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NUMERICAL AND GRAPHICAL RESULTS OF FINITE SYMMETRIC INVERSE (I_n) AND FULL (T_n) TRANSFORMATION SEMIGROUPS

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ABSTRACT

Supposed $\chi_n = \{1,2,3,\cdots,n\}$ is a finite set, then a function $f: \chi_n \to \chi_n$ is called a finite partial transformation semigroup (PT_n), which moves elements of χ_n from its domain to its co-domain by a distance of $|a-b|$, where $a, b \in \chi_n$. The total work done by the function is therefore the sum of these distances. It is a known fact that $I_n \subseteq PT_n$ and $T_n \subseteq PT_n$. In this this research paper, we have mainly presented the numerical solutions of the total work done, the average work done by functions on the finite symmetric inverse semigroup of degree n, I_n and the finite full transformation semigroup of degree n, T_n as well as their respective powers for a given fixed time t in space. We used an effective methodology and valid combinatorial results to generalize the total work done, the average work done and powers of each of the transformation semigroups. The generalized results were tested by substituting small values of n and a specified fixed times t in space. Graphs were plotted in each case to illustrate the nature of the total work done and the average work done. The results obtained in this research article have an important application in some branch of physics and theoretical computer science.

Keywords: Semigroup, Partial Transformation, Total work done, Average work done, Power.

INTRODUCTION

A semigroup is a pair $(S,*)$, where S is a nonempty set and $*$ is an associative binary operation on S. That is, $*$ is a function $S \times S \to S$ with $(a,b) \to a*b$ and $\forall a,b,c \in S$ we have $a*(b*c) = (a*b)*c$. It is customary to abbreviate "$(S,*)$" by "S" and write "ab" for "$a*b$" by omitting $*$. Semigroups may be considered as a generalization of groups without requiring the existence of identity element and the existence of inverses.

The semigroup operation needs not be commutative, that is, it is not necessary that $a*b = b*a$, if the semigroup is commutative, then the semigroup is called an abelian semigroup. Transformation semigroups are one of the most fundamental mathematical structures. They arise naturally as endomorphism semigroups of various mathematical structures. They also occur in theoretical computer science, where properties of languages depend on algebraic properties of various transformation semigroups related to them.

Finite transformation semigroups are of utmost importance in semigroup theory as every semigroup is isomorphic to a transformation semigroup. The theory of finite semigroups has been of particular importance in theoretical computer science since 1950s because of the natural link between finite semigroups and finite automata via the syntactic monoids. The study of semigroup trailed behind that of other algebraic structures with more complex axioms such as groups or rings. A number of sources attributes the first use of the term (in French) to J.A Seguier in Elements de la Theorie des Groupes Abstriats (Elements of the theory of Abstract Groups) in (1964).Anton Sushkevich obtained the first non-trivial results about semigroups. His 1928 paper "Uber die endlichen Gruppen Ohne das Gesetz der Eindeutigen umkehrbarkeit" ("On finite groups without the rule of unique invertibility") determined the structure of finite simple semigroups and showed that the minimal ideal (or Green's relations J-class) of a finite semigroups is simple. From that point on, the foundations of semigroup theory were further laid by David Rees, James Alexander Green, Alfred H. Clifford and Gordon Preston. The latter two published a two-volume monograph on semigroup theory in 1961 and 1967 respectively.

A new periodical called semigroup forum became one of the few mathematical journals devoted entirely to semigroup theory. The representation theory of semigroups was developed in 1963 by Boris Schein using binary relations on a set X and composition of relations for the semigroup product. At an algebraic conference in 1972, Shein surveyed the literature on the semigroups of relations. In 1997, Schein and Ralph Mckenzie proved that every semigroup is isomorphic to a transitive semigroup of binary relations.In recent years, researchers in the field have become more specialized with dedicated monographs appearing on important classes of semigroups as well as monographs focusing on applications in algebraic automata theory, particularly for finite automata and also Functional analysisIn 1815, inspired by Gauss' "theory of forms"(In Disquisitions Arithmeticae) Caushy published a memoir in which he introduced the (cycle) notation "(a,b)" to indicate the transposition of two letters a and b by a permutation. In the second part of the memoir, Cauchy also

introduced both the decomposition of a permutation into disjoint cycles and the alternating subgroup A_n of the symmetric group S_n.D. Daly and P. Vojte in their work "How Permutations Displace Points and Stretch Intervals." explained how points are stretched on an equally spaced interval. In 1922, perhaps inspired by the evolution of group theory, A.K.Suschkewitsch essentially expressed the idea that the basic content of group theory is its relationship with transformation group and that the basic content of semigroup theory should be its relationship with transformation semigroups. Gluskin and Scein defined transformation semigroup as: A transformation semigroup on a set X is a semigroup whose elements are partial transformations on X and whose multiplication is the usual composition of functions. Thirty years later, in 1952 V.V Wagner was the first to introduce inverse semigroups, where the semigroup of degree n on $N=\{1.\cdots,n\}$ plays the role that S_n plays in the theory of groups.(Formally, a semigroup S is an inversesemigroup if it satisfies the additional property that for each $a \in S$ there is a unique $b \in S$ such that both $\mathrm{aba} = \mathrm{a}$ and $\mathrm{bab} = \mathrm{b}$ A.Laradji and A. Umar. in 2004 worked on "Combinatorial Results for Semigroups of Order-Decreasing Partial Transformations.
"Most of their combinatorial results turned out to be useful in many applications.
In 2018, Wilf.A. Wilson worked on "Computational techniques in finite semigroup theory" as his Phd Thesis submitted to the university of St Andrews; In his work he provided an algorithm for dealing with computational semigroup of partial transformation among many other wonderful algorithms for computations on semigroup theory contained in his Phd Thesis. The finite partial transformation semigroup denoted by PT_n is the semigroup of all partial transformations on n; that is, all functions between subsets of n. Note that the word "**partial**" does not mean that the domain is necessarily a proper subset of n, this implies that PT_n also includes the full transformation of n(that is all functions $\ni n \to n$). A partial transformation τ moves a point x of its domain to a (possibly) new point $\tau(x) = y$ in its image. The finite partial transformation has different subsets as shown in the table below:

Table1: Some subsets of partial Transformation semigroups

The finite full transformation semigroup	T_n
The finite symmetric group of degree n	S_n
The finite symmetric inverse semigroup	I_n
The finite order-preserving injective partial transformation semigroup	POI_n
The finite order-preserving semigroup	O_n
Order-reversing partial transformation semigroup	POD_n
Orientation-preserving partial transformation semigroup	POP_n
Orientation-reversing partial transformation semigroup	POR_n

In this research work we are mainly concern with the finite symmetric inverse semigroup of degree n and the full transformation semigroup. We will be using valid lemma and combinatorial results to establish generalized formulas for the total work done which we will denote by $\psi(S)$,the average work done denoted by $\tilde{\psi}(S)$ and the power denoted by $\rho(S)$of the transformations by elements of S. The average work done and the power of the transformation are respectively defined by :

$$\tilde{\psi}(S) = \frac{\psi(S)}{|S|} \text{ and } \rho(S) = \frac{\psi(S)}{t}, t > 0$$

Where S is any of the two transformation semigroups(i.e I_n or T_n) and $|S|$ is the order of the semigroup. This paper is objectively written on four reasons, first to utilize the cardinality of each of the semigroups and come up with relations that represent the work done by the transformation semigroups, secondly to test the reliability of these relations, thirdly to plot the graphs in each case in order to investigate the nature of the work done and test for consistency with intuitive ideology and lastly to relate the degrees of the semigroup to the power of the functions in the two semigroups of transformation.

DEFINITIONS

Transformation Semi group

A transformation Semigroup is a pair (X, S), where X is a set and S is a Semigroup of transformations of X. Here a transformation X is just a partial function from subset of X to X, not necessarily invertible, and therefore S is simply a set of transformations of X which is closed under composition of functions. The set of all partial functions on a given base set X, forms a regular Semigroup called the Semigroup of all partial transformations (or the partial transformation Semigroup on X), typically denoted by PT_X.

Partial Transformation

A partial transformation is a function $\{1,2,3,\ldots,n\} \rightarrow \{1,2,3,\ldots,n\}$ for some $n \in \mathbb{N}$.

If $\tau$is a partial transformation of degree n, then we define the rank (τ), the rank of $\tau$to be $|im(\tau)|$. More generally we can say that a transformation is a partial transformation that is a function. We may donate a partial transformation of degree $n$in two-line notation.

This uses a $2 \times n$ matrix, where i^{th} entry in the first row contains the number i and the i^{th} entry in the second row contains $\tau(i)$ when i is in the domain $(\tau.)$, otherwise it contains a dash. For example if σ is the partial transformation of degree 9 whose domain is $\{2,4,6,8\}$ and write $\sigma(i) = \frac{i}{2}$ for any $i \in dom\,(\tau)$, then σ is written in two-line notation as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ - & 1 & - & 2 & - & 3 & - & 4 & - \end{pmatrix}$$

Symmetric Inverse Semi Group

The symmetric inverse semi group on a set X are the partial bijections on X, that is, bijections $: Y \rightarrow Z$, where $Y\ and\ Z$ are subset of X. If $f : Y \rightarrow Z$ and $g : U \rightarrow V$, then the composition fg is define on the points of Y whose image under f happens to lie in U, so that g can be applied to it. Compose in symmetric inverse semi group is shown below:

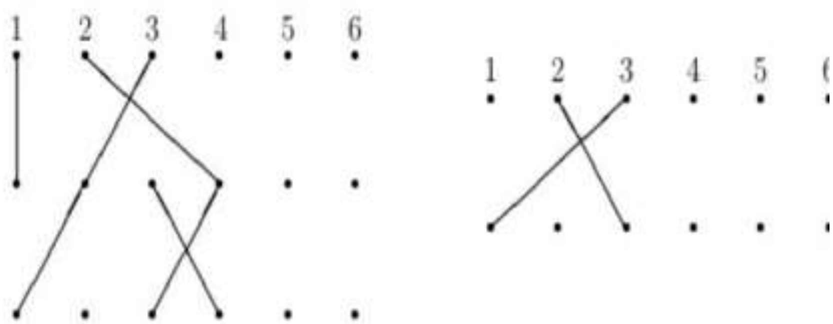

Fig1: Symmetric inverse composition

Work done

In science, work is the product of force and displacement. A force is said to do work if, when acting, there is a displacement of the point of application in the direction of the force.

Power

Power is defined as the amount of energy transferred or converted per unit time. We can recall that given any set, the number of subsets in the set is given by 2^n, where $n$is the cardinality of the set. (note that the set must be finite). Given set $A = \{1,2,3\}$, there are $2^3 = 8$ distinct subsets of the set A which are given by $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. By definition a partial transformation semigroup is set of functions that maps these subsets together.For example,

$\alpha: \{1,2\} \longrightarrow \{2,1\} \Longrightarrow \alpha(1) = 2$ and $\alpha(2) = 1.$

$\beta: \{1,2,3\} \longrightarrow \{1,2,3\} \Longrightarrow \beta(1) = 1, \beta(2) = 2$ and $\beta(3) = 3.$

$\gamma: \{\phi, \{2,3\}\} \longrightarrow \{\phi, \{1,2\}\} \Longrightarrow \gamma(\phi) = \phi, \gamma(2) = 1$ and $\gamma(3) = 2.$

The collections $\alpha, \beta, \gamma, \cdots$ are functions between subsets of the given set A.

Let us supposed that in each case of I_n and T_n, the elements of $\chi_n = \{1,2,3,4,\cdots,n\}$ for any $n \geq 1$ are thought of as points that are equally spaced on a real line $\mathbb{R}$ as shown in the figure below

Fig2: Equally spaced point

Then we can consider the total work done by an arbitrary function in moving a point in the domain to a point in the co-domain for each of the transformation semigroups studied in this research article.

Now let $\sigma \in PT_n$ and $x \in n$, we define the work done by σ in moving x as:

$$\psi_x(\sigma) = \begin{cases} |x - \sigma(x)|, & if x \in dom(\sigma) \\ 0, & otherwise \end{cases} \quad (1)$$

The total work done by σ will then be given as

$$\psi(\sigma) = \sum_{x \in n} \psi_x(\sigma) \quad (2)$$

Now for each of the subsets S of PT_n, the total work done by a transformation σ on $S$is defined as:

$$\psi(S) = \sum_{\sigma \in S} \psi(\sigma) \quad (3)$$

Consequently, the average work done and the power of the transformation for some given time in space are respectively given by:

$$\tilde{\psi}(S) = \frac{\sum_{\sigma \in S} \psi(\sigma)}{|S|} \quad (4)$$

And $\rho(S) = \frac{\sum_{\sigma \in S} \psi(\sigma)}{t}$ (5)

where $|S|$ is the order of S as contained in [9],[10] and [11] Now for each $x, y \in n$ let

$$N_{xy}(S) = \{\sigma \in S \mid \sigma(x) = y\} \quad (6)$$

be the set of all elements of S which moves x to y and write $n_{xy}(S) = |N_{xy}(S)|$ which represents the cardinality.

Note that $\psi_x(\sigma) = |x - \tau(x)| \forall \sigma \in N_{xy}(S)$

Lemma

Let $S \subseteq PT_n$, then $\psi(S) = \sum_{x,y \in n} |x - y| \cdot n_{xy}(S)$ (7)

it is well known that

$$\sum_{1 \leq x \leq y \leq n} |x - y| = \binom{n+1}{3}$$

this follows that $\sum_{x,y \in n} |x - y| = 2 \times \binom{n+1}{3}$, further simplification gives

$$\sum_{x,y \in n} |x - y| = \frac{n(n^2 - 1)}{3} \quad (8)$$

equation (8) and (7) are facts that will prove very useful in deriving the generalized formulas for the total work done by each of the transformation semigroupsIn the case of this research we are looking at the condition in which $S = I_n$ and $S = T_n$.

Applying the above methodologies we have the following analysis:

For finite symmetric inverse transformation semigroup, I_n, we have;$S = I_n \subseteq PT_n$ and define I_n as

$$I_n = \{\sigma \in PT_n \mid \sigma is injective\}$$

Now for all $x, y \in n$ we have that $N_{xy}(I_n) = \{\sigma \in I_n \mid \sigma(x) = y\}$, this immediately follows that

$n_{xy}(PT_n) = |I_{n-1}| = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \times k!$ (A.Umar, 2014) (9)

Substituting into equation (7) and simplifying we have

$$\psi(I_n) = \sum_{x,y\in n} |x-y|\cdot|I_{n-1}|$$

$$\psi(I_n) = \frac{n^3-n}{3} \times \sum_{k=0}^{n-1}\binom{n-1}{k}^2 \times k! \qquad (10)$$

The average work done by element of I_n is given as:

$$\tilde{\psi}(I_n) = \frac{\psi(I_n)}{|I_n|}$$

$$\tilde{\psi}(I_n) = \frac{(n^3-n)\cdot|I_{n-1}|}{3\times|I_n|}$$

Substituting we obtain that

$$\tilde{\psi}(I_n) = \frac{n^3-n}{3\times\sum_{l=0}^{n}\binom{n}{l}^2\times l!} \times \sum_{k=0}^{n-1}\binom{n-1}{k}^2 \times k! \qquad (11)$$

and the power of the transformation is therefore given by:

$$\rho(I_n) = \frac{\psi(I_n)}{t}, t > 0$$

$$\rho(I_n) = \frac{\frac{n^3-n}{3}\times\sum_{k=0}^{n-1}\binom{n-1}{k}^2\times k!}{t}, t > 0 \qquad (12)$$

The formulas in equation(10),(11) and (12) represents the total work done, the average work done and the power of the symmetric inverse transformation semigroup

respectively, they are all expressed in terms of the degree n. This relations is curiouslyobtained by the combination of the lemma and the cardinality of I_n as contained in [11] and [12]

For full transformation semigroup, T_n, we have the following analogous analysis$S = T_n \subseteq PT_n$ and we define T_n as

$T_n = \{\sigma \in PT_n \ni \ dom(\sigma) = n\}$ and for each $x, y \in n$, let

$N_{xy}(T_n) = \{\sigma \in T_n \ |\sigma(x) = y\}$ be the set of all elements of T_n which moves x to y

and write $n_{xy}(T_n) = n^{n-1}$ [11]

By substituting into the lemma in equation(7) and using the results of the cardinality from [12] we have the total work done, the average work done by elements of T_n respectively given by

$$\psi(T_n) = \frac{n^n\times(n^2-1)}{3} \qquad (13)$$

$$\tilde{\psi}(T_n) = \frac{n^2-1}{3} \qquad (14)$$

The power of the transformation for a given fixed t is therefore given below:

$$\rho(T_n) = \frac{n^n\times(n^2-1)}{t}, t > 0 \qquad (15)$$

RESULTS AND DISCUSSIONS

In this section we are going to substitute small values of n, say $n \le 10$, where $n \in \mathbb{N}$and a fixed time in space of $t = 5seconds$ into the respective relations for each case of S=I_n or T_n**finite symmetric inverse transformation semigroup**, I_n, here we will use the formulas in (10),(11) and (12)

When $n = 1$

$$\psi(I_1) = 0 \times \sum_{k=0}^{1-1}\binom{1-1}{k}^2 \times k!$$

$= 0.00 \times 10^0$J

$$\tilde{\psi}(I_1) = \frac{1^3-1}{3\times\sum_{l=0}^{1}\binom{1}{l}^2\times l!} \times \sum_{k=0}^{1-1}\binom{1-1}{k}^2 \times k!$$

$\tilde{\psi}(I_1) = 0.00 \times 10^0$J

$\rho(I_1) = 0.00 \times 10^0$J/s

When $n = 2$

$$\psi(I_2) = 2 \times \sum_{k=0}^{1}\binom{1}{k}^2 \times k!$$

$= 4.00 \times 10^1$J

$$\tilde{\psi}(I_2) = \frac{2^3 - 2}{3 \times \sum_{l=0}^{2} \binom{2}{l}^2 \times l!} \times \sum_{k=0}^{1} \binom{1}{k}^2 \times k!$$

$$\sum_{l=0}^{2} \binom{2}{l}^2 \times l! = 7$$

$$\tilde{\psi}(I_2) = \frac{6}{21} \times \sum_{k=0}^{1} \binom{1}{k}^2 \times k!$$

$= 5.71 \times 10^{-1}$J

$\rho(I_2) = 8.00 \times 10^{-1}$J/s

When $n = 3$

$$\psi(I_3) = 8 \times \sum_{k=0}^{2} \binom{2}{k}^2 \times k!$$

$= 5.60 \times 10^{1}$J

$$\tilde{\psi}(I_3) = \frac{3^3 - 3}{3 \times \sum_{l=0}^{3} \binom{3}{l}^2 \times l!} \times \sum_{k=0}^{2} \binom{2}{k}^2 \times k!$$

$$\sum_{l=0}^{3} \binom{3}{l}^2 \times l! = 34$$

$$\tilde{\psi}(I_3) = \frac{24}{102} \times \sum_{k=0}^{2} \binom{2}{k}^2 \times k!$$

$= 1.65 \times 10^{0}$J

$\rho(I_3) = 1.12 \times 10^{1}$J/s

When $n = 4$

$$\psi(I_4) = 20 \times \sum_{k=0}^{3} \binom{3}{k}^2 \times k!$$

$= 6.80 \times 10^{2}$J

$$\tilde{\psi}(I_4) = \frac{4^3 - 4}{3 \times \sum_{l=0}^{4} \binom{4}{l}^2 \times l!} \times \sum_{k=0}^{3} \binom{3}{k}^2 \times k!$$

$$\sum_{l=0}^{4} \binom{4}{l}^2 \times l! = 200$$

$$\tilde{\psi}(I_4) = \frac{60}{627} \times \sum_{k=0}^{3} \binom{3}{k}^2 \times k!$$

$= 3.30 \times 10^{0}$J

$\rho(I_4) = 1.36 \times 10^{2}$J/s

When $n = 5$

$$\psi(I_5) = 40 \times \sum_{k=0}^{4} \binom{4}{k}^2 \times k!$$

$= 8.36 \times 10^{3}$J

$$\tilde{\psi}(I_5) = \frac{5^3 - 5}{3 \times \sum_{l=0}^{5} \binom{5}{l}^2 \times l!} \times \sum_{k=0}^{4} \binom{4}{k}^2 \times k!$$

$$\sum_{l=0}^{5} \binom{5}{l}^2 \times l! = 1546$$

$$\tilde{\psi}(I_5) = \frac{120}{4636} \times \sum_{k=0}^{4} \binom{4}{k}^2 \times k!$$

$= 5.41 \times 10^{0}$J

$\rho(I_5) = 1.67 \times 10^{3}$J/s

When $n = 6$

$$\psi(I_6) = 70 \times \sum_{k=0}^{5} \binom{5}{k}^2 \times k!$$

$= 1.08 \times 10^{5}$J

$$\tilde{\psi}(I_6) = \frac{6^3 - 6}{3 \times \sum_{l=0}^{6} \binom{6}{l}^2 \times l!} \times \sum_{k=0}^{5} \binom{5}{k}^2 \times k!$$

$$\sum_{l=0}^{6} \binom{6}{l}^2 \times l! = 13327$$

$$\tilde{\psi}(I_6) = \frac{210}{39981} \times \sum_{k=0}^{5} \binom{5}{k}^2 \times k!$$

$= 8.12 \times 10^{0}$J

$\rho(I_6) = 2.16 \times 10^{4}$J/s

When $n = 7$

$$\psi(I_7) = 112 \times \sum_{k=0}^{6} \binom{6}{k}^2 \times k!$$

$= 1.49 \times 10^{6}$J

$$\tilde{\psi}(I_7) = \frac{7^3 - 7}{3 \times \sum_{l=0}^{7} \binom{7}{l}^2 \times l!} \times \sum_{k=0}^{6} \binom{6}{k}^2 \times k!$$

$$\sum_{l=0}^{7} \binom{7}{l}^2 \times l! = 130922$$

$$\tilde{\psi}(I_7) = \frac{336}{392760} \times \sum_{k=0}^{6} \binom{6}{k}^2 \times k!$$

$= 1.14 \times 10^1$J
$\rho(I_7) = 2.98 \times 10^5$J/s
When $n = 8$

$$\psi(I_8) = 168 \times \sum_{k=0}^{7} \binom{7}{k}^2 \times k!$$

$= 2.20 \times 10^7$J

$$\tilde{\psi}(I_8) = \frac{8^3 - 8}{3 \times \sum_{l=0}^{8} \binom{8}{l}^2 \times l!} \times \sum_{k=0}^{7} \binom{7}{k}^2 \times k!$$

$$\sum_{l=0}^{8} \binom{8}{l}^2 \times l! = 14411729$$

$$\tilde{\psi}(I_8) = \frac{504}{4325187} \times \sum_{k=0}^{7} \binom{7}{k}^2 \times k!$$

$= 1.53 \times 10^1$J
$\rho(I_8) = 4.40 \times 10^6$J/s
When $n = 9$

$$\psi(I_9) = 240 \times \sum_{k=0}^{8} \binom{8}{k}^2 \times k!$$

$= 3.46 \times 10^8$J

$$\tilde{\psi}(I_9) = \frac{9^3 - 9}{3 \times \sum_{l=0}^{9} \binom{9}{l}^2 \times l!} \times \sum_{k=0}^{8} \binom{8}{k}^2 \times k!$$

$$\sum_{l=0}^{9} \binom{9}{l}^2 \times l! = 17572114$$

$$\tilde{\psi}(I_9) = \frac{720}{52716342} \times \sum_{k=0}^{8} \binom{8}{k}^2 \times k!$$

$= 1.97 \times 10^1$J
$\rho(I_9) = 6.92 \times 10^7$J/s
When $n = 10$

$$\psi(I_{10}) = 330 \times \sum_{k=0}^{9} \binom{9}{k}^2 \times k!$$

$= 5.80 \times 10^9$J

$$\tilde{\psi}(I_{10}) = \frac{10^3 - 10}{3 \times \sum_{l=0}^{10} \binom{10}{l}^2 \times l!} \times \sum_{k=0}^{9} \binom{9}{k}^2 \times k!$$

$$\sum_{l=0}^{10} \binom{10}{l}^2 \times l! = 231396311$$

$$\tilde{\psi}(I_{10}) = \frac{990}{3 \times 231396311} \times \sum_{k=0}^{9} \binom{9}{k}^2 \times k!$$

$= 2.50 \times 10^1$J
$\rho(I_{10}) = 1.16 \times 10^9$J/s

The results obtained are shown in the table below:

Table 2: Total work done and average work done by elements of I_n

n	$\psi(I_n)$ J	$\tilde{\psi}(I_n)$ J
1	0.00×10^0	0.00×10^0
2	4.00×10^1	5.71×10^{-1}
3	5.60×10^1	1.65×10^0
4	6.80×10^2	3.30×10^0
5	8.36×10^3	5.41×10^0
6	1.08×10^5	8.12×10^0
7	1.49×10^6	1.14×10^1
8	2.20×10^7	1.53×10^1
9	3.46×10^8	1.97×10^1
10	5.80×10^9	2.50×10^1

Table 3: power of the transformation

	J\s
$\rho(I_1)$	0.00×10^0
$\rho(I_2)$	8.00×10^{-1}
$\rho(I_3)$	1.12×10^1
$\rho(I_4)$	1.36×10^2
$\rho(I_5)$	1.67×10^3
$\rho(I_6)$	2.16×10^4
$\rho(I_7)$	2.98×10^5
$\rho(I_8)$	4.40×10^6
$\rho(I_9)$	6.92×10^7
$\rho(I_{10})$	1.16×10^9

The graph of the total work done and the average work done corresponding to the tabulated results in **Table 2** above are as shown below

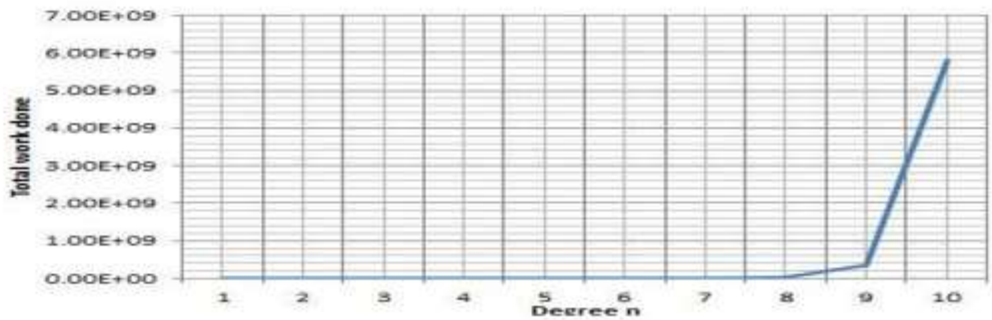

Fig 3: Total work done by elements of I_n

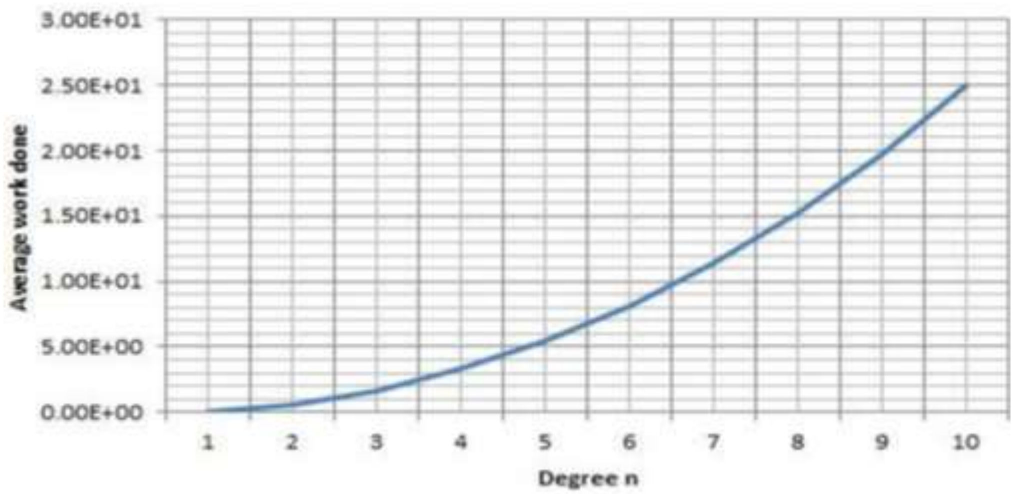

Fig 4: Average work done by elements of I_n

full transformation semigroup, T_n here we will use the formulas in (13),(14) and (15)

when $n = 1$

$$\psi(\mathrm{T}_1) = \frac{(1)^1 \times [(1)^2 - 1]}{3}$$

$\psi(\mathrm{T}_1) = 0.00 \times 10^0$J

$$\tilde{\psi}(\mathrm{T}_1) = \frac{(1)^2 - 1}{3}$$

$\tilde{\psi}(\mathrm{T}_1) = 0.00 \times 10^0$J

$\rho(T_1) = 0.00 \times 10^0$J/s

when $n = 2$

$$\psi(\mathrm{T}_2) = \frac{(2)^2 \times [(2)^2 - 1]}{3}$$

$\psi(\mathrm{T}_2) = 4.00 \times 10^0$J

$$\tilde{\psi}(\mathrm{T}_2) = \frac{(2)^2 - 1}{3}$$

$\tilde{\psi}(\mathrm{T}_2) = 1.00 \times 10^0$J

$\rho(T_2) = 8.00 \times 10^{-1}$J/s

when $n = 3$

$$\psi(\mathrm{T}_3) = \frac{(3)^3 \times [(3)^2 - 1]}{3}$$

$\psi(\mathrm{T}_3) = 7.20 \times 10^1$J

$$\tilde{\psi}(\mathrm{T}_3) = \frac{(3)^2 - 1}{3}$$

$\tilde{\psi}(\mathrm{T}_3) = 2.70 \times 10^0$J

$\rho(T_3) = 1.40 \times 10^1$J/s

when $n = 4$

$$\psi(\mathrm{T}_4) = \frac{(4)^4 \times [(4)^2 - 1]}{3}$$
$\psi(\mathrm{T}_4) = 1.30 \times 10^3$ J
$$\tilde{\psi}(\mathrm{T}_4) = \frac{(4)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_4) = 5.00 \times 10^0$ J
$\rho(T_4) = 2.60 \times 10^2$ J/s
when $n = 5$
$$\psi(\mathrm{T}_5) = \frac{(5)^5 \times [(5)^2 - 1]}{3}$$
$\psi(\mathrm{T}_5) = 2.50 \times 10^4$ J
$$\tilde{\psi}(\mathrm{T}_5) = \frac{(5)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_5) = 8.00 \times 10^0$ J
$\rho(T_5) = 5.00 \times 10^3$ J/s
when $n = 6$
$$\psi(\mathrm{T}_6) = \frac{(6)^6 \times [(6)^2 - 1]}{3}$$
$\psi(\mathrm{T}_6) = 5.40 \times 10^5$ J
$$\tilde{\psi}(\mathrm{T}_6) = \frac{(6)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_6) = 1.20 \times 10^1$ J
$\rho(T_6) = 1.10 \times 10^4$ J/s
when $n = 7$
$$\psi(\mathrm{T}_7) = \frac{(7)^7 \times [(7)^2 - 1]}{3}$$
$\psi(\mathrm{T}_7) = 1.30 \times 10^7$ J
$$\tilde{\psi}(\mathrm{T}_7) = \frac{(7)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_7) = 1.60 \times 10^1$ J
$\rho(T_7) = 2.60 \times 10^6$ J/s
when $n = 8$
$$\psi(\mathrm{T}_8) = \frac{(8)^8 \times [(8)^2 - 1]}{3}$$
$\psi(\mathrm{T}_8) = 3.50 \times 10^8$ J
$$\tilde{\psi}(\mathrm{T}_8) = \frac{(8)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_8) = 2.10 \times 10^1$ J
$\rho(T_8) = 7.00 \times 10^7$ J/s
when $n = 9$
$$\psi(\mathrm{T}_9) = \frac{(9)^9 \times [(9)^2 - 1]}{3}$$
$\psi(\mathrm{T}_9) = 1.03 \times 10^{10}$ J
$$\tilde{\psi}(\mathrm{T}_9) = \frac{(9)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_9) = 2.70 \times 10^1$ J
$\rho(T_9) = 2.10 \times 10^9$ J/s
when $n = 10$
$$\psi(\mathrm{T}_{10}) = \frac{(10)^{10} \times [(10)^2 - 1]}{3}$$
$\psi(\mathrm{T}_{10}) = 3.30 \times 10^{11}$ J
$$\tilde{\psi}(\mathrm{T}_{10}) = \frac{(10)^2 - 1}{3}$$
$\tilde{\psi}(\mathrm{T}_{10}) = 3.30 \times 10^1$ J
$\rho(T_{10}) = 6.60 \times 10^{10}$ J/s

The results obtained are shown in the table below:

Table 4: Total work done and average work done by elements of T_n

n	$\psi(T_n)$ J	$\tilde{\psi}(T_n)$ J
1	0.00×10^0	0.00×10^0
2	4.00×10^0	1.00×10^0
3	7.20×10^1	2.70×10^0
4	1.30×10^3	5.00×10^0
5	2.50×10^4	8.00×10^0
6	5.40×10^5	1.20×10^1
7	1.30×10^7	1.60×10^1
8	3.50×10^8	2.10×10^1
9	1.03×10^{10}	2.70×10^1
10	3.30×10^{11}	3.30×10^1

Table 5: Power of the transformation

	J/s
$\rho(T_1)$	0.00×10^0
$\rho(T_2)$	8.00×10^{-1}
$\rho(T_3)$	1.40×10^1
$\rho(T_4)$	2.60×10^2
$\rho(T_5)$	5.00×10^3
$\rho(T_6)$	1.10×10^4
$\rho(T_7)$	2.60×10^6
$\rho(T_8)$	7.00×10^7
$\rho(T_9)$	2.10×10^9
$\rho(T_{10})$	6.60×10^{10}

The graph of the total work done and the average work done corresponding to the tabulated results in **Table 4** above are as shown below:

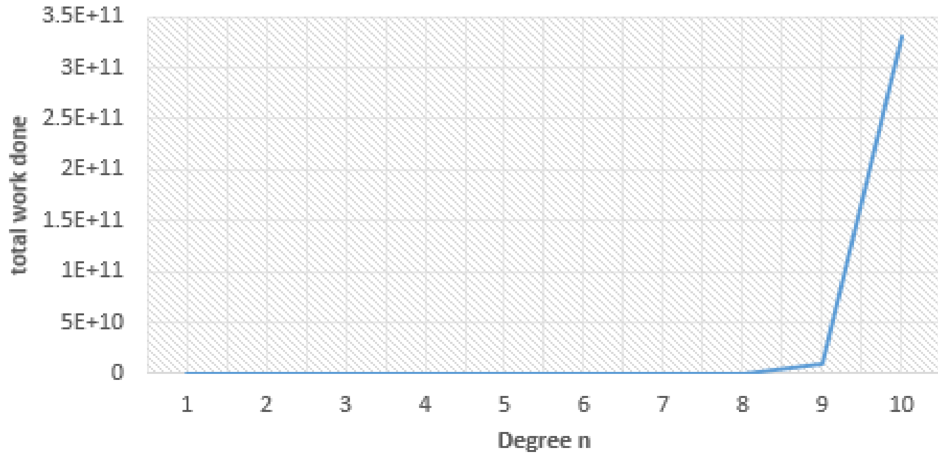

Fig 5: Total work done by elements of T_n

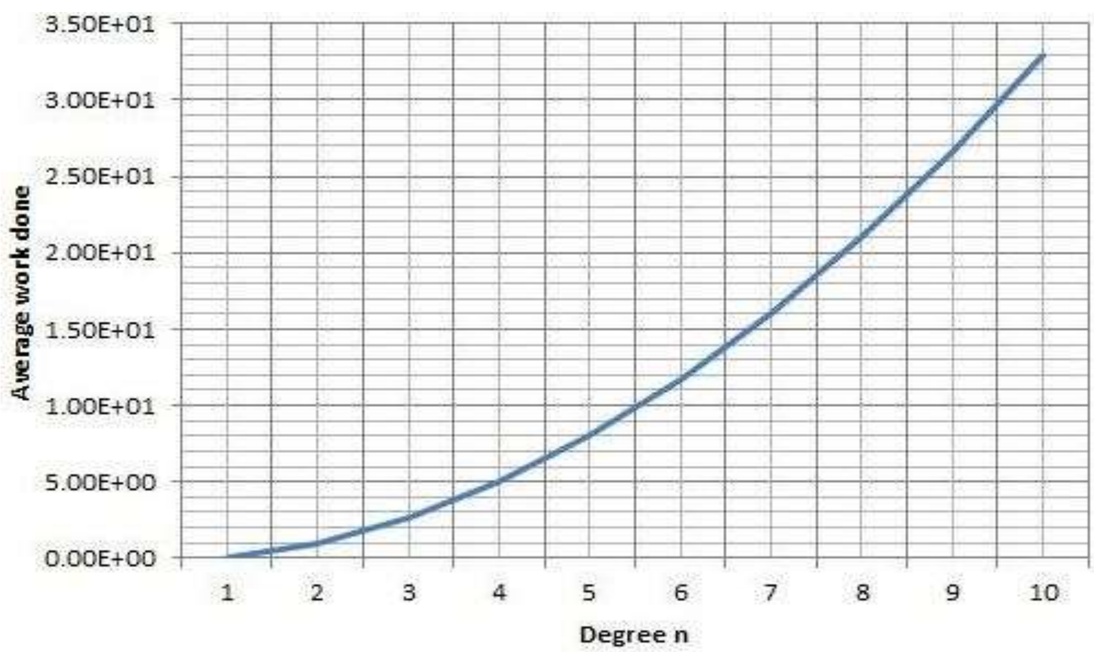

Fig 6: Average work done by elements of T_n

DISCUSSIONS

We have used simple MATLAB code to execute the numerical calculations as demonstrated above, the results obtained in each case of $S = I_n \ or \ T_n$ are expressed in standard form in order to maintain consistency. It is clear from the tables that the numerical values of $\psi(S)$,$\tilde{\psi}(S)$ and $\rho(S)$ for each of the semigroups increase as n is made as large as possible, this literarily means that the higher the degree n the higher the total work done on it and the higher the power of the transformation. As we clearly stated we have assumed that n is a set of points displaced on $\mathbb{R}$ and having equal distances, then for any transformation it moves a point in the domain of n to the co-domain of n and by this intuitive idea, the transformation has performed work on the numerical scale since it moved from one point to another and therefore covering a distance.

CONCLUSION

In this research work we have used valid methodology to formulate the generalize relations for calculating the total work done, the average work done and the power of the inverse transformation I_n as well as full transformation semigroup T_n, this was achieved by assuming elements of n to be set of points that are equally spaced on a real line $\mathcal{R}$.We tested the reliability and effectiveness of the generalized formula by substituting some values of n into the respective formulas and the results obtained are shown in **Table 2** to **Table 5** above. Graphs of the solution obtained are shown in **Figure 1** to **Figure 6** above.

It is clear from the tables and the graphs that the value of $\psi(S)$,$\tilde{\psi}(S)$ and $\rho(S)$ for each S approached ∞ as $n \to \infty$.

We also discovered that as $n = 1, \psi(S)$=$\tilde{\psi}(S)$ =$\rho(S)$=0 for each S which literarily means that there is no work performed when the degree of the transformation semigroup is 1 and this makes sense because the transformation has not covered any distance as it moves a point to itself. The generalized results and the tabulated solutions have an important application to some branches of physics and theoretical computer science. We will be looking at this application and the significant of the slopes of the graphs obtained in this research work in our subsequent research.

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