FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 4 No. 4, December, 2020, pp 477 - 483
DOI: https://doi.org/10.33003/fjs-2020-0404-505

DERIVATION AND IMPLEMENTATION OF A THREE-STEP HYBRID BLOCK ALGORITHM (TSHBA) FOR DIRECT SOLUTION OF LINEAR AND NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

[1]ABOLARIN, O. E., *[2]OGUNWARE, B. G., [3]ADEBISI, A. F. and [4]AYINDE, S. O.

[1, 2]Mathematics Department, Federal University Oye-Ekiti, Ekiti State
[3]Department of Mathematical Sciences, Osun State University, Osogbo, Osun State
[4]Department of Mathematics, Ekiti State University, Ado-Ekiti, Ekiti State

*Corresponding Author: ogunwaregbenga@gmail.com

ABSTRACT
The development and application of an implicit hybrid block method for the direct solution of second order ordinary differential equations with given initial conditions is shown in this research. The derivation of the three-step scheme was done through collocation and interpolation of power series approximation to give a continuous linear multistep method. The evaluation of the continuous method at the grid and off grid points formed the discrete block method. The basic properties of the method such as order, error constant, zero stability, consistency and convergence were properly examined. The new block method produced more accurate results when compared with similar works carried out by existing authors on the solution of linear and non-linear second order ordinary differential equations.

Keywords: Power Series, Block method, Implicit, Interpolation and Collocation, Linear and Non-linear Ordinary Differential Equations (ODEs)

INTRODUCTION

The reduction of second order ordinary differential equations (ODEs) to a system of first order ODEs and then solve using any appropriate method for first order ODEs was the principal approach used for solving higher order initial value problems specifically second order ODEs. Predictor-corrector method was later used by some researchers for the direct computational solution of second order ODEs. Authors such as Brugnano and Trigiante (1998), Jator (2001), Awoyemi (2003), Awoyemi and Kayode (2005), Butcher (2008), Kayode (2008), Adesanya et al (2008), Kayode and Adeyeye (2011) have discussed these reduction and predictor-corrector methods widely. The major challenges familiar with the reduction and predictor-corrector approaches include low level of accuracy, complicated programming process, inefficiency and high cost in terms of system time and resources.

In order to rise above these drawbacks of the reduction and predictor-corrector methods and bring improvement on numerical analysis, authors like Jator (2007), Adeyeye and Omar (2016), Omole and Ogunware (2018), Omar and Raft (2016), and many more proposed the block method for the direct solution of second order ordinary differential equations independently. The aforementioned authors maintained that the block method generates more accurate result than both reduction and predictor-corrector methods. The distinctiveness of the block method is that in each usage, the solution value will be obtained concurrently at several different points and it is found to be cost effective because of the evaluation of few functions involved.

Hence, the focus of this work is to develop and implement a three-step hybrid block method for the solution of second order ODEs directly.

DERIVATION OF THE METHOD

Generally, second order ODEs with initial value problems is of the form

$$y'' = f(x, y(x), y'(x)), \quad y(0) = \eta_0, \quad y'(a) = \eta_1 \tag{1}$$

Power series approximate solution of (1) is of the form:

$$y(x) = \sum_{j=0}^{(k+6)} a_j x^j \tag{2}$$

Where k is the end-point.
The second derivative of (1) is obtained as

$$y'' = \sum_{j=2}^{(k+6)} j(j-1)a_j x^{j-2} \quad (3)$$

The combination of equations (2) and (3) gives the differential system

$$y'' = \sum_{j=0}^{(k+6)} j\left(j-1\right)a_j x^{j-2} = f\left(x, y, y'\right) \quad (4)$$

Collocating (4) at $x = x_{n+j},\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ and interpolating (2) at $x = x_{n+j},\ j = \frac{1}{2}, \frac{3}{2}$ gives a structure of non-linear equation of the form

$$\begin{bmatrix} 1 & x_{n+1/2} & x^2_{n+1/2} & x^3_{n+1/2} & x^4_{n+1/2} & x^5_{n+1/2} & x^6_{n+1/2} & x^7_{n+1/2} & x^8_{n+1/2} \\ 1 & x_{n+3/2} & x^2_{n+3/2} & x^3_{n+3/2} & x^4_{n+3/2} & x^5_{n+3/2} & x^6_{n+3/2} & x^7_{n+3/2} & x^8_{n+3/2} \\ 0 & 0 & 2 & 6x_n & 12x^2_n & 20x^3_n & 30x^4_n & 42x^5_n & 56x^6_n \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} & 42x^5_{n+\frac{1}{2}} & 56x^6_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x^2_{n+\frac{3}{2}} & 20x^3_{n+\frac{3}{2}} & 30x^4_{n+\frac{3}{2}} & 42x^5_{n+\frac{3}{2}} & 56x^6_{n+\frac{3}{2}} \\ 0 & 0 & 2 & 6x_{n+2} & 12x^2_{n+2} & 20x^3_{n+2} & 30x^4_{n+2} & 42x^5_{n+2} & 56x^6_{n+2} \\ 0 & 0 & 2 & 6x_{n+\frac{5}{2}} & 12x^2_{n+\frac{5}{2}} & 20x^3_{n+\frac{5}{2}} & 30x^4_{n+\frac{5}{2}} & 42x^5_{n+\frac{5}{2}} & 56x^6_{n+\frac{5}{2}} \\ 0 & 0 & 2 & 6x_{n+3} & 12x^2_{n+3} & 20x^3_{n+3} & 30x^4_{n+3} & 42x^5_{n+3} & 56x^6_{n+3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+1/2} \\ y_{n+3/2} \\ f_n \\ f_{n+1/2} \\ f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \\ f_{n+5/2} \\ f_{n+3} \end{bmatrix} \quad (5)$$

Gaussian elimination technique is employed to equation (5) in finding the unknown values $a_j,\ j = 0(1)8$ which are then substituted into equation (2) to produce a continuous implicit scheme of the form

$$y(t) = \alpha_{\frac{1}{2}}(t)\, y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}}(t)\, y_{n+\frac{3}{2}} + h^2\left[\beta_0(t) f_n + \beta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \beta_1(t) f_{n+1} + \beta_{\frac{3}{2}}(t) f_{n+\frac{3}{2}} + \beta_2(t) f_{n+2} + \beta_{\frac{5}{2}}(t) f_{n+\frac{5}{2}} + \beta_3(t) f_{n+3}\right] \quad (6)$$

Then, using the transformation

$$t = \frac{x - x_{n+k-1}}{h} \text{ and } \frac{dt}{dx} = \frac{1}{h}$$

The coefficients of y_{n+j} *and* f_{n+j} are obtained in terms of t as follows:

$$\alpha_{\frac{1}{2}}(t) = \left(-t - \frac{1}{2}\right)$$

$$\alpha_{\frac{3}{2}}(t) = \left(t + \frac{3}{2}\right)$$

$$\beta_0(t) = h^2\left(\frac{1}{630}t^8 - \frac{1}{270}t^6 - \frac{1}{120}t^5 + \frac{1}{540}t^4 + \frac{1}{180}t^3 - \frac{5}{3456}t + \frac{1}{315}t^7 - \frac{53}{161280}\right)$$

$$\beta_{\frac{1}{2}}(t) = h^2\left(-\frac{8}{315}t^7 + \frac{1}{45}t^6 + \frac{1}{15}t^2 - \frac{1}{90}t^4 - \frac{2}{45}t^3 + \frac{137}{5040}t - \frac{1}{105}t^8 + \frac{277}{26880}\right)$$

$$\beta_1(t) = h^2\left(-\frac{1}{30}t^6 - \frac{29}{120}t^5 + \frac{1}{72}t^4 + \frac{1}{6}t^3 + \frac{9311}{40320}t + \frac{5}{63}t^7 + \frac{1}{42}t^8 + \frac{6931}{53760}\right)$$

$$\beta_{\frac{3}{2}}(t) = h^2\left(-\frac{8}{63}t^7 - \frac{2}{135}t^6 + \frac{2}{5}t^5 + \frac{5}{27}t^4 - \frac{4}{9}t^3 + \frac{2011}{3780}t - \frac{2}{63}t^8 + \frac{8497}{40320}\right)$$

$$\beta_2(t) = h^2\left(\frac{1}{9}t^7 + \frac{7}{90}t^6 - \frac{7}{24}t^5 - \frac{7}{18}t^4 + \frac{7}{36}t^3 + \frac{1}{2}t^2 + \frac{9137}{40320} + \frac{1}{42}t^8 + \frac{1471}{53760}\right)$$

$$\beta_{\frac{5}{2}}(t)=h^2\left(-\frac{1}{105}t^8-\frac{1}{15}t^6+\frac{1}{15}t^5+\frac{19}{90}t^4+\frac{2}{15}t^3-\frac{17}{1008}t-\frac{16}{315}t^7-\frac{59}{26880}\right)$$

$$\beta_3(t)=h^2\left(\frac{1}{630}t^8+\frac{1}{54}t^6+\frac{1}{120}t^5-\frac{13}{1080}t^4-\frac{1}{90}t^3+\frac{191}{120960}t+\frac{1}{105}t^7+\frac{31}{161280}\right)$$

Evaluating the continuous method at the end point gives the discrete scheme

$$y_{n+3}=\frac{5}{2}y_{n+\frac{3}{2}}-\frac{3}{2}y_{n+\frac{1}{2}}+\frac{h^2}{32256}\left[-53f_n+22889f_{n+1}+15501f_{n+2}+535f_{n+3}+1158f_{n+\frac{1}{2}}+22900f_{n+\frac{3}{2}}+8550f_{n+\frac{5}{2}}\right] \quad (7)$$

While the evaluation of the first derivative of the continuous scheme at all points yields

$$y'_n=-\frac{1}{120960h}\left(\begin{array}{l}120960y_{n+\frac{1}{2}}-120960y_{n+\frac{3}{2}}+18479h^2f_n-3357h^2f_{n+1}-16275h^2f_{n+2}\\-703h^2f_{n+3}+85800h^2f_{n+\frac{1}{2}}+31904h^2f_{n+\frac{3}{2}}+5112h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (8)$$

$$y'_{n+\frac{1}{2}}=\frac{1}{7560h}\left(\begin{array}{l}7560y_{n+\frac{3}{2}}-7560y_{n+\frac{1}{2}}+38h^2f_n-2694h^2f_{n+1}-246h^2f_{n+2}\\-10h^2f_{n+3}-1293h^2f_{n+\frac{1}{2}}+350h^2f_{n+\frac{3}{2}}+75h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (9)$$

$$y'_{n+1}=-\frac{1}{40320h}\left(\begin{array}{l}40320y_{n+\frac{3}{2}}-40320y_{n+\frac{1}{2}}+85h^2f_n-1295h^2f_{n+1}-1121h^2f_{n+2}\\-37h^2f_{n+3}-1480h^2f_{n+\frac{1}{2}}+3552h^2f_{n+\frac{3}{2}}+296h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (10)$$

$$y'_{n+\frac{3}{2}}=\frac{1}{7560h}\left(\begin{array}{l}7560y_{n+\frac{3}{2}}-7560y_{n+\frac{1}{2}}+h^2f_n+2169h^2f_{n+1}-213h^2f_{n+2}\\-5h^2f_{n+3}+105h^2f_{n+\frac{1}{2}}+1678h^2f_{n+\frac{3}{2}}+45h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (11)$$

$$y'_{n+2}=-\frac{1}{120960h}\left(\begin{array}{l}120960y_{n+\frac{1}{2}}-120960y_{n+\frac{3}{2}}+175h^2f_n-27933h^2f_{n+1}-27411h^2f_{n+2}\\-191h^2f_{n+3}-3288h^2f_{n+\frac{1}{2}}-64352h^2f_{n+\frac{3}{2}}+2040h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (12)$$

$$y'_{n+\frac{5}{2}}=\frac{1}{2520h}\left(\begin{array}{l}2520y_{n+\frac{3}{2}}-2520y_{n+\frac{1}{2}}+2h^2f_n+734h^2f_{n+1}+1550h^2f_{n+2}\\-14h^2f_{n+3}+25h^2f_{n+\frac{1}{2}}+1002h^2f_{n+\frac{3}{2}}+481h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (13)$$

$$y'_{n+3}=-\frac{1}{120960h}\left(\begin{array}{l}120960y_{n+\frac{1}{2}}-120960y_{n+\frac{3}{2}}+767h^2f_n-15021h^2f_{n+1}-27939h^2f_{n+2}\\-18415h^2f_{n+3}-7512h^2f_{n+\frac{1}{2}}-85600h^2f_{n+\frac{3}{2}}-88200h^2f_{n+\frac{5}{2}}\end{array}\right) \quad (14)$$

These schemes in equations (7) - (14) are combined together in matrix form and by using the matrix inversion technique, a block method of the following form is produced

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1/2}\\y_{n+1}\\y_{n+3/2}\\y_{n+2}\\y_{n+5/2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-1/2}\\y_{n-1}\\y_{n-3/2}\\y_{n-2}\\y_{n-5/2}\\y_{n}\end{bmatrix}+h\begin{bmatrix}0&0&0&0&0&1/2\\0&0&0&0&0&1\\0&0&0&0&0&3/2\\0&0&0&0&0&2\\0&0&0&0&0&5/2\\0&0&0&0&0&3\end{bmatrix}\begin{bmatrix}y'_{n-1/2}\\y'_{n-1}\\y'_{n-3/2}\\y'_{n-2}\\y'_{n-5/2}\\y'_{n}\end{bmatrix}+$$

$$h^2\begin{bmatrix}\frac{275}{2304}&-\frac{5717}{53760}&\frac{10621}{120960}&-\frac{7703}{161280}&\frac{403}{26880}&-\frac{199}{96768}\\ \frac{97}{210}&-\frac{2}{9}&\frac{197}{945}&-\frac{97}{840}&\frac{23}{630}&-\frac{19}{3780}\\ \frac{1485}{1792}&-\frac{2403}{17920}&\frac{45}{128}&-\frac{3267}{17920}&\frac{513}{8960}&-\frac{141}{17920}\\ \frac{376}{315}&-\frac{2}{105}&\frac{656}{945}&-\frac{2}{9}&\frac{8}{105}&-\frac{2}{189}\\ \frac{8375}{5376}&\frac{3125}{32256}&\frac{25625}{24192}&-\frac{625}{10752}&\frac{275}{2304}&-\frac{1375}{96768}\\ \frac{27}{14}&\frac{27}{140}&\frac{51}{35}&\frac{27}{280}&\frac{27}{70}&0\end{bmatrix}\begin{bmatrix}f_{n+1/2}\\f_{n+1}\\f_{n+3/2}\\f_{n+2}\\f_{n+5/2}\\f_{n+3}\end{bmatrix}+h^2\begin{bmatrix}0&0&0&0&0&\frac{28549}{483840}\\0&0&0&0&0&\frac{1027}{7560}\\0&0&0&0&0&\frac{759}{3584}\\0&0&0&0&0&\frac{272}{945}\\0&0&0&0&0&\frac{35225}{96768}\\0&0&0&0&0&\frac{123}{280}\end{bmatrix}\begin{bmatrix}f_{n-1/2}\\f_{n-1}\\f_{n-3/2}\\f_{n-2}\\f_{n-5/2}\\f_{n}\end{bmatrix} \quad (15)$$

Substituting the schemes that made up the block in (15) into equations (8) to (14), gives

$$y'_{n+\frac{1}{2}}=y'_n+h\left[\frac{19087}{120960}f_n-\frac{15487}{40320}f_{n+1}-\frac{6737}{40320}f_{n+2}-\frac{863}{120960}f_{n+3}+\frac{2713}{5040}f_{n+\frac{1}{2}}+\frac{293}{945}f_{n+\frac{3}{2}}+\frac{263}{5040}f_{n+\frac{5}{2}}\right] \quad (16)$$

$$y'_{n+1}=y'_n+h\left[\frac{1139}{7560}f_n+\frac{11}{2520}f_{n+1}-\frac{269}{2520}f_{n+2}-\frac{37}{7560}f_{n+3}+\frac{47}{63}f_{n+\frac{1}{2}}+\frac{166}{945}f_{n+\frac{3}{2}}+\frac{11}{315}f_{n+\frac{5}{2}}\right] \quad (17)$$

$$y'_{n+\frac{3}{2}}=y'_n+h\left[\frac{137}{896}f_n+\frac{1161}{4480}f_{n+1}-\frac{729}{4480}f_{n+2}-\frac{29}{4480}f_{n+3}+\frac{81}{112}f_{n+\frac{1}{2}}+\frac{17}{35}f_{n+\frac{3}{2}}+\frac{27}{560}f_{n+\frac{5}{2}}\right] \quad (18)$$

$$y'_{n+2}=y'_n+h\left[\frac{143}{945}f_n+\frac{64}{315}f_{n+1}+\frac{29}{315}f_{n+2}-\frac{4}{945}f_{n+3}+\frac{232}{315}f_{n+\frac{1}{2}}+\frac{752}{945}f_{n+\frac{3}{2}}+\frac{8}{315}f_{n+\frac{5}{2}}\right] \quad (19)$$

$$y'_{n+\frac{5}{2}}=y'_n+h\left[\frac{3715}{24192}f_n+\frac{2125}{8064}f_{n+1}+\frac{3875}{8064}f_{n+2}-\frac{275}{24192}f_{n+3}+\frac{725}{1008}f_{n+\frac{1}{2}}+\frac{125}{189}f_{n+\frac{3}{2}}+\frac{235}{10082}f_{n+\frac{5}{2}}\right] \quad (20)$$

$$y'_{n+3}=y'_n+h\left[\frac{41}{280}f_n+\frac{27}{280}f_{n+1}+\frac{27}{280}f_{n+2}+\frac{41}{280}f_{n+3}+\frac{27}{35}f_{n+\frac{1}{2}}+\frac{34}{35}f_{n+\frac{3}{2}}+\frac{27}{35}f_{n+\frac{5}{2}}\right] \quad (21)$$

ANALYSIS OF THE BASIC PROPERTIES OF THE METHOD

Order and Error Constant:

The Lambert (1973)'s method for finding the order of a numerical scheme is also applied to equation (15). Hence, the new hybrid block method is of uniform order $\rho = 7$ with error constants $\left[\frac{6637}{78331}\quad -\frac{1129}{32552}\quad -\frac{2243}{5377}\quad \frac{5221}{8745}\quad \frac{1667}{2353}\quad \frac{141}{723}\right]^T$

Zero Stability of the Block

Definition: The block is said to be zero stable if the roots $z_{s,} s = 1, 2, 3, ..., n$ of the characteristics polynomial $\rho(z)$ defined by $\rho(z) = \det(zA - E)$ satisfies $|z_s| \le 1$ and the roots $|z_s| = 1$ is simple. (Kuboye and Omar 2015)

For our hybrid method,

$$A = \left[z \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$A = z^6 - z^5 = 0, z = 0, 0, 0, 0, 0, 0$$

Hence the block is zero stable. See Abolarin *et al* (2020)

Consistency
Our new hybrid block method is consistent since the order of the method is greater than one.
Convergence

In tandem with the theory of Lambert (1973), the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Hence the new hybrid block method is convergent.

NUMERICAL EXPERIMENTS

In this section, the performance of the three-step hybrid method is examined on some test examples. The results obtained from the test examples are displayed in tabular form. We used MATLAB codes for the computational purposes.

Problem 1: $y'' - x(y')^2 = 0, \quad y(0) = 1, \; y'(0) = \frac{1}{2}, h = 0.003125$

Exact solution: $y(x) = 1 + \frac{1}{2} In\left[\frac{2+x}{2-x}\right]$

Table 1: Comparison of the result of the developed three-step hybrid block method for test problem 1 with the errors in Kuboye *et al* (2018) and Awari and Abada (2014)

x	Exact solution	TSHBA (Computed solution)	Error in TSHBA	Error in Kuboye et al (2018)	Error in Awari and Abada (2014)
0.1	1.050041729278491400	1.050041729278489400	1.998401E-15	1.113847E-10	1.440E-08
0.2	1.099546268422664900	1.099546268422647400	1.754152E-14	5.078077E-10	3.850E-08
0.3	1.151140435936466800	1.151140435936399500	6.727952E-14	1.516145E-09	6.330E-08
0.4	1.202732554054082300	1.202732554053906400	1.758593E-13	4.193464E-09	8.800E-08
0.5	1.254579651931831100	1.254579651931447400	3.836931E-13	1.143373E-08	1.151E-07
0.6	1.309519604203112100	1.309519604202333600	7.784884E-13	3.109587E-08	1.427E-07
0.7	1.365443754271396900	1.365443754269914000	1.482814E-12	8.453424E-08	1.716E-07
0.8	1.422719216277534600	1.422719216274822300	2.712275E-12	2.297908E-07	1.796E-07
0.9	1.484700278594052600	1.484700278589056600	4.996004E-12	6.246375E-07	1.941E-07
1.0	1.549306144334055900	1.549306144324915000	9.140910E-12	1.697941E-06	2.109E-07

Problem 2: $y'' + \lambda^2 y = 0, y(0) = 1, \; y'(0) = 2, \quad \lambda = 2, h = 0.01$

Exact solution: $y(x) = \cos 2x + \sin 2x,$

Table 2: Comparison of the result of the developed three-step hybrid block method for test problem 2 with the errors in Abhulimen and Okunuga (2008)

x	Exact solution	TSHBA (Computed solution)	Error in TSHBA	Error in Abhulimen and Okunuga (2008)
0.01	1.019798673359910900	1.019798673280353500	7.955747E-11	-
0.02	1.039189440847612100	1.039189438315828900	2.531783E-09	0.26E-05
0.03	1.058164546414648700	1.058164527294756600	1.911989E-08	0.40E-05
0.04	1.076716400271792200	1.076716349318852700	5.095294-08	0.53E-05
0.05	1.094837581924853900	1.094837496836278800	8.508858E-08	0.66E-05
0.06	1.112520843142785500	1.112520710115289000	1.330275E-07	0.79E-05
0.07	1.129759110856873600	1.129758915016439300	1.958404E-07	0.93E-05
0.08	1.146545489989872800	1.146545229144277000	2.608456E-07	0.11E04
0.09	1.162873266213945600	1.162872926945781100	3.392682E-07	0.12E-04
0.1	1.178735908636302700	1.178735476496691100	4.321396E-07	0.13E-04

DISCUSSION OF RESULTS

The results generated by the developed three-step hybrid block method are displayed in tables (1) and (2). Table 1 shows the result of the new three-step hybrid method when applied to a non-linear second order ODE problem 1. The newly developed hybrid method produces more accurate result when compared with the errors generated by the method of Kuboye *et al* (2018) and Awari and Abada (2014). The three-step hybrid scheme also gives better result when compared with the method of Abhulimen and Okunuga (2008) for solving the linear second order ODE in problem 2. The result is displayed in table 2.

CONCLUSION

The three-step implicit hybrid block method for the numerical solution of second order ODEs with initial value problems is developed in this research. The developed method is zero stable, consistent and convergent. It also produces more accurate result than the existing methods.

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