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APPLICATION OF THE VARIATIONAL ITERATION METHOD TO FOURTH ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

***Kolade, Ayodeji David, Alabi, Taiye John and Ajayi, Omolola Felicia**

Department of Statistics, Kogi State Polytechnic, Lokoja, Kogi State, Nigeria.

*Corresponding authors' email: koladedavid@gmail.com

ABSTRACT

This paper shows the application of Variational Iteration Method (VIM) to fourth-order Volterra Integro-Differential Equations (VIDEs). The method was used to obtain approximate analytical solutions to the considered problems, and the obtained results were compared with the corresponding exact solutions in order to examine the accuracy and efficiency of the technique. The variational iteration correction functional was constructed using appropriate Lagrange multipliers, leading to rapidly convergent successive approximations. Numerical computations were carried out for selected examples with the aid of Maple software, and the results obtained from VIM showed excellent agreement with the exact solutions. The errors revealed extremely small absolute errors of order 10^{-9}, demonstrating the high level of precision and convergence of the method. Also, graphical comparisons indicated that the VIM solutions almost completely overlap with the exact solutions throughout the interval considered, confirming the reliability and stability of the approach.

Keywords: Variational Iteration Method (VIM), Volterra Integro-Differential Equation (VIDE), Fourth-Order Differential Equation, Numerical Solution

INTRODUCTION

Integro-differential equations constitute an important class of mathematical models that arise naturally in diverse fields such as physics, engineering, biology, economics, and control theory. In particular, Volterra Integro-Differential Equations (VIDEs) are widely used to describe systems in which the current state depends on both local derivatives and integral effects over a finite domain. These equations are commonly encountered in problems involving viscoelasticity, heat conduction, fluid dynamics, and population dynamics, Badeye et al. (2023).

Analytical solutions of higher-order VIDEs, especially fourth-order forms, are often difficult or impossible to obtain due to their complexity and nonlinearity. As a result, various numerical and semi-analytical methods have been developed to approximate their solutions. Among these, methods such as the Adomian Decomposition Method (ADM), Homotopy Analysis Method (HAM), finite difference methods, and collocation techniques have been widely applied, Hosry et al. (2018), Yousefi et al. (2017). In recent years, the Variational Iteration Method (VIM), introduced by He, has gained significant attention due to its simplicity, efficiency, and ability to handle both linear and nonlinear problems without requiring discretization or linearization. The method constructs correction functionals using Lagrange multipliers, enabling rapid convergence to accurate approximate solutions, He (2007), Wazwaz (2010). The application of VIM to Integro-differential equations has demonstrated promising results. Studies have shown that VIM produces highly accurate solutions with reduced computational effort compared to traditional numerical methods, Lanlege et al. (2023).

Despite these advances, there remains limited work specifically addressing the application of VIM to fourth-order Volterra integro-differential equations, which are particularly relevant in modeling beam deflection, plate theory, and higher-order physical systems. This study therefore aims to bridge this gap by applying the VIM to obtain approximate solutions of fourth-order VIDEs and analyzing its effectiveness.

The study of integro-differential equations has attracted considerable attention over the years due to their applicability in modeling real-world phenomena. Early approaches focused on analytical and classical numerical techniques; however, these methods often encounter difficulties when dealing with nonlinear or higher-order equations. He (2007) introduced the Variational Iteration Method as a powerful analytical technique for solving differential and integro-differential equations. The method utilizes correction functionals and optimally identified Lagrange multipliers to generate successive approximations that converge rapidly to the exact solution.

Wazwaz (2010) extended the application of VIM to linear and nonlinear VIDEs, demonstrating that the method can handle both first- and second-kind equations efficiently without restrictive assumptions. The study showed that VIM significantly reduces computational complexity while maintaining high accuracy. While Fariborzi et al. (2011) applied VIM to nonlinear VIDEs and established the convergence, existence, and uniqueness of the solution. Their results also confirmed that VIM produces accurate approximate solutions that compare favorably with the Adomian Decomposition Method.

Behzadi (2012) further demonstrated the applicability of VIM in solving nonlinear VIDEs. The study highlighted that VIM provides solutions in series form with easily computable components and improved accuracy over traditional techniques.

In addition, hybrid approaches combining VIM with other numerical techniques have been proposed. For instance, Hosry et al. (2019) introduced the Variational Adomian Decomposition Method (VADM), which combines VIM with ADM to handle nonlinear terms effectively and improve convergence rates. Recent studies have also focused on comparing VIM with other numerical methods. Okafor and Oladipo (2025) conducted a comparative analysis between VIM and the Series Expansion Method (SEM) for solving Fredholm integro-differential equations. Their findings revealed that VIM yields more accurate solutions with faster convergence and less computational effort.

MATERIALS AND METHODS

Problem Formulation

Consider the general fourth-order VIDE of the form:

$$g^{iv}(h) = f(h) + \int_a^b M(h,t)g(t)\,dt, a \le h \le b \qquad (1)$$

Where:

$g^{iv}(h)$ is the fourth derivative of the unknown function,
$f(h)$ is a known continuous function,
$M(h,t)$ is the kernel of the integral equation.

Overview of Variational Iteration Method (VIM)

Consider the differential equation

$$Lu + Nu = g(t) \qquad (2)$$

Where L and N are linear and nonlinear operators respectively, $g(t)$ is the source inhomogeneous term.

The Variational Iteration Method constructs a sequence of approximations $g_n(h)$ using a correction functional defined as:

$$g_{n+1}(h) = g_n(h) + + \int_a^x \lambda(t)[L(g_n(t)) + N(g_n(t)) - f(t)]\,dt \qquad (3)$$

Where:

$\lambda(t) = \frac{(-1)^i(t-h)^{i-1}}{(i-1)!}$ Is a Lagrange multiplier.

Where i is the order of the equation

Wazwaz (2010) proposed the Lagrange multiplier which is optimally determined via variational theory.

RESULTS AND DISCUSSION

Example 1

Consider the fourth order VIDE.

$$g^{iv}(h) = 1 + h - \int_0^h (h-t)g(t)dt\,, g(0) = g'(0) = 1,\ g''(0) = g'''(0) = -1 \qquad (4)$$

$$\lambda = \frac{(t-h)^3}{6}$$

Exact solution $g(h) = cos\,h + sin\,h$

Correctional Functional

$$g_{n+1}(h) = g_n(h) + \int_0^h \lambda[g_n^{iv}(t) - 1 - t + \int_0^t (t-r)\,g_n(r)dr]dt$$

$$g_0 = 1 + h - \tfrac{1}{2}h^2 - \tfrac{1}{6}h^3$$

$$g_1(h) = 1 + h - \frac{1}{2}h^2 - \frac{1}{6}h^3 + \frac{1}{24}h^4 + \frac{1}{120}h^5 - \frac{1}{720}h^6 - \frac{1}{5040}h^7 + \frac{1}{40320}h^8 + \frac{1}{362880}h^9$$

$$g_2(h) = 1 + h - \frac{1}{2}h^2 - \frac{1}{6}h^3 + \frac{1}{24}h^4 + \frac{1}{120}h^5 - \frac{1}{720}h^6 - \frac{1}{5040}h^7 + \frac{1}{40320}h^8 + \frac{1}{362880}h^9 - \frac{1}{3628800}h^{10} - \frac{1}{39916800}h^{11} + \frac{1}{479001600}h^{12} + \frac{1}{6227020800}h^{13} - \frac{1}{87178291200}h^{14} - \frac{1}{1307674368000}h^{15}$$

$$g_3(h) = 1 + h - \frac{1}{2}h^2 - \frac{1}{6}h^3 + \frac{1}{24}h^4 + \frac{1}{120}h^5 - \frac{1}{720}h^6 - \frac{1}{5040}h^7 + \frac{1}{40320}h^8 + \frac{1}{362880}h^9 - \frac{1}{3628800}h^{10} - \frac{1}{39916800}h^{11} + \frac{1}{479001600}h^{12} + \frac{1}{6227020800}h^{13} - \frac{1}{87178291200}h^{14} - \frac{1}{1307674368000}h^{15} + \frac{1}{20922789888000}h^{16} + \frac{1}{355687428096000}h^{17} - \frac{1}{6402373705728000}h^{18} - \frac{1}{121645100408832000}x^{19} + \frac{1}{2432902008176640000}h^{20} + \frac{1}{51090942171709440000}h^{21}$$

$$g_4(h) = 1 + h - \frac{1}{2}h^2 - \frac{1}{6}h^3 + \frac{1}{24}h^4 + \frac{1}{120}h^5 - \frac{1}{720}h^6 - \frac{1}{5040}x^7 + \frac{1}{40320}h^8 + \frac{1}{362880}h^9 - \frac{1}{3628800}h^{10} - \frac{1}{39916800}h^{11} + \frac{1}{6227020800}h^{13} - \frac{1}{87178291200}h^{14} - \frac{1}{1307674368000}h^{15} + \frac{1}{20922789888000}h^{16} + \frac{1}{355687428096000}h^{17} - \frac{1}{6402373705728000}h^{18} - \frac{1}{121645100408832000}h^{19} + \frac{1}{2432902008176640000}h^{20} + \frac{1}{51090942171709440000}h^{21} - \frac{1}{1124000727777607680000}h^{22} - \frac{1}{25852016738884976640000}h^{23} + \frac{1}{620448401733239439360000}h^{24} + \frac{1}{15511210043330985984000000}h^{25} - \frac{1}{403291461126605635584000000}h^{26} - \frac{1}{10888869450418352160768000000}h^{27}$$

$$g_5(x) = 1 + h - \frac{1}{2}h^2 - \frac{1}{6}h^3 + \frac{1}{24}h^4 + \frac{1}{120}h^5 - \frac{1}{720}h^6 - \frac{1}{5040}h^7 + \frac{1}{40320}h^8 + \frac{1}{362880}h^9 - \frac{1}{3628800}h^{10} - \frac{1}{39916800}h^{11} + \frac{1}{479001600}h^{12} + \frac{1}{6227020800}h^{13} - \frac{1}{87178291200}h^{14} - \frac{1}{1307674368000}h^{15} + \frac{1}{20922789888000}h^{16} + \frac{1}{355687428096000}h^{17} - \frac{1}{6402373705728000}h^{18} - \frac{1}{121645100408832000}h^{19} + \frac{1}{2432902008176640000}h^{20} + \frac{1}{51090942171709440000}h^{21} - \frac{1}{1124000727777607680000}h^{22} - \frac{1}{25852016738884976640000}h^{23} + \frac{1}{620448401733239439360000}h^{24} - \frac{1}{403291461126605635584000000}h^{26} - \frac{1}{10888869450418352160768000000}h^{27} + \frac{1}{304888344611713860501504000000}h^{28} + \frac{1}{8841761993739701954543616000000}h^{29} - \frac{1}{265252859812191058636308480000000}h^{30} - \frac{1}{8222838654177922817725562880000000}h^{31} + \frac{1}{263130836933693530167218012160000000}h^{32} + \frac{1}{8683317618811886495518194401280000000}h^{33}$$

Table 1: Numerical Result for Example 1 showing VIM, Exact and Error

h	VIM	Exact	ERROR
0.0	1.000000000	1.000000000	0
0.1	1.094837582	1.094837582	0
0.2	1.178735909	1.178735909	0
0.3	1.250856696	1.250856696	0
0.4	1.310479336	1.310479336	0
0.5	1.357008102	1.357008100	2×10^{-9}
0.6	1.389978089	1.389978088	1×10^{-9},
0.7	1.409059875	1.409059874	1×10^{-9}
0.8	1.414062801	1.414062800	1×10^{-9}
0.9	1.404936879	1.404936878	0
1.0	1.381773290	1.381773291	-1×10^{-9}

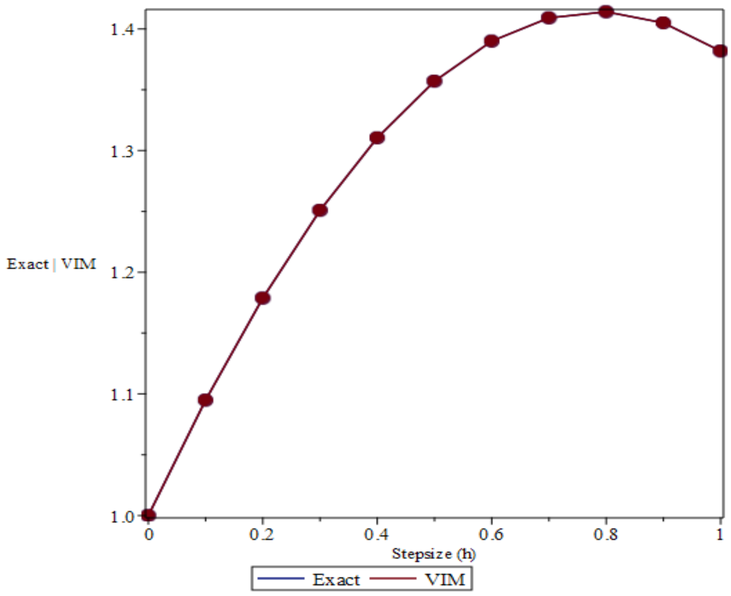

Figure 1: Graphical Illustration of VIM Compared To the Exact Solution in Example 1

Discussion of Result for Example 1

The results from table 1 clearly demonstrate the effectiveness, accuracy, and convergence capability of the VIM in solving fourth-order VIDEs. It is observed that the numerical solutions generated by VIM are almost identical to the exact solutions at all grid points within the interval $0 \leq h \leq 1$. At several points, particularly from $h = 0.0$ to $h = 0.4$, the error values are exactly zero up to the displayed decimal places, indicating perfect agreement between the approximate and exact solutions. This confirms the high precision of the method. For the remaining points, the computed errors are extremely small, ranging approximately between 10^{-9} and2×10^{-9}. Such negligible errors indicate that the VIM possesses rapid convergence and strong numerical stability. The small magnitude of the errors also validates the reliability of the method for solving higher-order integro-differential equations.

Figure 1 shows that the VIM solution curve almost completely overlaps with the exact solution curve throughout the interval considered. The coincidence of the two curves demonstrates the excellent approximation property of the method. There is no visible deviation between the curves, which further confirms the accuracy of the obtained numerical solution. In addition, the behavior of the solution indicates that the method effectively captures the dynamics of the problem without oscillation or divergence. The smooth nature of the graph reveals that the VIM preserves the continuity and structure of the exact solution. This makes the method computationally efficient and suitable for practical applications involving fourth-order VIDEs.

Example 2

Consider the fourth order Volterra Integro-differential equation.

$$g^{iv}(h) = e^{2h} - \int_0^h e^{2(h-t)}\, g(t)dt\,, g(0) = g'(0) = g''(0) = g'''(0) = 1 \qquad (5)$$

$$\lambda = \frac{(t-h)^3}{6}$$

Exact solution = e^h

Correctional Functional

$$g_{n+1}(h) = g_n(h) + \int_0^h \lambda[g_n^{iv}(t) - e^{2t} + \int_0^t e^{2(h-t)}\, g_n(r)dr]dt$$

$$g_0(x) = 1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3$$

$$g_1(h) = \frac{255}{256} + \frac{127h}{128} + \frac{63h^2}{128} + \frac{31h^3}{192} + \frac{5h^4}{128} + \frac{7h^5}{960} + \frac{h^6}{960} + \frac{h^7}{10080} + \frac{e^{2h}}{256}$$

$$g_2(h) = \frac{4089}{4096} + \frac{4083h}{4096} + \frac{509h^2}{1024} + \frac{1013h^3}{6144} + \frac{251h^4}{6144} + \frac{247h^5}{30720} + \frac{h^6}{768} + \frac{19h^7}{107520} + \frac{13h^8}{645120} + \frac{11h^9}{5806080} + \frac{h^{10}}{7257600} + \frac{h^{11}}{159667200} + \frac{7e^{2h}}{4096} - \frac{e^{2h}h}{4096}$$

$$g_3(h) = +\frac{65493}{65536} + \frac{65459h}{65536} + \frac{65399h^2}{131072} + \frac{32647h^3}{196608} + \frac{2713h^4}{65536} + \frac{135h^5}{16384} + \frac{1339h^6}{983040} + \frac{283h^7}{1474560} + \frac{121h^8}{5160960} + \frac{233h^9}{92897280} + \frac{73h^{10}}{309657600} + \frac{h^{11}}{51609600} + \frac{h^{12}}{729907200} + \frac{h^{13}}{12454041600} + \frac{h^{14}}{278970531840} + \frac{h^{15}}{10461394944000} + \frac{43e^{2h}}{65536} - \frac{9e^{2h}h}{65536} + \frac{e^{2h}h^2}{131072}$$

$$g_4(h) = \frac{4095}{4096} + \frac{2096255h}{2097152} + \frac{1047795h^2}{2097152} + \frac{349075h^3}{2097152} + \frac{21797h^4}{524288} + \frac{21763h^5}{2621440} + \frac{65119h^6}{47185920} + \frac{21613h^7}{110100480} + \frac{503h^8}{20643840} + \frac{7957h^9}{2972712960} + \frac{3907h^{10}}{14863564800} + \frac{3797h^{11}}{163499212800} + \frac{227h^{12}}{122624409600} + \frac{53h^{13}}{398529331200} + \frac{191h^{14}}{22317642547200} + \frac{163h^{15}}{334764638208000} + \frac{h^{16}}{41845579776000} + \frac{h^{17}}{1034727063552000} + \frac{h^{18}}{34145993097216000} + \frac{h^{19}}{1946321606541312000} + \frac{e^{2h}}{4096} - \frac{127e^{2h}h}{2097152} + \frac{11e^{2h}h^2}{2097152} - \frac{e^{2h}h^3}{6291456}$$

$$g_5(h) = \frac{134205649}{134217728} + \frac{67098437h}{67108864} + \frac{33545481h^2}{67108864} + \frac{8384789h^3}{50331648} + \frac{16764265h^4}{402653184} + \frac{1675541h^5}{201326592} + \frac{279013h^6}{201326592} + \frac{19901h^7}{100663296} + \frac{5791h^8}{234881024} + \frac{43271h^9}{15854469120} + \frac{5377h^{10}}{19818086400} + \frac{63929h^{11}}{2615987404800} + \frac{5729h^{12}}{2853804441600} + \frac{30827h^{13}}{204047017574400} + \frac{1657h^{14}}{158703235891200} + \frac{7099h^{15}}{10712468422656000} + \frac{127h^{16}}{3296144130048000} + \frac{743h^{17}}{364223926370304000} + \frac{29h^{18}}{298001394302976000} + \frac{h^{19}}{243290200817664000} + \frac{31h^{20}}{20760763803107328000} +$$

$$\frac{29h^{21}}{653964059797880832000}+\frac{h^{22}}{10276578082538127360000}+\frac{h^{23}}{827264535644319252480000}+\frac{12079e^{2h}}{134217728}-\frac{413e^{2h}h}{16777216}+\frac{11e^{2h}h^2}{4194304}-\frac{13e^{2h}h^3}{100663296}+\frac{e^{2h}h^4}{402653184}$$

$$g_6(h)=+\frac{4294825159}{4294967296}+\frac{4294724471h}{4294967296}+\frac{67102415h^2}{134217728}+\frac{1073567387h^3}{6442450944}+\frac{134181079h^4}{3221225472}+\frac{134156509h^5}{16106127360}+\frac{8382259h^6}{6039797760}+\frac{4787513h^7}{24159191040}+\frac{2391859h^8}{96636764160}+\frac{2388823h^9}{869730877440}+\frac{49667h^{10}}{181193932800}+\frac{198041h^{11}}{7972533043200}+\frac{12317h^{12}}{5979399782400}+\frac{85573h^{13}}{544125380198400}+\frac{1627h^{14}}{146495294668800}+\frac{10397h^{15}}{14283291230208000}+\frac{12167h^{16}}{274239191619993600}+\frac{58651h^{17}}{23310331287699456000}+\frac{1739h^{18}}{13112061349330944000}+\frac{12911h^{19}}{1993033325098303488000}+\frac{1453h^{20}}{4982583312745758720000}+\frac{251h^{21}}{20926849913532186624000}+\frac{h^{22}}{2248001455555215360000}+\frac{193h^{23}}{13236232570309108039680000}+\frac{h^{24}}{2443612166826296868864000}+\frac{37h^{25}}{397086977109273241190400000}+\frac{h^{26}}{6452663378025690169344000000}+\frac{h^{27}}{696887644826774538289152000000}+\frac{142137e^{2h}}{4294967296}-\frac{41449e^{2h}h}{4294967296}+\frac{39e^{2h}h^2}{33554432}-\frac{233e^{2h}h^3}{3221225472}+\frac{5e^{2h}h^4}{2147483648}-\frac{e^{2h}h^5}{32212254720}$$

Table 2: Numerical Result for Example 2 Showing VIM, Exact and Error

x	VIM	Exact	ERROR
0.0	1.000000000	1.000000000	0
0.1	1.105170918	1.105170918	0
0.2	1.221402759	1.221402758	1×10^{-9}
0.3	1.349858808	1.349858808	0
0.4	1.491824699	1.491824698	1×10^{-9}
0.5	1.648721270	1.648721271	-1×10^{-9}
0.6	1.822118799	1.822118800	-1×10^{-9}
0.7	2.013752708	2.013752707	1×10^{-9}
0.8	2.225540928	2.225540928	0
0.9	2.459603112	2.459603111	1×10^{-9}
1.0	2.718281827	2.718281828	-1×10^{-9}

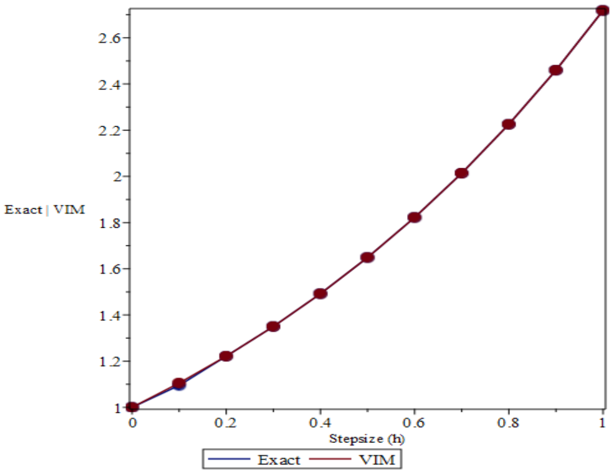

Figure 2: Graphical Illustration of VIM Compared to the Exact Solution in Example 2

Discussion of Results for Example 2

From table 2, it is evident that the numerical solutions generated by the VIM are in excellent agreement with the exact solutions at all selected grid points. The computed errors are either zero or extremely small, with magnitudes of approximately 10^{-9}. Such negligible deviations indicate a very high level of numerical accuracy and stability of the method. At several points, including, $h = 0.0, 0.1, 0.3$ and 0.8, the VIM solution coincides exactly with the exact solution up to the displayed decimal places, producing zero error. Even at points where slight discrepancies occur, such as $h = 0.2, 0.4, 0.5, 0.6, 0.7, 0.9$, and 1.0, the errors remain extremely insignificant. The alternating positive and negative signs of the errors further suggest that the approximation does not exhibit systematic overestimation or underestimation, which confirms the consistency of the method.

The graphical illustration in Figure 2 further validates the accuracy of the VIM. The curve overlaps almost perfectly with the exact solution curve throughout the interval considered. The absence of visible separation between the two curves demonstrates the rapid convergence property of the method and its capability to approximate the analytical solution with remarkable precision. In addition, the graph exhibits a smooth and continuously increasing behavior of the solution from $h = 0$ to $h = 1$. The VIM successfully preserves the qualitative characteristics of the exact solution without introducing oscillations or numerical instability. This confirms that the method is efficient and reliable for solving fourth-order VIDEs.

CONCLUSION

The numerical and graphical results obtained in both Example 1 and example 2 demonstrate that the VIM is a highly effective technique for solving fourth-order VIDEs. The method produced approximate solutions that are almost identical to the exact solutions, with errors of order 10^{-9}, indicating excellent accuracy and convergence. The close overlap between the VIM and exact solution curves confirms the reliability and computational efficiency of the method. The method maintains solution stability across the entire interval considered. Based on these findings, the Variational Iteration Method can be regarded as a powerful and dependable analytical-numerical approach for solving higher-order integro-differential equations arising in applied mathematics, physics, and engineering problems.

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