



EFFECTIVE METHOD OF SOLVING ONE-THIRD THIRD-ORDER IVPs WITH TWO OFF-GRID POINTS ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This article explores the effectiveness of a technique for solving one-third third-order initial value problems (IVPs) in ordinary differential equations by incorporating two off-grid points. A truncated power series was used as the basis function to construct the approximate solution. The resulting multistep methods were obtained through interpolation at selected grid points, while the third derivatives of the approximate solution were enforced at non-interpolating points through collocation. The system of linear equations arising from the collocation and interpolation of the approximate solution was solved to obtain the unknown coefficients. These computed values were then substituted back into the approximate expression to produce a continuous linear multistep method, from which the proposed scheme was derived through extrapolation. The method’s performance was tested on nonlinear problems involving general third-order ordinary differential equations, and the results showed good agreement with several existing methods documented in the literature.

Keywords: Power series expansion, Third order ODEs, Non-grid points, Collocation, Interpolation technique

INTRODUCTION

This work is a linear hybrid multistep block method for solving one-third general third-order IVPs of ordinary differential equations of the form:

$$y^{(3)} = f(x, y, y', y''), y(0) = y_0, y'(0) = y'_0, y''(0) = y''_0 \quad (1)$$

Third-order ordinary differential equations of (1) were traditionally solved by reducing them to the first-order initial value problem of ordinary differential equations and then solving the resulting equations using any of the available numerical methods. This numerical approach has been extensively discussed by several authors, such as [7] and many others. Even though this approach has been reason- ably practical, there are some drawbacks. For instance, the computer program used to implement the technique is frequently complex, particularly the subroutines used to give the method’s initial starting values, increasing the computational effort and computer time required. As a result,

alternative solutions for these problems need to be developed. Due to this, researchers developed alternate approaches to solving higher-order differential equations that did not involve first reducing the problem to a set of first-order ordinary differential equations such as [1], [3], [5], [6] and [8] just to mention but a few. This method includes the linear multi-step method (LMM), consisting of a backward differential formula: In the equation above, $y(x_{n+j})$ is the $(n + j)$ th approximation to the exact solution $y(x_{n+j})$.

According to [4] developed and applied a hybrid numerical approach with block extension to directly solve initial value problems in third-order ordinary differential equations, which can be integrated using numerical techniques. The approach had a broad form:

$$y_{\frac{1}{3}}(x) = \sum_{j=1}^{k-1} \alpha_{\frac{1}{3}}(x) y_{n+\frac{j}{3}} + h^3 \left(\sum_{j=0}^k \beta_{\frac{1}{3}}(x) f_{n+\frac{j}{3}} \right) \quad (2)$$

The method has order of accuracy of five. The focus of the present work is to provide method with improved order of accuracy.

MATERIALS AND METHODS

The Method

Numerical schemes hold profound significant in computational mathematics and scientific simulations , as they enable the efficient and accurate solution of complex differential equations by participating the domain into discrete blocks, facilitating parallel processing and enhancing computational scalability. This work entirely focused on numerical solution of initial value problems of general third order ordinary differential equations of the type:

$$y^3 = f(x, y, y', y''), y(0) = y_0, y'(0) = y'_0, y''(0) = y''_0, f \in R^n[a, b] \quad (3)$$

using power series approximation of the form

$$y(x) = \sum_{j=0}^{(c+i)-1} a_j x^j \quad (4)$$

as a basis function, where $a_j, j = 0, 1, 2, \dots, k$ are the coefficients to be determined. where $k = (c + i) - 1$

The third derivatives of (4) is given as

$$y^{(3)}(x) = \sum_{j=3}^k j(j-1)(j-2)a_j x^{j-3} \quad (5)$$

collocating equation (5) at $x = x_{n+r}, r = 0, \frac{1}{3}, 2, 3$

and interpolating equation (4) at $x = x_{n+r}, r = 1, 2, \frac{7}{3}$, where k denotes step number.

$$y_n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 + a_9 t^9 \quad (6)$$

$$y'''_n = 504t^6 a_9 + 336t^5 a_8 + 210t^4 a_7 + 120t^3 a_6 + 60t^2 a_5 + 24t a_4 + 6a_3 \quad (7)$$

$$y_{n+1} = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \quad (8)$$

$$y_{n+2} = a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 + 64a_6 + 128a_7 + 256a_8 + 512a_9 \quad (9)$$

$$y_{n+\frac{7}{3}} = a_0 + \frac{7}{3}a_1 + \frac{49}{9}a_2 + \frac{343}{27}a_3 + \frac{2401}{81}a_4 + \frac{16807}{243}a_5 + \frac{117649}{729}a_6 + \frac{823543}{2187}a_7 + \frac{5764801}{6561}a_8 + \frac{40353607}{19683}a_9 \quad (10)$$

$$f_n = 6a_3 \quad (11)$$

$$f_{n+\frac{1}{3}} = 6a_3 + 88a_4 + \frac{20}{3}a_5 + \frac{40}{9}a_6 + \frac{70}{27}a_7 + \frac{112}{81}a_8 + \frac{56}{81}a_9 \quad (12)$$

$$f_{n+1} = 6a_3 + 24a_4 + 60a_5 + 120a_6 + 210a_7 + 336a_8 + 504a_9 \quad (13)$$

$$y_{n+\frac{5}{3}} = 6a_3 + 40a_4 + \frac{500}{3}a_5 + \frac{5000}{9}a_6 + \frac{43750}{27}a_7 + \frac{350000}{81}a_8 + \frac{875000}{81}a_9 \quad (14)$$

$$f_{n+2} = 6a_3 + 48a_4 + 240a_5 + 960a_6 + 3360a_7 + 10750a_8 + 32256a_9 \quad (15)$$

$$y_{n+\frac{8}{3}} = 6a_3 + 64a_4 + \frac{1280}{3}a_5 + \frac{20480}{9}a_6 + \frac{286720}{27}a_7 + \frac{3670016}{81}a_8 + \frac{14680064}{81}a_9 \quad (16)$$

$$f_{n+3} = 6a_3 + 72a_4 + 540a_5 + 3240a_6 + 17010a_7 + 81648a_8 + 367416a_9 \quad (17)$$

The parameters $j = 0, \frac{1}{3}, \dots, 3$ which when substituted in Eq.(4), yield after some simplification, a hybrid linear multi-step method with continuous coefficient of the form:

$$y_{\frac{j}{3}}(t) = \sum_{i=1}^{k-1} \alpha_i(t) y_{n+\frac{i}{3}} + h^3 \left(\sum_{j=0}^k \beta_j(t) f_{n+\frac{j}{3}} \right) \quad (18)$$

Where the coefficients of $\alpha_j(t)$ and $\beta_j(t)$ are given as:

$$\alpha_1(t) = \frac{3}{4}t^2 + \frac{7}{2} - \frac{13t}{4}$$

$$\alpha_2(t) = -3t^2 + 10t - 7$$

$$\alpha_7(t) = \frac{9}{2} + \frac{9}{4}t^2 - \frac{27t}{4}$$

$$\beta_0 = \frac{33403}{16329600}th^3 + \frac{101}{4200}t^7h^3 + \frac{t^9h^3}{4480} - \frac{697}{2880}t^4h^3 + \frac{61}{320}t^5h^3 + \frac{4147}{3499200}h^3 - \frac{t^8h^3}{280} - \frac{1271}{14400}t^6h^3 + \frac{1}{6}t^3h^3 - \frac{1147633}{24494400}t^2h^3$$

$$\beta_{\frac{1}{3}} = -\frac{10071}{156800}t^7h^3 + \frac{81}{224}t^4h^3 - \frac{9099}{22400}t^5h^3 + \frac{2511}{250880}t^8h^3 - \frac{9130601}{33868800}t^2h^3 - \frac{81}{125440}t^9h^3 + \frac{131273}{705600}th^3 + \frac{19521}{89600}t^6h^3 - \frac{42169}{1209600}h^3$$

$$\beta_1 = -\frac{197399}{583200}h^3 - \frac{1591}{4800}t^6h^3 + \frac{3}{2240}t^9h^3 - \frac{5948101}{16329600}t^2h^3 - \frac{1}{4}t^4h^3 + \frac{577}{4200}t^5h^3 + \frac{317}{2800}t^7h^3 - \frac{87}{4480}t^8h^3 + \frac{192749}{272160}th^3$$

$$\beta_{\frac{5}{3}} = -\frac{188699}{1036800}t^2h^3 - \frac{2421}{11200}t^7h^3 - \frac{27}{8960}t^9h^3 + \frac{166357}{302400}th^3 - \frac{45}{64}t^5h^3 + \frac{3591}{6400}t^6h^3 + \frac{27}{80}t^4h^3 - \frac{99817}{259200}h^3$$

$$\beta_2 = -\frac{527}{1200}t^6h^3 - \frac{1}{4}t^7h^3 - \frac{39}{1120}t^8h^3 - \frac{3881}{291600}h^3 + \frac{3}{1120}t^9h^3 - \frac{104927}{4082400}t^2h^3 + \frac{637}{1200}t^5h^3 + \frac{31}{175}t^7h^3 + \frac{71231}{1360800}th^3$$

$$\beta_{\frac{8}{3}} = -\frac{2801}{6350400}t^2h^3 + \frac{5543}{846720}th^3 + \frac{27}{448}t^4h^3 + \frac{81}{7840}t^8h^3 + \frac{1269}{11200}t^6h^3 - \frac{27}{31360}t^9h^3 - \frac{1467}{11200}t^5h^3 - \frac{8611}{907200}h^3 - \frac{8611}{907200}t^7h^3$$

$$\beta_3 = \frac{73}{1920}t^5h^3 + \frac{6361}{2799360}h^3 - \frac{503}{408240}th^3 - \frac{769}{23040}t^6h^3 - \frac{5}{288}t^4h^3 + \frac{t^9h^3}{3584} + \frac{197}{13440}t^7h^3 - \frac{23}{7168}t^8h^3 - \frac{803}{15676416}t^2h^3$$

Evaluating at $t = 0, \frac{1}{3}, \frac{5}{3}, \frac{8}{3}$ and 3 yield the discrete three step formulas below:

$$y_n = \frac{7}{2}y_{n+1} - 7y_{n+2} + \frac{9}{2}y_{n+\frac{7}{3}} + \frac{4147}{3499200}h^3f_n - \frac{197399}{583200}h^3f_{n+1} + \frac{3881}{291600}h^3f_{n+2} + \frac{6361}{2799360}h^3f_{n+3}$$

$$- \frac{42169}{1209600}h^3f_{n+\frac{1}{3}} - \frac{99817}{259200}h^3f_{n+\frac{5}{3}} - \frac{8611}{907200}h^3f_{n+\frac{8}{3}}$$

$$y_{n+\frac{1}{3}} = \frac{20675}{11757312}h^3f_{n+3} - \frac{3343}{476280}h^3f_{n+\frac{8}{3}} - \frac{89}{612360}h^3f_{n+2} - \frac{47899}{217728}h^3f_{n+\frac{5}{3}} - \frac{353671}{2449440}h^3f_{n+1} + \frac{1333}{5080320}h^3f_{n+\frac{1}{3}} - \frac{155}{183708}h^3f_n$$

$$+ \frac{5}{2}y_{n+1} - 4y_{n+2} + \frac{5}{2}y_{n+\frac{7}{3}}$$

$$y_{n+\frac{5}{3}} = -\frac{9167}{176359680}h^3f_{n+3} + \frac{1579}{7144200}h^3f_{n+\frac{8}{3}} + \frac{71741}{9185400}h^3f_{n+2} + \frac{253859}{16329600}h^3f_{n+\frac{5}{3}} + \frac{48163}{36741600}h^3f_{n+1} - \frac{14857}{76204800}h^3f_{n+\frac{1}{3}} +$$

$$\frac{691}{13778100}h^3f_n + \frac{1}{6}y_{n+1} + \frac{1}{3}y_{n+2} + \frac{1}{2}y_{n+\frac{7}{3}}$$

$$y_{n+\frac{8}{3}} = \frac{42037}{35271936}h^3f_{n+3} + \frac{85507}{11430720}h^3f_{n+\frac{8}{3}} + \frac{182369}{3674160}h^3f_{n+2} + \frac{1553}{653184}h^3f_{n+\frac{5}{3}} + \frac{31043}{7348320}h^3f_{n+1}$$

$$- \frac{17099}{15240960}h^3f_{n+\frac{1}{3}} + \frac{2857}{8817984}h^3f_n + \frac{1}{6}y_{n+1} + \frac{5}{3}y_{n+2} + \frac{5}{2}y_{n+\frac{7}{3}}$$

$$y_{n+3} - \frac{1}{2}y_{n+1} + 4y_{n+2} + \frac{9}{2}y_{n+\frac{7}{3}} = \frac{91067}{19595520}h^3f_{n+3} + \frac{38119}{793800}h^3f_{n+\frac{8}{3}} + \frac{176321}{1020600}h^3f_{n+2} + \frac{12001}{1814400}h^3f_{n+\frac{5}{3}} +$$

$$\frac{64543}{4082400}h^3f_{n+1} - \frac{37117}{8467200}h^3f_{n+\frac{1}{3}} + \frac{1951}{1530900}h^3f_n \quad (19)$$

Analysis of the Method

Order and Error Constant

The conventional linear multi-step method of the form:

$$\sum_{i=0}^k \alpha_i y_{n+i} = h^3 \sum_{j=0}^k \beta_j f_{n+j} \quad (20)$$

According to Lambert (1973), the difference linear operator L and the associated continuous linear multi-step method (19) are said to be of order p, if $c_0 = c_1 = \dots = c_p = c_{p+2} = 0$, and $c_{p+3} \neq 0$. Where c_{p+3} denoted the error constant

Therefore, the method is of order $p = 7$, with error constant $c_{p+3} = \frac{119509}{299987815680}$

Zero Stability

Linear multi-step method is said to be zero stable if the root of the characteristics polynomial satisfy

$|z| \leq 1$ and the root of $|z|=1$ has multiplicity not exceeding the order of the differential equation. Moreover, $h^\mu \Rightarrow 0$, $\rho(z) = z^{\mu}(\lambda - 1)^\mu$, where μ is the order of differential equation for the block method. This implies that $\rho(\lambda) = \det[\lambda A^{(0)} - A^{(1)}]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho(\lambda) = \det\left[\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \lambda^3 (\lambda - 1) = 0 \quad (21)$$

Since $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$, the block method is zero stable.

Consistency

The main method (20) is consistent if

- i. The order of the method is $p = 7 \geq 1$, which is obvious

$$\sum_{r=0}^k \alpha_r = 1 + 4 - \frac{9}{2} - \frac{1}{2} = 0$$

- ii. $\rho(1) = \rho'(1) = 0$

$$\rho(r) = r^3 + 4r^2 - \frac{9}{2}r - \frac{1}{2}r^1$$

where ρ is defined as first characteristic polynomial

$$\text{Then, } \rho(r = 1) = 1 + 4 - \frac{9}{2} - \frac{1}{2} = 0$$

$$\text{Also, } \rho'(r) = 3r^2 + 8r - \frac{21}{2}r - \frac{1}{2}$$

$$\text{Where } \rho'(r = 1) = 3 + 8 - \frac{21}{2} - \frac{1}{2} = 0$$

It follows from here that $\rho(1) = \rho'(1) = 0$ shown that condition (iii) satisfied

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- iii. The first characteristics polynomial :

$$\rho(r) = r^3 + 4r^2 - \frac{9}{2}r - \frac{1}{2}r^1$$

$$\rho'(r) = 3r^2 + 8r - \frac{21}{2}r - \frac{1}{2}$$

$$\rho''(r) = 6 - \frac{14}{3}r$$

$$\rho''(r = 1) = \frac{4}{3}$$

The second characteristics polynomial :

$$\sigma(r) = -\frac{91067}{19595520}r^3 + \frac{38119}{793800}r^2 + \frac{176321}{1020600}r - \frac{12001}{1814400}r^{\frac{5}{3}} + \frac{64543}{4082400}r^1 - \frac{37117}{8467200}r^{\frac{1}{3}} + \frac{1951}{1530900}r^0$$

$$\sigma(r = 1) = -\frac{91067}{19595520}(1)^3 + \frac{38119}{793800}(1)^2 + \frac{176321}{1020600}(1) - \frac{12001}{1814400}(1)^{\frac{5}{3}} + \frac{64543}{4082400}(1)^1 - \frac{37117}{8467200}(1)^{\frac{1}{3}} + \frac{1951}{1530900}(1)^0 = \frac{2}{9}$$

$$\text{a. } \rho'''(1) = 3!\sigma(1) \text{ for } r = 1.$$

$$\rho'''(1) = 3!\sigma(1) \text{ for } r = 1.$$

The conditions are satisfied therefore the method is consistent.

Convergence

The necessary and sufficient condition for a linear multi-step method to be convergent is for it to be consistent and zero stable.

Thus, it has been successfully shown from the above conditions, it could be seen that the method is convergent. (Henrici (1962))

Region of Absolute Stability

We consider the stability polynomial written in general form

$\pi(r, h) = \rho(r)$ and $\sigma(r)$ are first characteristics polynomial and second characteristics polynomial respectively.

$\bar{h} = h^2\lambda$ and $\lambda = \frac{\delta f}{\delta y}$ is assumed constant. The stability polynomial of the main method

$$y_{n+3} + 4y_{n-2} - \frac{9}{2}y_{n+\frac{7}{3}} - \frac{1}{2}y_{n+1} = \frac{h^3}{685843200}(-3187345f_{n+3} + 32934816f_{n+\frac{8}{3}} + 118487712f_{n+2} - 4536378f_{n+\frac{5}{3}} + 10843224f_{n+1} - 3006477f_{n+\frac{1}{3}} + 874048f_n)$$

$$\text{Becomes } (r^3 + 4r^2 - \frac{9}{2}r - \frac{1}{2}r^1) = \bar{h}\left(-\frac{91067}{19595520}r^3 + \frac{38119}{793800}r^2 + \frac{176321}{1020600}r - \frac{12001}{1814400}r^{\frac{5}{3}} + \frac{64543}{4082400}r^1 - \frac{37117}{8467200}r^{\frac{1}{3}} + \frac{1951}{1530900}r^0\right)$$

Obviously, the first characteristic polynomial is $\rho(r) = r^3 + 4r^2 - \frac{9}{2}r^{\frac{7}{3}} - \frac{1}{2}r^1$ and the second characteristic polynomial is

$$\sigma(r) = -\frac{91067}{19595520}r^3 + \frac{38119}{793800}r^{\frac{8}{3}} + \frac{176321}{1020600}r^2 - \frac{12001}{1814400}r^{\frac{5}{3}} + \frac{64543}{4082400}r^1 - \frac{37117}{8467200}r^{\frac{1}{3}} + \frac{1951}{1530900}r^0$$

Adopting the boundary locus method whose equation is given by

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} \tag{21}$$

By inserting the value of $\rho(r)$ and $\sigma(r)$ into equation (22), we obtain boundary locus equation for the method as:

$$\bar{h}(r) = \frac{r^3 + 4r^2 - \frac{9}{2}r^{\frac{7}{3}} - \frac{1}{2}r^1}{-\frac{91067}{19595520}r^3 + \frac{38119}{793800}r^{\frac{8}{3}} + \frac{176321}{1020600}r^2 - \frac{12001}{1814400}r^{\frac{5}{3}} + \frac{64543}{4082400}r^1 - \frac{37117}{8467200}r^{\frac{1}{3}} + \frac{1951}{1530900}r^0} \tag{22}$$

By setting $r = \exp^{i\theta}$, $0 \leq \theta \leq \pi$, where $t^{i\theta} = \cos\theta + i\sin\theta$, then in equation (23), we have By expanding the above trigonometrically and equating the imaginary part to zero

$\bar{h}(\theta) = \frac{a}{b}$ where :

$$a = 685843200 \left[\left(465341891 - 666612072\cos\frac{\theta}{3} + 1702677445\cos\frac{2\theta}{3} + 89430348\cos\theta - 53330886170267\cos\frac{4\theta}{3} - 28493316\cos\frac{5\theta}{3} + 83798263\cos2\theta - 3933216\cos\frac{7\theta}{3} - 3006477\cos\frac{8\theta}{3} + 874048\cos3\theta \right) \right]$$

$$b = 152821564028718 - 1290214952599104\cos\frac{\theta}{3} + 761164253232944\cos\frac{2\theta}{3} + 1515464577641301\cos\theta + 56195035713432\cos\frac{4\theta}{3} - 6152011483968\cos\frac{5\theta}{3} + 138005703795792\cos2\theta - 198035533606464\cos\frac{7\theta}{3} + 76738578977466\cos\frac{8\theta}{3} - 5571785045120\cos3\theta]$$

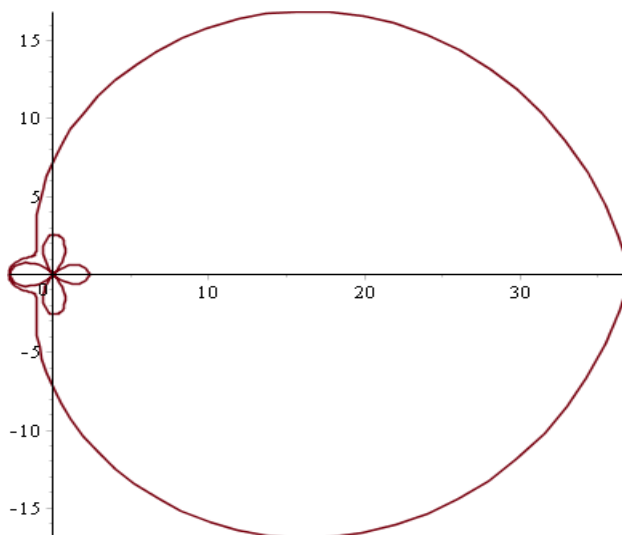


Figure 1: Region of Absolute Stability of the Method

Numerical Applications

In this section, the derived methods were used to solve three problems, the exact and numerical results of each case were represented in tabular form including the absolute error.

The table showing $Erc = |y_{ex} - y_c|$, where y_{ex} is the exact solution, y_c is the computed result and A_c

absolute error defined as $Erc = |y_c - y_{ex}|$ was also reported.

Problem 1

$$y''' - 4y' - x = 0, h = 0.1$$

$$y(0) = 0, y'(0) = 0, y''(0) = 1, h = \frac{1}{10}$$

$$\text{Exact solution: } y(x) = \frac{3}{16}(1 - \cos2x) + \frac{1}{18}x^2$$

Source: [9] et al. (2021)

Table 1: Result of Problem 1 Computed with 3-step Block Method, k = 3, h = 0.1

x-Value	y-Exact-Solution	y-Computed Solution	Error in Newly Method, P=7	Error in [9] (2021), P=7
0.10	0.0049875166547671941600	0.0049875166546846171245	8.25770355e-14	1.6654e-08
0.20	0.019801063624459046980	0.019801063624408837675	5.0209305e-14	3.8095e-07
0.30	0.043999572204435319270	0.043999572204382563068	5.2756202e-14	1.5664e-06
0.40	0.076867491997406483580	0.076867491997097711680	3.08771900e-13	3.9865e-06
0.50	0.11744331764972380299	0.11744331764911057518	6.1322781e-13	7.9597e-06
0.60	0.16455792103562370419	0.16455792103457560881	1.04809538e-12	1.3680e-05
0.70	0.21688116070620482401	0.21688116070449470116	1.71012285e-12	2.1195e-05
0.80	0.27297491043149163616	0.27297491042897697538	2.51466078e-12	3.0396e-05
0.90	0.33135039275495382287	0.33135039275151741382	3.43640905e-12	4.10008e-5
1.00	0.39052753185258919756	0.39052753184818091594	4.40828162e-12	5.2605e-05

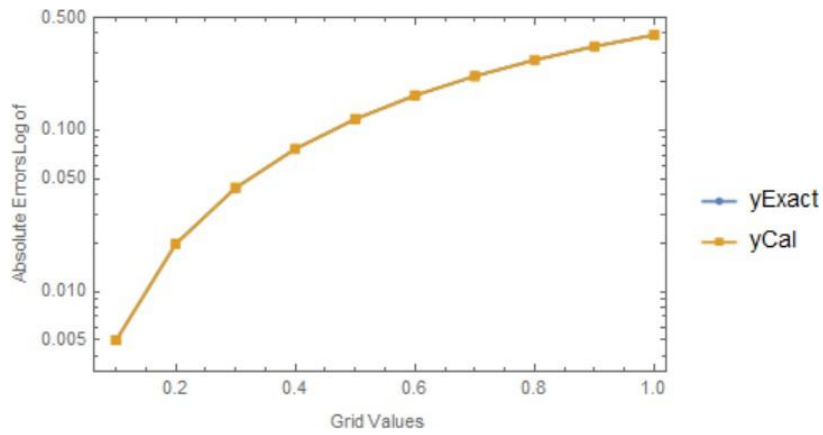


Figure 2: Solution Curve of Problem 1 as Compared with Exact Solution

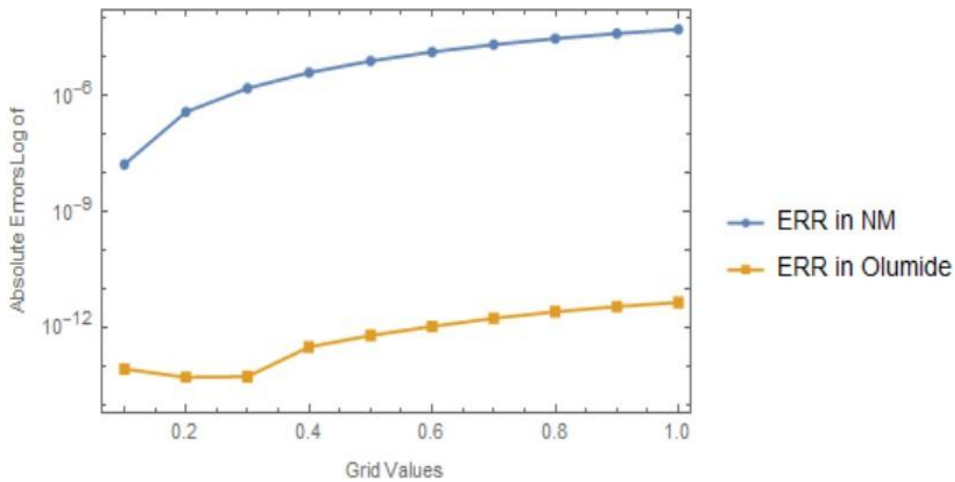


Figure 3: Comparison of Absolute Error of the Proposed Method on Problem 1 as Compared with [9]

Problem 2

$y''' = 3\sin(x), h = 0.01$

$y(0) = 1, y'(0) = 0, y''(0) = -2, h = \frac{1}{100}$

Exact solution: $y(x) = 3 \cos(x) + \frac{x^2}{2} - 2$

Source: [2] (2018)

Table 2: Result of Problem 2, Computed with 3-Step Block Method, $k = 3, h = 0.01$

x-value	y-exact-solution	y-computed solution	Error in Newly Method , P=7	Error in [2] (2018), P=
0.10	0.9999000012499958333	0.99990000124999583334	4.0e-20	1.7282e-12
0.20	0.9996000199997333352	0.99960001999973333524	4.0e-20	6.3179e-12
0.30	0.9991001012469625488	0.99910010124696254880	0.0e-00	1.4295e-11
0.40	0.9984003199829338209	0.99840031998293382093	3.03e-20	2.5020e-11
0.50	0.9975007811848987397	0.99750078118489873967	3.0e-20	3.8928e-11
0.60	0.9964016198056124966	0.99640161980561249661	1.0e-20	5.5360e-11
0.70	0.9951030007598387239	0.99510300075983872383	7.0e-20	7.4644e-11
0.80	0.9936051189078581549	0.99360511890785815490	0.0e-00	9.6128e-11
0.90	0.9919081990359827593	0.99190819903598275924	6.0e-20	1.2002e-10
1.00	0.9900124958340772983	0.99001249583407729825	5.0e-20	1.4570e-10

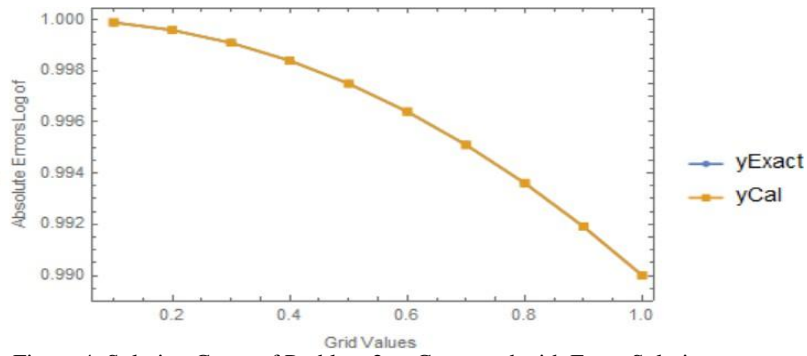


Figure 4: Solution Curve of Problem 2 as Compared with Exact Solution

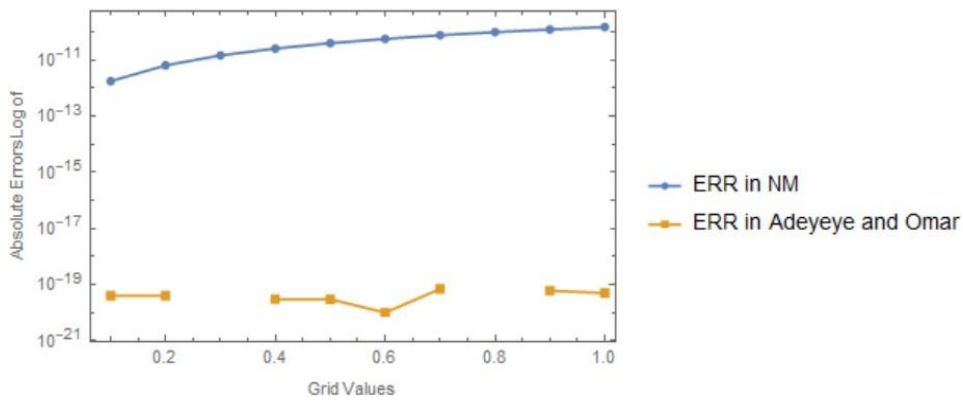


Figure 5: Comparison of Absolute Error of the Proposed Method on Problem 2 as Compared with [2]

Problem 3

$y''' = -y, h= 0.05$

$y(0) = 1, y'(0) = -1, y''(0) = 1, h = \frac{1}{10}$

Exact solution: $y(x) = t^{-x}$

Source: [2] (2018)

Table 3: Result of Problem 3, Computed with 3-Step Block Method, k = 3, h = 0.1

x-value	y-Exact-Solution	y-Computed Solution	Error in Newly Method , p=7	Error in [2] (2018), P=8
0.10	0.90483741803595957316	0.90483741803595917985	3.9331e-16	5.2981e-13
0.20	0.81873075307798185867	0.81873075307798160072	2.5795e-16	1.9341e-12
0.30	0.74081822068171786607	0.74081822068171757255	2.9352e-16	4.3127e-12
0.40	0.67032004603563930074	0.67032004603563773968	1.56106e-15	7.4462e-12
0.50	0.60653065971263342360	0.60653065971263036233	3.06127e-15	1.1436e-11
0.60	0.54881163609402643263	0.54881163609402112225	5.31038e-15	1.6124e-11
0.70	0.49658530379140951470	0.49658530379140042431	9.09039e-15	2.1614e-11
0.80	0.44932896411722159143	0.44932896411720793691	1.365452e-14	2.7797e-11
0.90	0.40656965974059911188	0.40656965974057973638	1.937550e-14	3.4785e-11
1.00	0.36787944117144232160	0.36787944117141550230	2.681930e-14	4.2511e-11

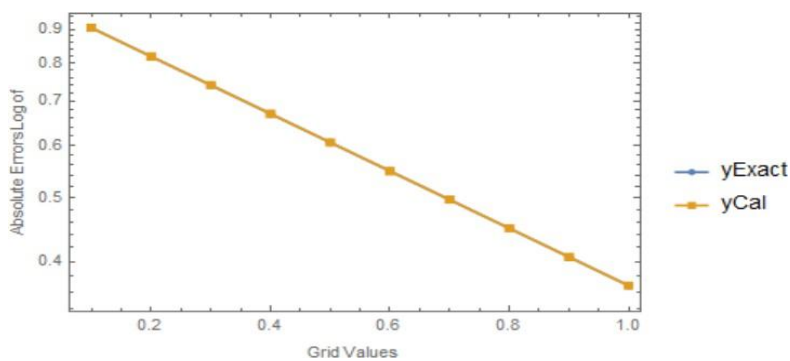


Figure 6: Solution Curve of Problem 3 as Compared with Exact Solution

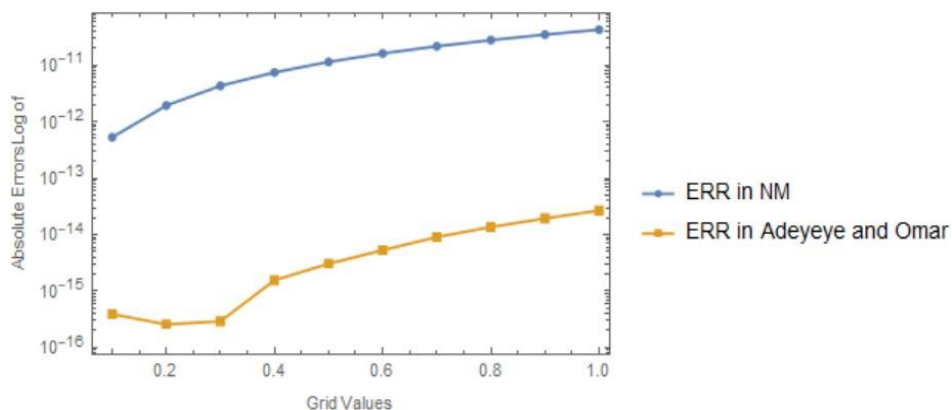


Figure 7: Comparison of Absolute Error of the Proposed Method on Problem 3 as Compared with [2]

Discussion

It is important to emphasize that the seventh-order method, characterized by its zero stability and consistency, achieves a high level of computational accuracy and reliability. Its elevated order of accuracy demonstrates a strong ability to closely approximate the exact solution of a differential equation, with errors diminishing significantly as the step size is reduced.

Zero stability ensures that, even in the presence of stiff or oscillatory problems, the numerical method remains reliable and prevents the amplification of errors as the computation advances.

Moreover, consistency ensures that as the step size tends toward zero, the method converges to the true solution, confirming its adherence to the underlying mathematical model.

This study employed collocation and interpolation techniques to construct a linear hybrid multistep method for solving the initial value problem of third-order ordinary differential equations, as presented in Tables 1, 2, and 3. The results demonstrate that the proposed approach achieves greater accuracy than the methods introduced in [9] and [2]. It can be observed from Figures 2 to 7 that the proposed method produces superior results compared to those reported in [9] and [2], even though their approaches are of higher order.

CONCLUSION

The three-step method was formulated for the direct solution of third-order differential equations. The methods were generated from the same continuous schemes constructed using interpolation and collocation techniques. The methods were implemented in block form. The numerical results produced by the block schemes demonstrate that they are both efficient and suitable for solving general third-order ordinary differential equations. The seventh-order method is both zero-stable and consistent. A comparison of its accuracy with that of existing methods shows that the proposed methods achieve superior accuracy.

REFERENCES

Adeyeye O. and Omar Z. (2018), New Self-Starting approach for solving special third order initial value problems. *International Journal of Pure and Applied Mathematics*, Vol. 118, Nob.3, 2018, 511-517. doi: <https://doi.org/10.12732/ijpam.v118i3.2>

Awoyemi D.O. and Idowu O. M. (2005). A class of hybrid collocation methods for third-order ordinary differential equations. <https://doi.org/10.1080/00207160500112902>

Duromola M.K.(2016),An accurate five off-step points implicit block method for direct solution of fourth order differential equations. *Open Access Library Journal* 3(6): 1-14 <http://dx.doi.org/10.4236/oalib.1102667>

Duromola, M. K. and Momoh, A. L. (2019), Hybrid Numerical Method with Block Extension for Direct Solution of Third Order Ordinary Differential Equations. *American Journal of Computational Mathematics*, 9:68-80. doi: <https://doi.org/10.4236/ajcm.2019.92006>

Duromola M. K. (2022). Single-step Block Method of P-Stable For Solving Third Order Differential Equations (IVPs): Ninth Order of Accuracy, *American Journal of Applied Mathematics and Statistics*.Vol.10,No.1,4-13. doi: <https://doi.org/10.12691/ajams-10-1-2>

Duromola M. K., Kayode S. J. and Lawal R. S. (2023). Linear Hybrid Multistep Block Method for Direct Solution of Initial Value Problems of Third Order Ordinary Differential Equations, *Asian Research Journal of Mathematics*, <https://doi.org/10.9734/arjom/2023/v19i11764>

Fatunla S.O.(1995): Block method for second order differential equation. *International journal for computer Mathematics*,9: 68-80. <https://doi.org/10.1080/00207169508804368>

M. k. Duromola, R. S. Lawal and O.M. Akinmoladun (2024),Numerical Integration of linear hybrid multistep block method for third-order ordinary differential equations. *Journal of Scientific African*. <https://doi.org/10.1016/j.sciaf.2024.e02129>.

Olumide F. (2021), Computational study of some three-step hybrid integrator for solution of third order ordinary differential equations. *Journal of the Nigeria Society of Physical Science*, 2, 459-468. doi: <https://doi.org/10.46481/jnsps.2021.323>



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