0.1463	9.82e-06	12	51	0.1885	7.17e-07
	x_5	9	30	0.1781	4.45e-06	17	51	0.1200	5.80e-06	14	45	0.1343	9.68e-06	12	50	0.1563	6.12e-07
100,000	x_1	8	28	0.2234	5.94e-06	15	46	0.1574	9.05e-06	13	42	0.1711	7.90e-06	13	55	0.4519	4.51e-07
	x_2	8	28	0.3070	5.35e-06	15	45	0.1771	7.41e-06	13	41	0.1579	6.97e-06	13	55	0.4346	4.36e-07
	x_3	7	25	0.2508	1.69e-06	15	45	0.1688	7.07e-06	13	41	0.1497	6.65e-06	18	75	0.7301	2.65e-07
	x_4	8	28	0.3197	7.69e-06	17	51	0.2117	5.71e-06	14	45	0.1710	9.53e-06	16	66	0.5210	4.79e-07
	x_5	9	30	0.3945	6.29e-06	17	51	0.1846	8.20e-06	15	47	0.1756	5.33e-06	16	66	0.4918	4.97e-07

To provide a comprehensive visualization of algorithmic performance, we employ the performance profiles introduced by Dolan and Moré (2002). Figures 1–3 illustrate the comparative efficiency of the four methods across all test problems

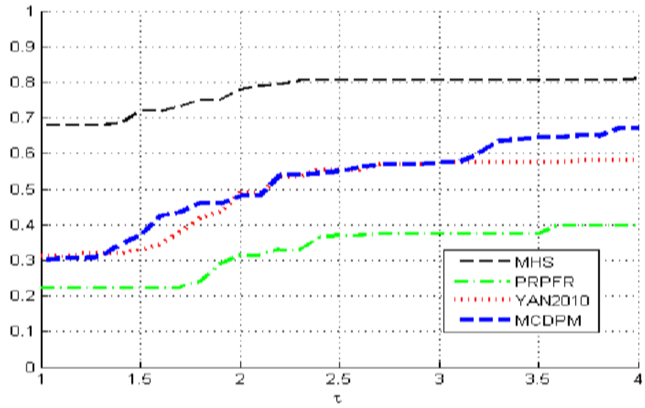

Figure 1: Performance Profile Comparison Based On Number of Iterations

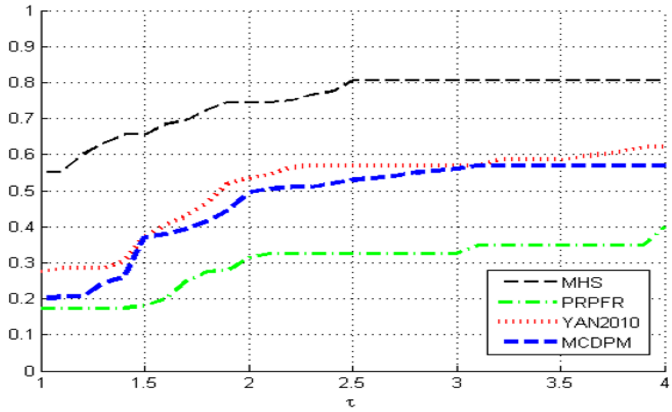

Figure 2: Performance Profile Comparison Based On Function Evaluations

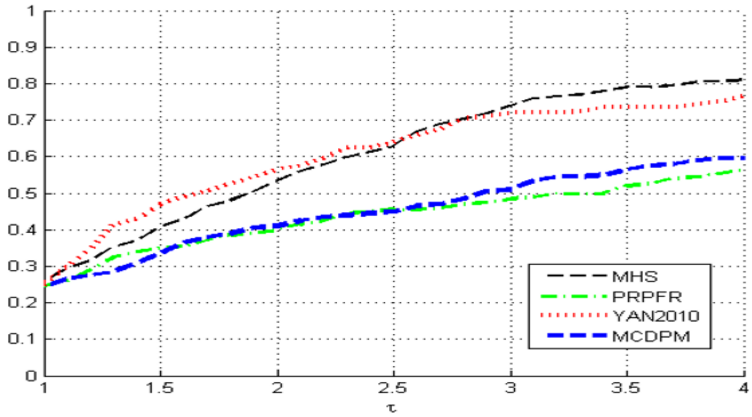

Figure 3: Performance Profile Comparison Based on CPU Time

Analysis of performance profiles shows that the MHS algorithm achieves the lowest number of iterations in approximately 68% of the test problems, compared to 22%, 30% and 30% for PRPFR, Yan2010 and MCDPM, respectively as shown in Figure 1. Regarding function evaluations (Figure 2), MHS proves most efficient in about 55% of the test problems, while the competing methods succeed in less than 18%, 29% and 20% of same problems. In terms of computational time (Figure 3), MHS exhibits superior performance in 25% of the test problems, whereas PRPFR, MHS and MCD achieve timing in approximately 25%, 25% and 25% of same problems. These results collectively demonstrate the robustness of the TTHS algorithm compared to existing methods, establishing it as a viable alternative for solving nonlinear monotone equations.

CONCLUSION

This paper has presented a novel three-term derivative-free projection method for solving constrained nonlinear monotone equations, extending the conjugate gradient framework of Jie and Zhong (2022) from unconstrained optimization to the monotone equation setting. The proposed MHS algorithm offers several advantages: it eliminates the need for Jacobian matrix computations, avoids solving linear systems, and consequently scales efficiently to large-scale problems. The search directions automatically satisfy a sufficient descent condition independent of line search procedures. Under standard assumptions, we have rigorously established the global convergence of the algorithm. Furthermore, extensive numerical experiments across diverse test problems demonstrate that MHS consistently outperforms existing methods (PRPFR, Yan2010, MCDPM) in terms of iteration count, function evaluations, and computational time. Performance profile analysis confirms the enhanced robustness of the proposed approach.

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