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A Mathematical Modeling and Global Stability Analysis of An Integrated Control Measures For Diphtheria Diseases, Incorporating Immunization, Surveilllance, Prompt Case Management Strategies

Nasiru Mohammed Mangga and Musa Kida

Department of Mathematics and Computer Science, Kashim Ibrahim University, Maiduguri KIU.

* Corresponding authors' email: nasiru.mangga@kiu.edu.ng

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ABSTRACT

This study develops and analyzes a mathematical model for diphtheria transmission dynamics incorporating immunization, surveillance, and prompt case management as control measures. Analytical results include the disease-free and endemic equilibria, the basic reproduction number (R_0), global stability using Lyapunov function and a sensitivity analysis showing that transmission rate, immunization coverage, surveillance and prompt case management effort have the greatest influence on disease spread. Numerical simulations using Matlab2024a demonstrate that combined interventions have drastically reduced both carriers and infected individuals, achieving Diphtheria elimination within 100 days. These results provide quantitative guidance for public health strategies, emphasizing the importance of immunization, surveillance and the prompt care management control measures for proper control of diphtheria.

Keywords: Diphtheria, Surveillance, Basic Reproduction Number, Prompt Care Management, Global Stability

INTRODUCTION

Diphtheria is a highly contagious bacterial infection that primarily affects the mucous membranes of the nose, throat, and airways (Abubakar, A. *et al.,* 2025). Corynebacterium diphtheriae is the main cause of this extremely contagious, toxin-mediated bacterial infection, with Corynebacterium ulcerans and Corynebacterium pseudotuberculosis being less common. It is primarily spread by respiratory droplets or intimate physical contact with infected people. Clinical symptoms include fever, sore throat, cervical lymphadenopathy, and the development of a grayish pseudomembrane in the upper respiratory tract, which can obstruct airways and cause death if left untreated (Hadfield et al., 2000; Nasiru & Kida, 2026; World Health Organization, WHO, 2017).

In many low- and middle-income countries, diphtheria is still a public health concern despite the availability of an effective vaccine for more than 70 years. This is especially true in areas where routine immunization coverage is either suboptimal or disrupted due to conflict, population displacement, and weak health systems (Clarke, 2019). The occurrence of recurring outbreaks in regions of Asia and Africa highlights the difficulties in maintaining herd immunity and the persistence of immunity gaps (WHO, 2023). Since immunity wanes with age, children and adults who receive insufficient booster shots are also at risk for illness reappearance (Truelove et al., 2020). Antibiotic therapy, early case diagnosis, preventive immunization, timely administration of diphtheria antitoxin, and efficient isolation of infectious individuals are all essential components of diphtheria control from a public health standpoint (Efstratiou & George, 1999). However, these control attempts are frequently hampered by logistical issues, delayed diagnosis, and restricted access to antitoxin in areas with limited resources. The aforementioned difficulties underscore the necessity of strong analytical instruments that can bolster evidence-based decision-making and maximize intervention tactics.

A potent paradigm for comprehending the dynamics of infectious disease transmission, including diphtheria, is mathematical modeling. Researchers can measure the effects of important epidemiological parameters on disease persistence and control, such as vaccination, waning immunity, recovery, and disease-induced mortality, using compartmental models (Hethcote, 2000). Additionally, the combination of numerical simulations, bifurcation theory, and stability analysis sheds light on the possibility for complicated dynamics like oscillatory outbreaks as well as the threshold conditions for disease elimination.

Through mathematical modeling, mathematicians like Nasiru M. Mangga and Musa Kida (2026) have contributed to the eradication of diphtheria; however, Hospitals and Treatment facilities, where the infected individuals should receive the appropriate medications before they can recover, were not included in that model. As a result, we will modify our model (Nasiru and Kida, 2026) in this study by incorporating Hospitals and Treatment facilities and using different control strategies than in the previous work. Here, we'll employ surveillance, immunization, and prompt case management as control measures.

To find economical methods of reducing disease burden while taking limited resources into consideration, optimal control theory has been used more and more to infectious disease models in recent years (Lenhart & Workman, 2007). According to them, when developing vaccine and treatment plans for diphtheria, these methods are especially pertinent since they strike a balance between the influence on public health and practicality from an economic standpoint. Therefore, developing sustainable control and eradication strategies—particularly in areas that are endemic and prone to outbreaks, it requires a thorough analytical and numerical study of diphtheria transmission dynamics.

According to WHO, 2023, the best defense against diphtheria is immunization, which is administered via vaccines like pentavalent or DPT. It reduces transmission by fostering the development of both individual and herd immunity. It includes booster shots and regular childhood vaccinations. During outbreaks, mass immunization is performed to stop the spread. The organization is also with the opinion that diphtheria cases are continuously monitored and reported as part of surveillance. They said, surveillance aids in the early identification of outbreaks which includes contact tracking and laboratory confirmation and guarantees prompt action to stop the spread of infection. WHO, 2023, showed that the goal of prompt care and management is early diagnosis and prompt treatment, which is mostly accomplished with antibiotics and diphtheria antitoxin. In order to stop the infection from spreading, it also entails isolating sick people, which lowers complications, mortality, and transmission.

MATERIALS AND METHODS

Mathematical Model

Model Assumptions

i) Every compartment is subject to natural death at a constant rate μ.
ii) The population is assumed to be homogeneous.
iii) Diphtheria is transmitted only through person-to-person contact; vertical transmission (from mother to child) is not considered
iv) Susceptible population can be vaccinated and move to vaccination class.
v) Waning of immunity on vaccinated class can force them to move to susceptible class.
vi) When susceptible individuals are contacted with carrier or infected individuals, they can be exposed to Diphtheria and later be detained.
vii) When detained individuals are vaccinated, they can be moved to vaccination or susceptible class.
viii) Infected individuals can be taken to the Hospital and treatment centers for medication and treatment.
ix) Individuals that are recovered from the Hospitals and treatment centers move to recovery class.
x) There is possibility for carriers to get recovered naturally and then move to recovery class.
xi) Individuals who recovered can also stand the risk to contact Diphtheria, thus can be vaccinated and move to vaccination class.
xii) Infectious individuals are categorized as Carrier (C) and infected (I).

Model Formulation

Nine epidemiological compartments—Susceptible, Vaccinated, Exposed, Detained, Carrier, Infected, Hospitalized, Treatment (T), and Recovered (R) classes—are used in the model diagram to show the dynamics of diphtheria transmission in a human population. Recruitment brings new members into the population at a steady pace (Λ), and they are presumed to join the susceptible compartment (S). A susceptible person can contract diphtheria by coming into close contact with an infected person and entering the exposed class (E) at a rate determined by the force of infection (λ). When susceptible individuals receive vaccinations at a rate (γ), they are placed in the vaccinated class (V). The vaccination approach, which lowers vulnerability to infection, improves this process, as given in Figure 1.

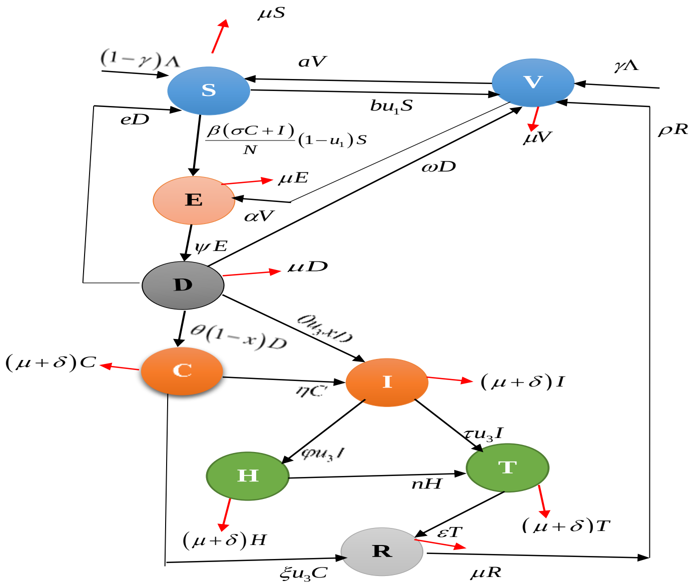

Figure 1: Schematic Diagram of the Model

After the pre-symptomatic period, exposed individuals become infected; a proportion x of these individuals are identified early and placed in the Detained class (D); the remaining proportion (1−x) are considered carriers. Even though they are detained, they may become infected and infectious, reflecting clinical progression. The carriers may recover naturally and enter the recovered class (R) at the rate ξ or can be infected and infectious and then moved to infected class at the rate η. These infected individuals may be transferred to the hospital at a rate φ or treatment center at a rate n for prompt medication. Finally, they may move to the recovery compartment at a rate ε.

The immunization control measure (u_1) transfers individuals from the susceptible class to vaccinated after they are being immunized at the rate b. If immunity declines, vaccinated individuals may return to the susceptible class at the rate a which allow possible reinfection. Immunization control measure also acts on the force of infection by reducing or stopping effective contact between susceptible and infectious individuals, resulting to reduction or eradication of movement from susceptible class to exposed compartment. The Surveillance control measure (u_2) allows early detection, monitoring and intervention which reduces the rate at which exposed individuals require detention. With surveillance control, the health workers are at the advantages of monitor or isolate individuals before they can be detained and thus reduce congestion at detention centers. This prompt care management (u_3) as a control measure, speed up diagnosis, treatment initiation and clinical management. It decreases the average time individuals remain in the infected class and transmission to others such as Hospitals, Treatment and recovery centers. It also increases recovery rate from Hospital compartment to recovery class through treatment. At recovery class, prompt care management stabilizes individuals and reduce movement back to infected class. It is considered that those who have received vaccinations and recovered are immune. All compartments are liable to natural death at rate (μ). The overall dynamics of the diphtheria and the possibility of its eradication are determined by the combined effects of immunization, surveillance, and timely care management.

Table 1: Model Variables and Descriptions

Variables	Descriptions
$S(t)$	Susceptible Population at time, t
$V(t)$	Vaccinated Individuals at time, t
$E(t)$	Exposed individuals at time, t
$D(t)$	Detained population at time, t
$C(t)$	Carrier compartment at time, t
$I(t)$	Infected class at time, t
$H(t)$	Hospitalized individuals at time, t
$T(t)$	Treatment class at time, t
$R(t)$	Recovered individuals at time, t

Table 2: Model Parameters and Description

Parameters	Descriptions
Λ	Recruitment rate into susceptible and vaccinated population
β	Rate of contact of susceptible class and the carrier and the infected class
μ	Natural death rate
δ	Death rate caused by disease
a	Progression rate from vaccinated class to susceptible class
b	Progression rate from susceptible class to vaccinated class
e	Transfer rate from detained class to susceptible individuals
ω	Progression rate from detained population to vaccinated individuals
θ	Rate of movement from detained class to infected and as well to carrier compartment
η	Movement rate from class of carriers to infected compartment
ξ	Transfer rate from carriers to Recovery class
σ	Rate at which exposed individuals leaves the latent stage
n	Transfer rate from hospitalized class to treatment compartment
ψ	Transfer rate from exposed class to detained compartment
φ	Rate of movement from infected class to Hospitalized compartment
α	Rate of movement from Hospital to Recovery compartment
ε	Movement rate from treatment class to Recovery compartment
ρ	Rate of movement from Recovery class to vaccinated individuals
τ	Transfer rate to treatment class from infected class
χ	Propossion of detained population that are infected
u_1	Immunization against Diphtheria transmission
u_2	Surveillance taken to control diphtheria spread
u_3	Prompt care management of Diphtheria disease.

Table 3: Model Parameters/ Variables Description and Value Source/Remark

Parameter /Variables	Meaning	Value	Source/Remark
Λ	Recruitment rate	1958	Nasiru & Kida, 2026
μ	Natural death rate	0.02	WHO, 2023
β	Transmission rate	0.6	Nasiru & Kida, 2026
a	Progression rate (V\to S)	0.5	Assumed
b	Vaccination rate (S\to V)	0.4	Lenhart & Workman, 2007.

Parameter /Variables	Meaning	Value	Source/Remark
e	Movement rate (D\to S)	0.7	Assume
γ	Proportion vaccinated at birth	0.6	Nasiru & Kida, 2026
ψ	Progression rate (E\to D)	0.25	Truelove *etal.,* 2020
θ	Progression (D\to I)	0.4	Estimated
x	Fraction of infected	0.65	WHO, 2017
ξ	Recovery rate of Carrier	0.3	Estimated
δ	Disease- induced death	0.05	WHO, 2023
ρ	Loss of immunity	0.1	Truelove *etal.,* 2020
α	Transfer rate (E\to V)	0.8	Eloho *et al,* 2023
ω	Progression rate (D\to V)	0.67	Afolabi & M. Miswanto, 2025
η	Progression rate (C\to I)	0.4	Assumed
φ	Movement rate (I\to H)	0.05	Eloho & Akindele, 2025
ε	Progression rate (T\to R)	1.5	Eloho & Akindele, 2025
n	Transfer rate (H\to T)	0.6	Assumed
τ	Movement rate (I\to T)	0.3	Assumed
V	Vaccinated Population	50,000	Afolabi & Miswanto, 2025
S	Susceptible Population	100,000	Afolabi & Miswanto, 2025
E	Exposed Population	10,000	Afolabi & Miswanto, 2025
D	Detained Individuals	2,000	Afolabi & Miswanto, 2025
C	Carrier Population	3,000	Assumed
I	Infected population	5,000	Afolabi & Miswanto, 2025
H	Hospitalized Individuals	500	Estimated
T	Treatment center	500	Estimated
R	Recovered individuals	500	Estimated

The Model Equations

$$\left.\begin{aligned}
\frac{dV}{dt} &= \gamma\Lambda + bu_1S + \omega D + \rho R - \alpha V - aV - \mu V \\
\frac{dS}{dt} &= (1-\gamma)\Lambda + aV + eD - bu_1S - \frac{\beta(\sigma C + I)}{N}(1-u_1)S - \mu S \\
\frac{dE}{dt} &= \frac{\beta(\sigma C + I)}{N}(1-u_1)S + \alpha V - \psi u_2E - \mu E \\
\frac{dD}{dt} &= \psi u_2E - \omega D - eD - \theta D - \mu D \\
\frac{dC}{dt} &= \theta(1-x)D - \eta C - \xi u_3C - (\mu+\delta)C \\
\frac{dI}{dt} &= \eta C + \theta xD - \varphi u_3I - \tau u_3I - (\mu+\delta)I \\
\frac{dH}{dt} &= \varphi u_3I - nH - (\mu+\delta)H \\
\frac{dT}{dt} &= nH + \tau u_3I - \varepsilon T - (\mu+\delta)T \\
\frac{dR}{dt} &= \xi u_3C + \varepsilon T - \rho R - \mu R
\end{aligned}\right\} \qquad (1)$$

with the initial conditions as follows

$V(0) \geq 0, S(0) \geq 0, E(0) > 0, D(0) > 0, C(0) > 0, I(0) > 0, H > 0, T > 0, R(0) > 0$

Model Analysis

Basic Properties and Invariant Region

All the solutions to system (1), the basic properties and invariant region must be shown to be positive for each instance in which $t > 0$. This is shown by the lemma 1:

Lemma 1

The system equation (1) has all possible solution $V(t)+S(t)+E(T)+D(T)+C(t)+I(t)+H(t)+T(t)+R(t)$ that are enclosed by the following region:

$$\Omega = \left\{ (V,S,E,C,I,D,H,T,R) \in \Re^9_+ : V,S,E,C,I,D,H,T,R \leq \frac{\Lambda}{\mu} \right\}$$

Proof: Assume that the system equation (1) gives

$$\frac{dN}{dt} = \frac{dV}{dt} + \frac{dS}{dt} + \frac{dE}{dt} + \frac{dD}{dt} + \frac{dC}{dt} + \frac{dI}{dt} + \frac{dH}{dt} + \frac{dT}{dtdt} + \frac{dR}{dt} \qquad (2)$$

Thus

$$\frac{dN}{dt} = \Lambda - \mu N \qquad (3)$$

It is important to remember that in the event that Diphtheria does not occur,

$$\frac{dN(t)}{dt} \le \Lambda - \mu N(t) \qquad (4)$$

and it is resolved to be

$$N(t) \le N(0)e^{-\mu t} \qquad (5)$$

This implies that $N(0)$ is called initial population, therefore,

$$\lim it \sup N(t) \le \frac{\Lambda}{\mu} \qquad (6)$$

$$t \to \infty$$

also,

$$V(t) + S(t) + E(T) + D(T) + C(t) + I(t) + H(t) + T(t) + R(t) \le \frac{\Lambda}{\mu}$$

Therefore, for the analysis of system equation (1), the region is given by the set

$$\Omega = \left\{ V + S + E + C + I + D + H + T + R \le \frac{\Lambda}{\mu} \right\} \qquad (7)$$

Positivity of the Model's system solution

The positivity of the solutions of the model is provided by the established theorem 1and its proof.

Theorem 1

The solution of system equation (1), $V, S, E, C, I, D, H, \mathrm{T}, R$ are positive for all $t > 0$ if and if $V(0) \ge 0, S(0) \ge 0, E(0) \ge 0, D(0) \ge 0, C(0) \ge 0, I(0) \ge 0, H(0) \ge 0, T(0) \ge 0, R(0) \ge 0.$

Proof

From System equation (1), initial equation is

$$\frac{dV}{dt} \ge -(\alpha + a + \mu)V \qquad (8)$$

which implies $\frac{dV}{V} \ge -(\alpha + a + \mu)dt$

and finally resulted to

$$V(t) \ge V(0)e^{-(\alpha + a + \mu)t} \ge 0 \qquad (9)$$

Thus, $V(t)$ is a positive solution.

In a similar way, the second system of equation (1) yields

$$S(t) \ge S(0)e^{-\mu(t)} \ge 0 \qquad (10)$$

Following the same techniques, the third, fourth up to ninth equations are found to be

$$\left.\begin{aligned} E(t) &\ge E(0)e^{-\mu(t)} \ge 0 \\ D(t) &\ge D(0)e^{-\mu(t)} \ge 0 \\ C(t) &\ge C(0)e^{-(\mu+\delta)t} \ge 0 \\ I(t) &\ge I(0)e^{-(\mu+\delta)t} \ge 0 \\ H(t) &\ge H(0)e^{-(\mu+\delta)t} \ge 0 \\ T(t) &\ge T(0)e^{-(\mu+\delta)(t)} \ge 0 \\ R(t) &\ge R(0)e^{-\mu(t)} \ge 0 \end{aligned}\right\} \qquad (11)$$

Thus,

$V(t) > 0, S(t) > 0, E(T) > 0, D(T) > 0, C(t) > 0, I(t) > 0, H(t) > 0, T(t) > 0, R(t) > 0,$

for all

$t \ge 0$

Diphtheria Equilibrium Points

Diphtheria-Free Equilibrium Point, E_0

In the absence of Diphtheria disease, we have

$$E = D = C = I = H = T = R = 0 \qquad (12)$$

Hence, the whole population consists of only the susceptible and vaccinated populations.

Substituting equation (12) into the system equation (1), we obtained

$$E_0 = \left(V^0, S^0, E^0, D^0, C^0, I^0, H^0, T^0, R^0\right) \qquad (13)$$

$$= \left(\frac{\gamma\Lambda + bu_1 S}{a + \mu}, \frac{NaV}{Nbu_1 + \beta(\sigma C + I)(1 - u_1) + N\mu}, 0, 0, 0, 0, 0, 0, 0 \right)$$

Diphtheria Presence Equilibrium Point, E_*

The Diphtheria presence equilibrium point is the point where $V \ne 0, S \ne 0, E \ne 0, D \ne 0, C \ne 0, I \ne 0, H \ne 0, T \ne 0, R \ne 0$.

Its equilibrium point is given by

$$E_* = \left(V^*, S^*, E^*, D^*, C^*, I^*, H^*, T^*, R^*\right) \qquad (14)$$

Where

$$V^* = \frac{\gamma\Lambda + bu_1 S^* + \omega D^* + \rho R^*}{a + \mu - \alpha};$$

$$S^* = \frac{Nbu_1 + \beta(\sigma C^* + I^*)(1 - u_1) + N\mu}{N\left[aV^* + eD^* + (1 - \gamma)\Lambda\right]}$$

$$E^* = \frac{\beta(\sigma C^* + I^*)(1 - u_1)S^* + \alpha V^*}{N(u_2\psi + \mu)};$$

$$D^* = \frac{\psi u_2 E^*}{\omega + e + \theta + + \mu};$$

$$C^* = \frac{\theta(1 - x)D^*}{\eta + \xi u_3 + \mu + \delta}$$

$$I^* = \frac{\eta C^* + \theta x D^*}{\varphi u_3 + \xi u_3 + \mu + \delta};$$

$$H^* = \frac{\varphi u_3 I^*}{n + \mu + \delta};$$

$$T^* = \frac{nH^* + \tau u_3 I^*}{\varepsilon + \mu + \delta};$$

$$T^* = \frac{\xi u_3 I^* + \varepsilon T^*}{\rho + \mu};$$

$$R^* = \frac{\xi u_3 C^* + \varepsilon T^*}{\rho + \mu}$$

Thus, the invariant set is positive within the region Ω and attracts all the solutions of the system equation (1). In this section, we obtain the following results which guarantee that the diphtheria disease is governed by the equation (1) and is mathematical meaningful in a region given by

$$S_0 = \left\{ \begin{array}{l} \{V, S, E, C, I, D, H, T, R\} \in \Re_+^9 V > S > E \ge 0, C \ge 0, \\ I \ge 0, D \ge 0, H \ge 0, T \ge 0, R \ge 0, \square_{(t)} \le \frac{\Lambda}{\mu} + \left(\mu_0 - \frac{\Lambda}{\mu}\right)e^{\mu t} \end{array} \right\} \qquad (15)$$

Reproduction Number for Diphtheria

Basic reproduction number, R_0 is the average number of secondary infections produced by one Diphtheria infected person introduced in a perfectly susceptible population. This number is denoted by R_0 and is obtained by the formula

$$R_0 = \rho(FV^{-1}) \qquad (16)$$

where ρ is the spectral radius of the Jacobian matrix, F a matrix representing the newly infected individuals and V a transferred individuals (van den Driessche & Watmough, 2002).

For this work, we apply next generation matrix on six infected compartments: Expoed, (E), Detained, (D), Carrier, (C), Infected, (I), Hospitalized, (H), and Treatment (T). Showing the newly infected rate of individuals, as given in system equation (1), we have

$$F=\begin{bmatrix}\frac{\partial f_1}{\partial E}&\frac{\partial f_1}{\partial D}&\frac{\partial f_1}{\partial C}&\frac{\partial f_1}{\partial I}&\frac{\partial f_1}{\partial H}&\frac{\partial f_1}{\partial T}\\ \frac{\partial f_2}{\partial E}&\frac{\partial f_2}{\partial D}&\frac{\partial f_2}{\partial C}&\frac{\partial f_2}{\partial I}&\frac{\partial f_2}{\partial H}&\frac{\partial f_2}{\partial T}\\ \frac{\partial f_3}{\partial E}&\frac{\partial f_3}{\partial D}&\frac{\partial f_3}{\partial C}&\frac{\partial f_3}{\partial I}&\frac{\partial f_3}{\partial H}&\frac{\partial f_3}{\partial T}\\ \frac{\partial f_4}{\partial E}&\frac{\partial f_4}{\partial D}&\frac{\partial f_4}{\partial C}&\frac{\partial f_4}{\partial I}&\frac{\partial f_4}{\partial H}&\frac{\partial f_4}{\partial T}\\ \frac{\partial f_5}{\partial E}&\frac{\partial f_5}{\partial D}&\frac{\partial f_5}{\partial C}&\frac{\partial f_5}{\partial I}&\frac{\partial f_5}{\partial H}&\frac{\partial f_5}{\partial T}\\ \frac{\partial f_6}{\partial E}&\frac{\partial f_6}{\partial D}&\frac{\partial f_6}{\partial C}&\frac{\partial f_6}{\partial I}&\frac{\partial f_6}{\partial H}&\frac{\partial f_6}{\partial T}\end{bmatrix}=\begin{bmatrix}0&0&X&Y&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix} \tag{17}$$

Where, $X=\frac{\beta\sigma}{N}(1-u_1)S$ and $Y=\frac{\beta}{N}(1-u_1)S$

Also,

$$V=\begin{bmatrix}\frac{\partial V_1}{\partial E}&\frac{\partial V_1}{\partial D}&\frac{\partial V_1}{\partial C}&\frac{\partial V_1}{\partial I}&\frac{\partial V_1}{\partial H}&\frac{\partial V_1}{\partial T}\\ \frac{\partial V_2}{\partial E}&\frac{\partial V_2}{\partial D}&\frac{\partial V_2}{\partial C}&\frac{\partial V_2}{\partial I}&\frac{\partial V_2}{\partial H}&\frac{\partial V_2}{\partial T}\\ \frac{\partial V_3}{\partial E}&\frac{\partial V_3}{\partial D}&\frac{\partial V_3}{\partial C}&\frac{\partial V_3}{\partial I}&\frac{\partial V_3}{\partial H}&\frac{\partial V_3}{\partial T}\\ \frac{\partial V_4}{\partial E}&\frac{\partial V_4}{\partial D}&\frac{\partial V_4}{\partial C}&\frac{\partial V_4}{\partial I}&\frac{\partial V_4}{\partial H}&\frac{\partial V_4}{\partial T}\\ \frac{\partial V_5}{\partial E}&\frac{\partial V_5}{\partial D}&\frac{\partial V_5}{\partial C}&\frac{\partial V_5}{\partial I}&\frac{\partial V_5}{\partial H}&\frac{\partial V_5}{\partial T}\\ \frac{\partial V_6}{\partial E}&\frac{\partial V_6}{\partial D}&\frac{\partial V_6}{\partial C}&\frac{\partial V_6}{\partial I}&\frac{\partial V_6}{\partial H}&\frac{\partial V_6}{\partial T}\end{bmatrix}=\begin{bmatrix}A&0&0&0&0&0\\G&B&0&0&0&0\\0&I&C&0&0&0\\0&J&-\eta&D&0&0\\0&0&0&L&E&0\\0&0&0&M&-n&F\end{bmatrix} \tag{18}$$

where,

$A=\psi u_2+\mu; B=\varepsilon+\omega+\theta+\mu; C=\eta+\xi u_3+\mu+\delta; D=\varphi u_3+\eta u_3+\mu+\delta;$
$E=n+\mu+\delta; F=\varepsilon+\mu+\delta; G=-\psi u_2; I=-\theta(1-x); J=-\theta x; L=-\varphi u_3; M=-\tau u_3$

Therefore,

$$V^{-1}=\begin{bmatrix}\frac{1}{A}&0&0&0&0&0\\ -\frac{G}{AB}&\frac{1}{B}&0&0&0&0\\ \frac{GI}{ABC}&-\frac{I}{BC}&\frac{1}{C}&0&0&0\\ \frac{G\eta I+CGI}{ABCD}&-\frac{\eta I+CJ}{BCD}&\frac{\eta}{CD}&\frac{1}{D}&0&0\\ -\frac{GL\eta+CGJL}{ABCDE}&\frac{L\eta I+CJL}{BCDE}&-\frac{L\eta}{CDE}&-\frac{L}{DE}&\frac{1}{E}&0\\ -\frac{CGJME+GLn\eta I+GM\eta EI+CGJLn}{ABCFDE}&\frac{CJME+Ln\eta I+M\eta EI+CJLn}{BCDEF}&-\frac{Ln\eta+M\eta E}{CFDE}&-\frac{ME+Ln}{FDE}&\frac{n}{FE}&\frac{1}{F}\end{bmatrix} \tag{19}$$

Multiplying equation (17) and (19), gives

$$FV^{-1}=\begin{bmatrix}\frac{GXI}{ABC}+\frac{Y(G\eta I+CGJ)}{ABCD}&-\frac{XI}{BC}-\frac{Y(\eta I+CJ)}{BCD}&\frac{X}{C}+\frac{Y\eta}{CD}&\frac{Y}{D}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix} \tag{20}$$

Therefore, we evaluate the characteristic equation $\left|FV^{-1}-\lambda I\right|=0$ of equation (20) to obtain

$$\left|FV^{-1}-\lambda I\right|=\begin{bmatrix}\frac{GXI}{ABC}+\frac{Y(G\eta I+CGJ)}{ABCD}-\lambda&-\frac{XI}{BC}-\frac{Y(\eta I+CJ)}{BCD}&\frac{X}{C}+\frac{Y\eta}{CD}&\frac{Y}{D}&0&0\\0&0-\lambda&0&0&0&0\\0&0&0-\lambda&0&0&0\\0&0&0&0-\lambda&0&0\\0&0&0&0&0-\lambda&0\\0&0&0&0&0&0-\lambda\end{bmatrix}=0 \tag{21}$$

Therefore, the basic reproduction number,

$$R_0=\frac{G\left[DXI+Y(\eta I+CJ)\right]}{ABCD}$$

$$R_0=\frac{(\psi u_2)\beta(1-u_1)S\left[\sigma\left(\theta(1-x)\right)(\varphi u_3+\eta u_3+\mu+\delta)+\left[\eta\left(\theta(1-x)\right)+(\eta+\xi u_3+\mu+\delta)\theta x\right]\right]}{N(\psi u_2)(\varepsilon+\omega+\theta+\mu)(\eta+\xi u_3+\mu+\delta)(\varphi u_3+\eta u_3+\mu+\delta)}$$

Since $S=N$, then

$$R_0=\frac{(\psi u_2)\beta(1-u_1)\left[\sigma\left(\theta(1-x)\right)(\varphi u_3+\eta u_3+\mu+\delta)+\left[\eta\left(\theta(1-x)\right)+(\eta+\xi u_3+\mu+\delta)\theta x\right]\right]}{(\psi u_2)(\varepsilon+\omega+\theta+\mu)(\eta+\xi u_3+\mu+\delta)(\varphi u_3+\eta u_3+\mu+\delta)} \tag{22}$$

Sensitivity Analysis

Sensitivity analysis is a critical tool for understanding how changes in model parameters impact key outcomes, such as the reproduction number R_0 and the stability of equilibrium points. This analysis helps identify which parameters have the most significant influence on the dynamics of disease transmission and can guide the design of effective control strategies. The normalized forward sensitivity index is a variable R_0 that is dependent on the differentiable parameter Λ. The sensitivity analysis of R_0 yields the analytical result by computing $X_{\vartheta}^{R_0}=\frac{\partial R_0}{\partial\vartheta}\times\frac{\vartheta}{R_0}$ for any parameter, say ϑ (Eloho B. A., & Akindele M, O. 2025)

Table 4: Table of the Sensitivity Index of the Parameters in the Basic Reproduction Number that Contribute to the Spread of Diphtheria

Parameter	Parameter Value	Sensitivity Value	Sensitivity Index
ψ	0.23	0.09	Positive
β	0.6	0.60	Positive
θx	0.26	-0.07	Negative
θ	0.4	0.00	Positive
φ	0.05	-0.03	Negative
η	0.4	-0.40	Negative
μ	0.02	-0.04	Negative
δ	0.05	-0.05	Negative
ξ	0.3	-0.17	Negative
ε	1.5	-1.50	Negative
ω	0.67	-0.67	Negative
σ	0.5	-0.33	Negative

Source: Authors

Sensitivity Indices Interpretation

The parameters' sensitivity indices are displayed in sensitivity index 3 of R_0. Parameters with positive indices (ψ, β and θ) have a significant impact on the spread of diphtheria in the community. As their values rise, the burden of diphtheria disease spread in the community is reduced by parameters with negative indices ($\theta x, \varphi, \eta, \delta, \zeta, \varepsilon, \omega$ and σ). Therefore, the model sensitivity analysis indicated that the government and other policymakers should increase the negative index parameters and lower the positive ones in order to eradicate diphtheria sickness in our community.

Global Stability of the Presence Equilibrium Point

If $R_0 > 1$ then the Diphtheria Presence equilibrium is globally asymptotically stable.

Proof

Consider the Lyapunov function defined by

$$\left.\begin{array}{l} L = (V - V^* - V^* \ln \frac{V^*}{V}) + (\mathrm{S} - \mathrm{S}^* - \mathrm{S}^* \ln \frac{S^*}{S}) + (E - E^* - E^* \ln \frac{E^*}{E}) + (D - D^* - D^* \ln \frac{D^*}{D}) + (C - C^* - C^* \ln \frac{C^*}{C}) \\ \quad + (I - I^* - I^* \ln \frac{I^*}{I}) + (H - H^* - H^* \ln \frac{H^*}{H}) + (T - T^* - T^* \ln \frac{T^*}{T}) + (R - R^* - R^* \ln \frac{R^*}{R}) \end{array}\right\} \tag{23}$$

The derivative of L along the solution of the system of the equation is

$$\left.\begin{array}{l} \frac{dL}{dt} = \left(1 - \frac{V^*}{V}\right)\frac{dV}{dt} + \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{E^*}{E}\right)\frac{dE}{dt} + \left(1 - \frac{D^*}{D}\right)\frac{dD}{dt} + \left(1 - \frac{C^*}{C}\right)\frac{dC}{dt} \\ \quad + \left(1 - \frac{I^*}{I}\right)\frac{dI}{dt} + \left(1 - \frac{T^*}{T}\right)\frac{dT}{dt} + \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt} \end{array}\right\} \tag{24}$$

Replacing the derivatives of the variables with respect to t gives

$$\left.\begin{array}{l} \frac{dL}{dt} = \left(1 - \frac{V^*}{V}\right)(\gamma\Lambda + bu_1 S + \omega D + \rho R - \alpha V - aV - \mu V) + \left(1 - \frac{S^*}{S}\right)((1-\gamma)\Lambda + aV + eD - bu_1 S - \frac{\beta(\sigma C + I)}{N}(1 - u_1)S - \mu S) \\ \quad + \left(1 - \frac{E^*}{E}\right)(\frac{\beta(\sigma C + I)}{N}(1 - u_1)S + \alpha V - \psi u_2 E - \mu E) + \left(1 - \frac{D^*}{D}\right)(\psi u_2 E - \omega D - eD - \theta D - \mu D) \\ \quad + \left(1 - \frac{C^*}{C}\right)(\theta(1-x)D - \eta C - \xi u_3 C - (\mu + \delta)C) + \left(1 - \frac{I^*}{I}\right)(\eta C + \theta x D - \varphi u_3 I - \tau u_3 I - (\mu + \delta)I) \\ \quad + \left(1 - \frac{H^*}{H}\right)(\varphi u_3 I - nH - (\mu + \delta)H) + \left(1 - \frac{T^*}{T}\right)(nH + \tau u_3 I - \varepsilon T - (\mu + \delta)T) + \left(1 - \frac{R^*}{R}\right)(\xi u_3 C + \varepsilon T - \rho R - \mu R) \end{array}\right\} \tag{25}$$

At Diphtheria presence equilibrium, vaccinated and susceptible classes are respectively

$$\Lambda_V = \alpha V + aV + \mu V - \omega D - \rho R \tag{26}$$

and $$\Lambda_S = bu_1 S + \frac{\beta(\sigma C + I)}{N}(1 - u_1)S + \mu S - eD \tag{27}$$

Substituting equations (26) and (27) into equation (25) gives

$$\left.\begin{aligned}\frac{dL}{dt}=&\left(1-\frac{V^*}{V}\right)(\gamma(\alpha V+aV+\mu V-\omega D-\rho R)+bu_1S+\omega D+\rho R-\alpha V-aV-\mu V)\\&+\left(1-\frac{S^*}{S}\right)((1-\gamma)(bu_1S+\frac{\beta(\sigma C+I)}{N}(1-u_1)S+\mu S-eD)+aV+eD-bu_1S-\frac{\beta(\sigma C+I)}{N}(1-u_1)S-\mu S)\\&+\left(1-\frac{E^*}{E}\right)(\frac{\beta(\sigma C+I)}{N}(1-u_1)S+\alpha V-\psi u_2E-\mu E)+\left(1-\frac{D^*}{D}\right)(\psi u_2E-\omega D-eD-\theta D-\mu D)\\&+\left(1-\frac{C^*}{C}\right)(\theta(1-x)D-\eta C-\xi u_3C-(\mu+\delta)C)+\left(1-\frac{I^*}{I}\right)(\eta C+\theta xD-\varphi u_3I-\tau u_3I-(\mu+\delta)I)\\&+\left(1-\frac{H^*}{H}\right)(\varphi u_3I-nH-(\mu+\delta)H)+\left(1-\frac{T^*}{T}\right)(nH+\tau u_3I-\varepsilon T-(\mu+\delta)T)+\left(1-\frac{R^*}{R}\right)(\xi u_3C+\varepsilon T-\rho R-\mu R)\end{aligned}\right\}\tag{28}$$

After rigorous expansion and rearranging, we have

$$\left.\begin{aligned}\frac{dL}{dt}=&\gamma aV\left(1-\frac{V^*}{V}\right)+\gamma\alpha V\left(1-\frac{V^*}{V}\right)+\gamma\mu V\left(1-\frac{V^*}{V}\right)+bu_1S\left(1-\frac{V^*}{V}\right)+V^*(\alpha+\mu)-\gamma(\omega D+\rho R)\\&+\frac{\beta(\sigma C+I)}{N}(1-u_1)S\gamma\left(1\text{-}\frac{E^*}{E}\right)+\alpha V\left(1\text{-}\frac{E^*}{E}\right)+\mu E\left(1\text{-}\frac{E^*}{E}\right)+\psi u_2E^*\\&+bu_1S\left(1\text{-}\frac{S}{S^*}\right)+\frac{\beta(\sigma C+I)}{N}(1-u_1)S\gamma\left(1\text{-}\frac{S}{S^*}\right)+\mu S\gamma\left(1\text{-}\frac{S}{S^*}\right)+eD\gamma\left(1\text{-}\frac{S}{S^*}\right)-2D\frac{S}{S^*}-\alpha V\frac{S}{S^*}\\&+\theta D\left(1\text{-}\frac{D}{D^*}\right)+\mu D\left(1\text{-}\frac{D}{D^*}\right)+(\omega+e)D^*-\psi u_2E\frac{D^*}{D}+\mu C\left(1\text{-}\frac{D}{D^*}\right)+\theta xD\left(\frac{C}{C^*}\text{-}\frac{I}{I^*}\right)+\eta C\left(\frac{C}{C^*}\text{-}\frac{I}{I^*}\right)\\&+\xi u_3C\left(\frac{C}{C^*}\text{-}\frac{R}{R^*}\right)-\theta D\frac{C^*}{C}-\delta C+\mu I\left(1\text{-}\frac{I^*}{I}\right)+\tau u_3I\left(\frac{I^*}{I}\text{-}\frac{T^*}{T}\right)+\delta I(\mu\left(1\text{-}\frac{I}{I^*}\right)+\varphi u_2I^*+\mu H\left(1\text{-}\frac{H^*}{H}\right)\\&+\delta H\left(1\text{-}\frac{H}{H^*}\right)-\varphi u_3I\frac{H^*}{H}+\mu T\left(1\text{-}\frac{T}{T^*}\right)+\varepsilon T\left(\frac{T^*}{T}\text{-}\frac{R^*}{R}\right)-\delta T+\mu R\left(1\text{-}\frac{R}{R^*}\right)+\rho R^*\end{aligned}\right\}\tag{29}$$

Since the geometric mean is less than the arithmetic mean (Maia, 2015), the following inequality from equation (29) holds

$$\left.\begin{aligned}&1\le\frac{V^*}{V},1\le\frac{V}{V^*},1\le\frac{S^*}{S},1\le\frac{S}{S^*},1\le\frac{E^*}{E},1\le\frac{E}{E^*},1\le\frac{D^*}{D},1\le\frac{D}{D^*},1\le\frac{C^*}{C},1\le\frac{C}{C^*},1\le\frac{I^*}{I},1\le\frac{I}{I^*},\\&1\le\frac{H^*}{H},1\le\frac{H}{H^*},1\le\frac{T^*}{T},1\le\frac{T}{T^*},1\le\frac{R^*}{R},1\le\frac{R}{R^*}\end{aligned}\right\}\tag{30}$$

Therefore, $\frac{dL}{dt}\le 0$ for $R_0>1$ and $\frac{dL}{dt}=0$ if and only if $V=V^*,S=S^*,E=E^*,D=D^*,C=C^*,I=I^*,H=H^*,T=T^*,R=R^*$

Hence, L is a Lyapunov function on the region of the system which agrees with the LaSalle's Principle, which states that every solution to the equations of the model tends to endemic equilibrium of the model as $t\to\infty$ for $R_0>1$. Therefore, the Diphtheria presence equilibrium point is globally asymptotically stable in the invariant region Ω if $R_0>1$.

RESULTS AND DISCUSSION

Numerical Simulations

This section talks about how intervention strategies affect how Diphtheria disease spreads among population. We adopted Matlab2024a as the software used for the numerical simulation of the model using the model equation (1) and the values of the variables and the parameters as indicated on table 3.

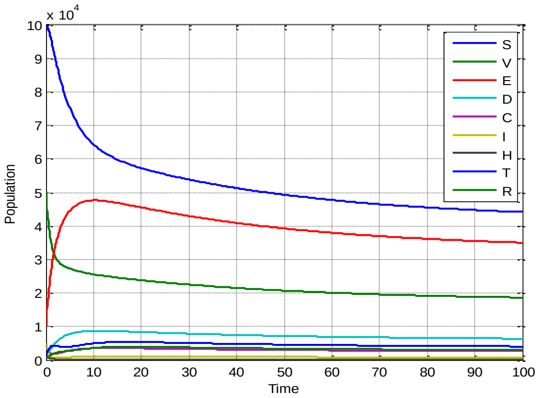

Figure 2: Graphical Solutions of the Representation of the Effects of Control Measures on the Variables

The blue curve in Figure 2 represents the Susceptible Population (S), which initially declines quickly as a result of infection and immunization. Reduced transmission and better control methods are indicated by the decline's gradual slowing and stabilization. Following an initial correction, the green curve representing the vaccinated population (V) rapidly decreases. This could indicate a decline in immunity, a low absorption of vaccines, or a shift into different compartments.

The Red Curve, which represents the Exposed Population (E), rises significantly in the early stages, indicating a rapid spread of the disease. As people move from infection to recovery, it peaks and then starts to fall. Due to early epidemics, the detained population (D), represented by the blue curve, originally increased but later stabilized. This is an indication of better intervention and medical response. The magenta curve representing carrier class (C) exhibits a slight increase followed by stabilization, indicating efficient management.

The Yellow Curve's Infected Population (I) is still comparatively small, suggesting that treatment, vaccination, or timely care management have effectively contained the infection. The Black Curve's Hospitalized Population (H) grows slightly before stabilizing, suggesting that the number of severe cases is small and controllable. The Dark Blue Curve's Treated Population (T) shows persistent healthcare intervention as it rises early and stabilizes. The Dark Green Curve's Recovered Population (R) continually increases, indicating effective illness control and the development of immunity. Lastly, the model indicates that immunization, surveillance, and prompt care management are effective ways to reduce the spread of the diphtheria disease.

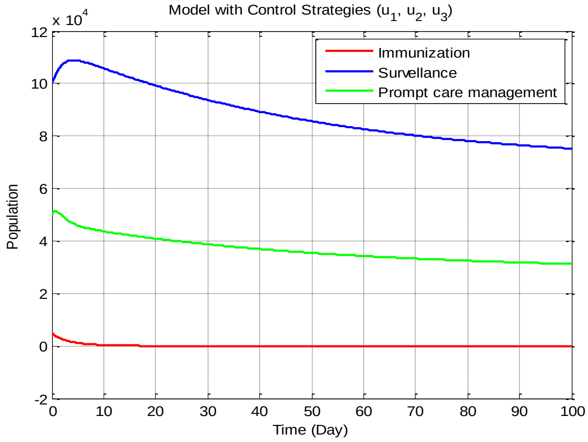

Figure 3: Graphical Solutions for Implementation of Control Measures (u_1, u_2, u_3) on the Entire Populations (Variables)

As seen in Figure 3 Immunization (Red Curve) begins at a modest level (1000) and quickly drops until stabilizing at a very low number (Less than 10 days). This suggests that it effectively slows the spread of disease. The sharp decline shows that immunization directly reduces the susceptible population. Long-term (100 days) illness control and stability are demonstrated by the curve attaining a nearly constant low level. As a result, it reduces disease prevalence the quickest and most significantly, making it the most successful stand-alone intervention of the three tactics.

The same Figure 3 shows that surveillance (the blue curve) first rises somewhat before progressively falling over time. This suggests that enhanced surveillance may detect more cases early in the intervention, leading to a brief increase. This means that surveillance helps improve monitoring and control over time, which results in a gradual decrease. It is less effective when employed alone, though, as it continues to be higher than the other curves. It can be said that surveillance is essential for tracking and identifying the spread of diseases, but its influence is gradual and contingent upon other actions.

Additionally, Figure 3 demonstrates that rapid Care Management (Green Curve) exhibits a consistent drop, although more slowly than immunization, which demonstrates that rapid treatment lowers consequences and transmission. The steady decline suggests a moderate level of effectiveness. It has less of an impact than vaccination, but it is more efficient than surveillance alone.

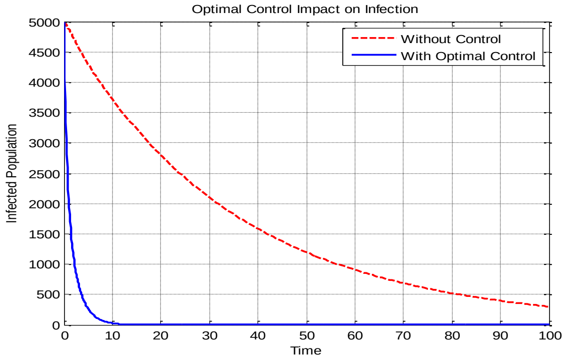

Figure 4: Graphical Solutions for Implementation of Control Measures (u_1, u_2, u_3) and Without Control Measure on the Infected Populations

There is a notable distinction between the controlled and uncontrolled settings, according to the simulation results. In the absence of management measures, the infected population steadily declines from 5000 on day 1 to 350 on day 100, indicating that the disease is still prevalent. On the other hand, the infected population quickly drops to almost zero within 10 days when the best control measures are put in place. This indicates that the combination of immunization, surveillance, and prompt care management is very successful in reducing the spread of disease and may even result in its elimination.

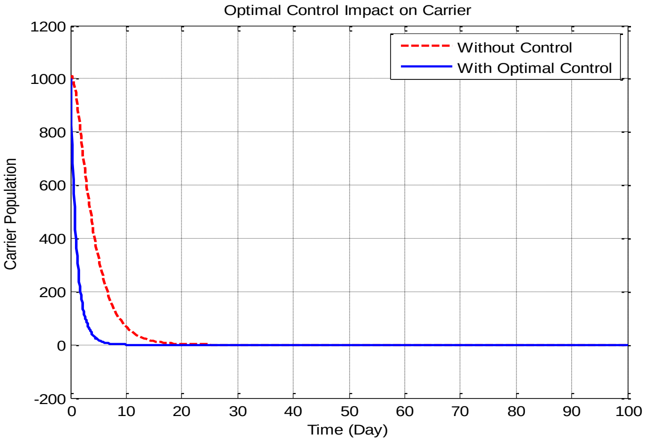

Figure 5: Graphical Solutions for Implementation of Control Measures (u_1, u_2, u_3) and Without Control Measure on the Carrier Populations

The simulation findings show a significant difference between the regulated and uncontrolled measurers. In the absence of control measures (red line), the carrier population drastically drops from 1000 people on day 1 to less than 20 people on day 25, suggesting that the disease is still widespread. On the other hand, when the best control measures are implemented (blue live), the carrier people rapidly decrease from 1000 to nearly zero on day 8. Thus, it was observed from the graph that after day 25, diphtheria disease appears to be stable even without control and with control, appears to be stable from day 9 upwards. This suggests that immunization, surveillance, and timely care management are highly effective in preventing the spread of disease and may even lead to its eradication.

Discussion

This study offered a thorough mathematical model for the dynamics of diphtheria transmission that included integrated control measures like immunization, surveillance, and prompt case management. It was demonstrated that the model was both mathematically and biologically well-posed, with all state variables left as bounded and non-negative, guaranteeing the system's validity and realism.

The basic reproduction number R_0 , a crucial threshold parameter, was derived and found to be a predictor of the persistence or eradication of the disease. Based on the analytical results, the diphtheria-free equilibrium (DFE) is locally asymptotically stable when $R_0 < 1$, suggesting that the disease can be managed if transmission is sufficiently decreased. In addition to that, the global asymptotic stability of the diphtheria-free equilibrium was established using suitable Lyapunov function techniques, suggesting that the disease will ultimately be eradicated irrespective of initial population conditions when control measures are properly utilized.

The analytical findings were further supported by the results of numerical simulations, which shown that the combination of immunization, surveillance, and prompt care management considerably lowers the spread of diphtheria when compared to the use of one intervention measure. These integrated control strategies exhibited a synergistic impact, resulting in fast reduction in infection prevalence within the population.

In summary, the study emphasizes the role of coordinated and sustained implementation of public health measures in controlling diphtheria transmission. The findings demonstrate that using a variety of interventions rather than just single control method is much more successful in lowering the spread of diphtheria. The results strongly imply that the eradication of diphtheria requires high immunization rates, efficient surveillance systems, and timely treatment of affected persons.

Therefore, in order to successfully manage and eventually eradicate diphtheria in endemic areas, policymakers and health authorities should give priority to these integrated methods.

CONCLUSION

(According to the graph's Comparative Analysis, immunization is the most successful tactic (fastest decline). The effectiveness of prompt care management is moderate. While crucial, surveillance is not very successful on its own.

This work offers a formal Mathematical structure for comprehending and managing the spread of diphtheria. Important findings consist of:

i. In less than 100 days, complete diphtheria eradication is achieved through the combination of immunization, surveillance, and timely care management, which significantly lowers both carrier and infected populations. This demonstrates how effective combined interventions are at eradicating diphtheria in a population.

ii. Sensitivity analysis reveals that the most influential criteria are transmission rate (β), progression rate (φ) from infected class to hospitalized compartment, and progression rate (θ) from detained to infected classes, which assist decision-makers in deciding which health issues should be taken first.

iii. It was found that implementing effective intervention techniques guarantees the greatest possible decrease in diphtheria infections while reducing expenses, providing health authorities with a useful tool for making decisions. This implies that control measures facilitate resource-efficient solutions.

iv. Diphtheria outbreaks can be successfully controlled with early, persistent, and flexible interventions based on the quantitative analysis, even in environments with limited resources. This method can be applied to other infectious diseases that can be prevented by vaccination.

v. The graphical results show that over time, there is a considerable decrease in illness prevalence when immunization, surveillance, and timely care management are integrated. The most effective of these is immunization, underscoring the significance of immunization programs in public health planning.

The paper concludes by emphasizing that mathematical modeling offers a strong basis for formulating research-informed public health responses to diphtheria when paired with immunization, surveillance, urgent care management, and sensitivity analysis.

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