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# A MODIFIED TRANSMUTED SINE DAGUM DISTRIBUTION: PROPERTIES, ESTIMATION, AND APPLICATIONS

***Ibrahim Abdullahi and Musa Chiwa Dalah**

Department of Mathematics and Statistics, Yobe State University, Damaturu, Nigeria

* Corresponding authors' email: ibworld82@yahoo.com

## ABSTRACT
A new flexible distribution, termed the Power Sine Sine Dagum (PSSD) distribution, is introduced by applying a power sine transformation to the classical Dagum model. The proposed construction incorporates an additional shape parameter, thereby enhancing the ability of the baseline distribution to capture complex tail behavior and varying degrees of skewness. Several structural properties of the model are derived, including explicit expressions for the probability density function, cumulative distribution function, survival and hazard functions, quantile function, and moments. Parameter estimation is developed within both maximum likelihood and Bayesian frameworks. A Monte Carlo simulation study is conducted to evaluate finite-sample performance, with results indicating that the maximum likelihood estimator may exhibit instability in small samples, whereas the Bayesian estimator provides more stable and accurate inference. The practical performance of the model is illustrated using real datasets. The results show that the PSSD model achieves higher log-likelihood values and lower AIC, together with smaller goodness-of-fit statistics (KS, AD, and CvM), compared to the classical Dagum distribution. These findings demonstrate that the proposed model provides a flexible and effective tool for modeling heavy-tailed data.

**Keywords**: Dagum distribution, Sine transformation, Heavy-tailed distribution, Maximum likelihood, Bayesian

## INTRODUCTION
Heavy-tailed distributions play a fundamental role in income modeling, actuarial science, finance, survival analysis, and reliability theory, where observed data often exhibit pronounced right skewness, heavy upper tails, and complex hazard-rate structures. Among classical models, the Dagum distribution has received considerable attention due to its ability to capture income inequality, scale heterogeneity, and heavy-tail behavior Kleiber and Kotz (2003). In recent years, several extensions of the Dagum distribution have been proposed to improve its flexibility, including generalized, inverted, and composite variants (Alotaibi et al., 2021; Silva et al., 2022; Koleoso, 2023; Alghamdi et al., 2023; Osi et al., 2024a).

Despite these developments, the classical Dagum model may still be inadequate for describing datasets characterized by intricate curvature, peak–tail interaction, and nonstandard hazard-rate shapes. This limitation has motivated the development of generator-based families of distributions, in which additional shape parameters are introduced to enhance model flexibility. In particular, trigonometric generators have attracted increasing interest due to their ability to produce highly flexible distributions while preserving analytical tractability. Notable examples include the alpha-sine-$G$ family, sine power-$G$ models, and other trigonometric-based constructions (Benchiha et al., 2023; Bakr et al., 2022; Alghamdi et al., 2025).

More recently, sine-based transformations have been shown to significantly improve the modeling of heavy-tailed data and complex hazard-rate behaviors. Several studies have demonstrated that sine-generated and related trigonometric distributions provide superior goodness-of-fit compared to classical models in reliability and survival contexts (Osi et al. ,2024b; Isa et al., 2023). These findings underscore the effectiveness of trigonometric generators as a powerful and versatile tool for constructing flexible probability models.

Motivated by these advances, this paper introduces a new flexible distribution, termed the Power Sine–Sine Dagum (PSSD) distribution. The proposed model is obtained by applying a power sine transformation to the classical Dagum distribution, resulting in an additional shape parameter that enhances its ability to capture heavy tails, skewness, and diverse hazard-rate patterns. To the best of our knowledge, the power sine–sine transformation applied to the Dagum distribution has not been previously investigated in the literature, thereby providing a novel contribution to the class of generator-based distributions.

The main contributions of this study are summarized as follows. First, the PSSD distribution is formally defined and its key structural properties are derived, including explicit expressions for the probability density function, cumulative distribution function, survival and hazard functions, quantile function, and moments. Second, parameter estimation is developed using both maximum likelihood and Bayesian approaches. Third, a Monte Carlo simulation study is conducted to examine the finite-sample performance of the proposed estimators. Finally, the practical applicability of the model is demonstrated using real datasets, where it is shown to outperform the classical Dagum distribution in terms of goodness-of-fit criteria.

## MATERIALS AND METHODS
### Model Construction
#### *Dagum Baseline Distribution*
Let $X$ follow the Dagum distribution with cumulative distribution function

$$G(x) = \left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}, x>0, \quad (1)$$

where $a>0$, $b>0$, and $p>0$.

Differentiating (1), the corresponding probability density function is:

$$g(x) = \frac{ap}{b}\left(\frac{x}{b}\right)^{-a-1}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p-1}, x>0. \quad (2)$$

#### *Power Sine–Sine Dagum Distribution*
*Definition 1.* The Power Sine–Sine Dagum (PSSD) distribution is defined through the transformation

$$F(x) = \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha}, x > 0, \quad (3)$$
where $\alpha > 0$ and $G(x)$ is the Dagum CDF given in (1).

*Proposition 1.* The probability density function of the PSSD distribution is
$$f(x) = \frac{\alpha\pi}{2}g(x)\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}G(x)\right), x > 0, \quad (4)$$
where $G(x)$ *and* $g(x)$ are given in (1) and (2), respectively.

*Proof.* From (3),
$$F(x) = \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha}.$$
Differentiating with respect to $x$ and applying the chain rule, we obtain
$$f(x) = \alpha\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}G(x)\right)\cdot\frac{\pi}{2}G'(x).$$
Since $G'(x) = g(x)$, it follows that
$$f(x) = \frac{\alpha\pi}{2}g(x)\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}G(x)\right),$$
which establishes (4).
Substituting (1) and (2) into (4), the PSSD density can be written explicitly as
$$\begin{aligned} f(x) &= \frac{\alpha\pi}{2}\frac{ap}{b}\left(\frac{x}{b}\right)^{-a-1}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p-1} \\ &\times\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha-1} \\ &\times\cos\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right), \quad x > 0. \end{aligned} \quad (5)$$
*Proposition 2:* The cumulative distribution function corresponding to the density in (5) is
$$F(x) = \left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}, \quad x > 0. \quad (6)$$
*Proof.* By definition,
$$F(x) = \int_0^x f(t)\,dt.$$
Substituting from (5), we obtain
$$\begin{aligned} F(x) &= \int_0^x \frac{\alpha\pi}{2}\frac{ap}{b}\left(\frac{t}{b}\right)^{-a-1}\left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p-1} \\ &\times\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha-1} \\ &\times\cos\left(\frac{\pi}{2}\left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p}\right)dt. \end{aligned}$$
Let
$$u = \left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p}.$$
Then
$$\frac{du}{dt} = \frac{ap}{b}\left(\frac{t}{b}\right)^{-a-1}\left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p-1},$$
so that
$$du = \frac{ap}{b}\left(\frac{t}{b}\right)^{-a-1}\left(1+\left(\frac{t}{b}\right)^{-a}\right)^{-p-1}dt.$$
Hence,
$$F(x) = \int_0^{\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}}\frac{\alpha\pi}{2}\left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}u\right)du.$$
Let
$$v = \sin\left(\frac{\pi}{2}u\right).$$
Then
$$dv = \frac{\pi}{2}\cos\left(\frac{\pi}{2}u\right)du.$$
Therefore,
$$\begin{aligned} F(x) &= \alpha\int_0^{\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)}v^{\alpha-1}dv \\ &= \left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}. \end{aligned}$$
This proves (6).

*Proposition 3:* The survival function corresponding to the cumulative distribution function in (6) is
$$S(x) = 1-\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}, \quad x > 0. \quad (7)$$
*Proof.* By definition, the survival function is
$$S(x) = 1 - F(x).$$
Using the cumulative distribution function in (6), we obtain
$$S(x) = 1-\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}.$$
Hence, the survival function is given by (7).

***Hazard Rate Function***

*Proposition 4:* The hazard rate function of the PSSD distribution is
$$h(x) = \frac{f(x)}{S(x)}, \quad x > 0, \quad (8)$$
where $f(x)$ and $S(x)$ are given in (5) and (7), respectively. Consequently,
$$\begin{aligned} h(x) &= \frac{\frac{\alpha\pi}{2}\frac{ap}{b}\left(\frac{x}{b}\right)^{-a-1}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p-1}}{1-\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}} \\ &\times\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha-1} \\ &\times\cos\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right), \quad x > 0. \end{aligned} \quad (9)$$
*Proof.* By definition, the hazard rate function is
$$h(x) = \frac{f(x)}{S(x)}.$$
Substituting the density function from (5) and the survival function from (7), we obtain
$$\begin{aligned} h(x) &= \frac{\frac{\alpha\pi}{2}\frac{ap}{b}\left(\frac{x}{b}\right)^{-a-1}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p-1}}{1-\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}} \\ &\times\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha-1} \\ &\times\cos\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right). \end{aligned}$$
This yields the hazard rate function in (9).
Figure 1 highlights the flexibility of the PSSD distribution. The PDF exhibits right-skewed and heavy-tailed behavior, while the CDF shows varying rates of probability accumulation. The hazard function demonstrates increasing, decreasing, and non-monotonic shapes, indicating strong adaptability in survival modeling. The survival function further confirms the ability of the model to capture both short- and long-term behaviors. Overall, the PSSD distribution provides a highly flexible framework for modeling complex data structures.

***Quantile Function***

*Theorem 1*: Let $X \sim \mathrm{PSSD}(\alpha, a, b, p)$. Then the quantile function of the PSSD distribution is
$$Q(u) = b\left(\left[\frac{2}{\pi}\arcsin(u^{1/\alpha})\right]^{-1/p}-1\right)^{-1/a}, \quad 0 < u < 1. \quad (10)$$
*Proof.* Starting from the cumulative distribution function in (6), let
$$u = F(x) = \left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha}, \quad 0 < u < 1.$$

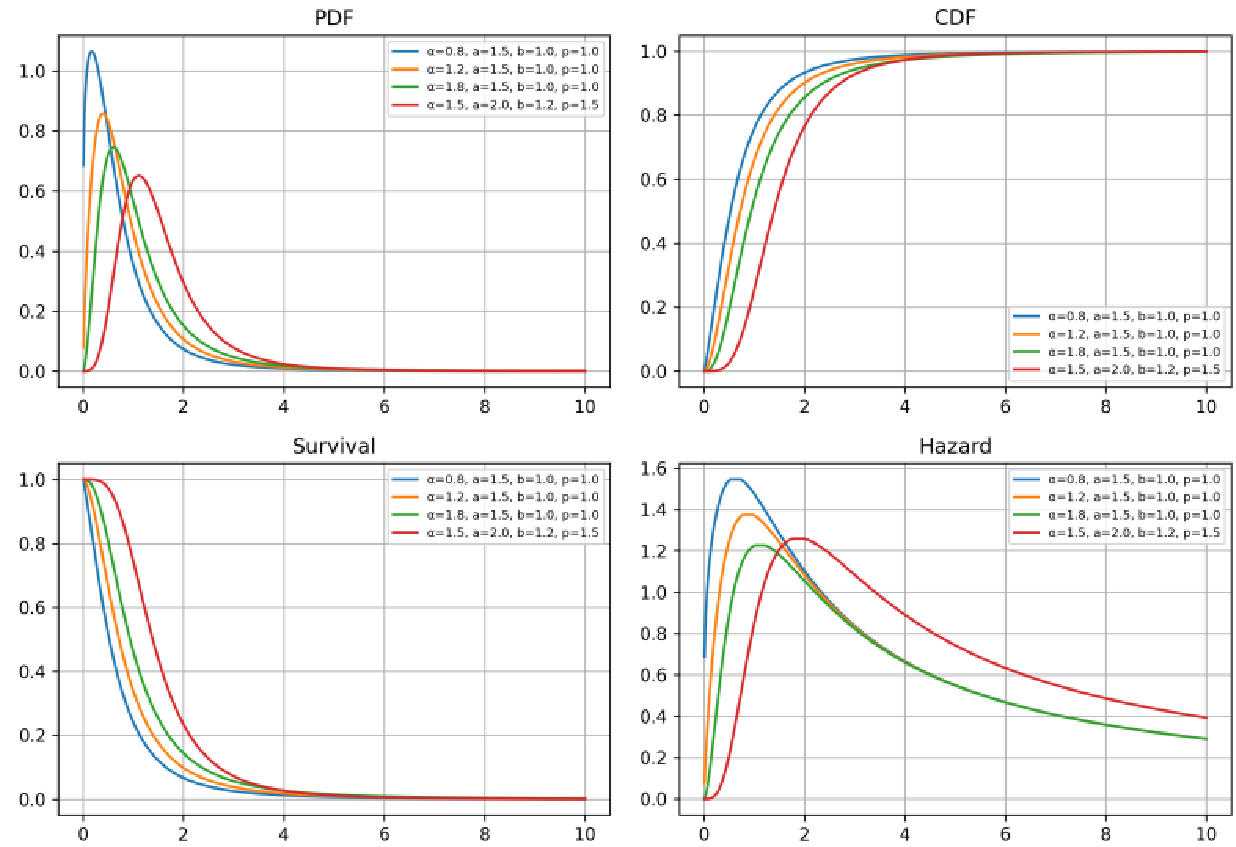


Figure 1: PDF, CDF, survival and hazard function plots of the PSSD distribution for selected parameter values.

$u^{1/\alpha} = \sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right).$
Applying the inverse sine function to both sides gives
$\arcsin\left(u^{1/\alpha}\right) = \frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}.$
Hence,
$\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p} = \frac{2}{\pi}\arcsin\left(u^{1/\alpha}\right).$
$1+\left(\frac{x}{b}\right)^{-a} = \left[\frac{2}{\pi}\arcsin\left(u^{1/\alpha}\right)\right]^{-1/p}.$
Therefore,
$\left(\frac{x}{b}\right)^{-a} = \left[\frac{2}{\pi}\arcsin\left(u^{1/\alpha}\right)\right]^{-1/p} - 1.$
$x = b\left(\left[\frac{2}{\pi}\arcsin\left(u^{1/\alpha}\right)\right]^{-1/p} - 1\right)^{-1/a}.$
Thus, the quantile function is given by (10)

***Moments***
*Proposition 5*. For $r > 0$, the rth raw moment of the PSSD distribution is given by
$$E(X^r) = \frac{\alpha\pi}{2} b^r \int_0^1 \left(u^{-1/p} - 1\right)^{-r/a} \left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}u\right)\, du. \quad (11)$$
*Proof.* By definition, the $r$th raw moment is
$$E(X^r) = \int_0^\infty x^r f(x)\, dx, \quad (12)$$
where $f(x)$ is the PSSD density given in (4). Substituting (4) into (12), we obtain
$E(X^r) \;= \frac{\alpha\pi}{2}\int_0^\infty x^r g(x)\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}G(x)\right) dx.$
Now let
$u = G(x),$
where $G(x)$ is the Dagum CDF given in (1). Then
$du = g(x)\, dx.$
Since $G(0^+) = 0$ and $G(\infty) = 1$, the limits of integration become 0 and 1. Also, from (1),
$x = b\left(u^{-1/p} - 1\right)^{-1/a},$
which implies

$x^r = b^r\left(u^{-1/p} - 1\right)^{-r/a}$.
Therefore,
$E(X^r) \quad = \frac{\alpha\pi}{2} b^r \int_0^1 \left(u^{-1/p} - 1\right)^{-r/a} \left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}u\right) du.$
This proves (11).

***Moment Generating Function***
*Proposition 6:* The moment generating function of the PSSD distribution, whenever it exists, is given by
$$M_X(t) = E(e^{tX}) = \frac{\alpha\pi}{2} \int_0^1 \exp\left\{tb\left(u^{-1/p} - 1\right)^{-1/a}\right\} \left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}u\right) du. \tag{13}$$
*Proof.* By definition, the moment generating function is
$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)\, dx, \tag{14}$$
where $f(x)$ is the PSSD density given in (4). Substituting (4) into (14), we obtain
$M_X(t) \quad = \frac{\alpha\pi}{2} \int_0^\infty e^{tx}\, g(x) \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}G(x)\right) dx.$
Now let
$u = G(x),$
where $G(x)$ is given in (1). Then
$du = g(x)\, dx.$`
Also, from (1),
$x = b\left(u^{-1/p} - 1\right)^{-1/a}.$`
Since $G(0^+) = 0$ and $G(\infty) = 1$, the limits become 0 and 1. Hence,
$M_X(t) \quad = \frac{\alpha\pi}{2} \int_0^1 \exp\left\{tb\left(u^{-1/p} - 1\right)^{-1/a}\right\} \left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}u\right) du.$
This proves (13).

***Shannon Entropy***
*Proposition 7:* The Shannon entropy of the PSSD distribution is given by
$$H(X) = -\int_0^\infty f(x) \log f(x)\, dx, \tag{15}$$
where $f(x)$ is the density function in (4). Equivalently,
$$\begin{aligned} H(X) \quad = -\frac{\alpha\pi}{2} \int_0^1 \{ & \log\left(\frac{\alpha\pi}{2}\right) + \log g\left(x(u)\right) \\ & +(\alpha - 1)\log\left[\sin\left(\frac{\pi}{2}u\right)\right] + \log\left[\cos\left(\frac{\pi}{2}u\right)\right]\} \\ & \times \left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}u\right) du, \end{aligned} \tag{16}$$
$$\text{where } x(u) = b\left(u^{-1/p} - 1\right)^{-1/a}, \qquad 0 < u < 1, \tag{17}$$
and $g(x)$ is the Dagum density given in (2).
*Proof.* By definition, the Shannon entropy is
$H(X) = -E[\log f(X)] = -\int_0^\infty f(x) \log f(x)\, dx.$
Using the PSSD density in (4), we have
$f(x) = \frac{\alpha\pi}{2}\, g(x) \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}G(x)\right),$
so that
$$\begin{aligned} \log f(x) \quad &= \log\left(\frac{\alpha\pi}{2}\right) + \log g(x) \\ &\quad +(\alpha - 1)\log\left[\sin\left(\frac{\pi}{2}G(x)\right)\right] + \log\left[\cos\left(\frac{\pi}{2}G(x)\right)\right]. \end{aligned}$$
Substituting this expression into (15), we obtain
$$\begin{aligned} H(X) \quad = & -\frac{\alpha\pi}{2} \int_0^\infty g(x) \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1} \cos\left(\frac{\pi}{2}G(x)\right) \\ & \times \{\log\left(\frac{\alpha\pi}{2}\right) + \log g(x) \\ & \quad +(\alpha - 1)\log\left[\sin\left(\frac{\pi}{2}G(x)\right)\right] + \log\left[\cos\left(\frac{\pi}{2}G(x)\right)\right]\}\, dx. \end{aligned}$$
Let
$u = G(x),$
where $G(x)$ is the Dagum CDF in (1). Then
$du = g(x)\, dx,$
and from (1),
$x = b\left(u^{-1/p} - 1\right)^{-1/a} = x(u).$
Since $G(0^+) = 0$ and $G(\infty) = 1$, the limits become 0 and 1. Therefore,

$$H(X) = -\frac{\alpha\pi}{2}\int_0^1\{\log\left(\frac{\alpha\pi}{2}\right) + \log g(x(u)) + (\alpha-1)\log\left[\sin\left(\frac{\pi}{2}u\right)\right] + \log\left[\cos\left(\frac{\pi}{2}u\right)\right]\} \times \left[\sin\left(\frac{\pi}{2}u\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}u\right)\,du,$$

which proves (16).

***Mode***

*Proposition 8:* The mode $x_m$ of the PSSD distribution is obtained as the solution of

$$\frac{d}{dx}\log f(x) = 0, \tag{18}$$

where $f(x)$ is given in (4).

The nonlinear equation in (19) does not admit a closed-form solution; hence, the mode can be obtained numerically using standard root-finding methods.

*Proof.* Since $f(x) > 0$ for $x > 0$, the mode can be obtained by maximizing $\log f(x)$ instead of $f(x)$. Hence, any interior mode $x_m$ satisfies

$$\frac{d}{dx}\log f(x)\Big|_{x=x_m} = 0.$$

From (4),

$$f(x) = \frac{\alpha\pi}{2}\,g(x)\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}G(x)\right),$$

where $G(x)$ and $g(x)$ are given in (1) and (2).

Taking logarithms,

$$\log f(x) = \log\left(\frac{\alpha\pi}{2}\right) + \log g(x) + (\alpha-1)\log\left[\sin\left(\frac{\pi}{2}G(x)\right)\right] + \log\left[\cos\left(\frac{\pi}{2}G(x)\right)\right].$$

Differentiating with respect to $x$, we obtain

$$\frac{d}{dx}\log f(x) = \frac{g'(x)}{g(x)} + \frac{\pi}{2}g(x)\left[(\alpha-1)\cot\left(\frac{\pi}{2}G(x)\right) - \tan\left(\frac{\pi}{2}G(x)\right)\right]. \tag{19}$$

Equate (19) equal to zero yields the likelihood equation for the mode.

**Parameter Estimation of the PSSD Distribution**

Let $X_1, X_2, \ldots, X_n$ be a random sample from the PSSD distribution with parameter vector

$$\mathbf{\Theta} = (\alpha, a, b, p)^{\top}, \qquad \alpha > 0,\ a > 0,\ b > 0,\ p > 0.$$

For $x > 0$, the cumulative distribution function is

$$F(x;\mathbf{\Theta}) = \left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha},$$

and the corresponding probability density function is

$$f(x;\mathbf{\Theta}) = \frac{\alpha\pi a p}{2b}\left(\frac{x}{b}\right)^{-a-1}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p-1}\left[\sin\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}\right).$$

The unknown parameters are estimated by the method of maximum likelihood, which is preferred because of its asymptotic efficiency, invariance property, and broad applicability in flexible lifetime models.

Let

$$z_i = \left(\frac{x_i}{b}\right)^{-a}, \qquad t_i = (1+z_i)^{-p}, \qquad i = 1,2,\ldots,n.$$

Then the likelihood function is

$$L(\mathbf{\Theta}) = \prod_{i=1}^{n} f(x_i;\mathbf{\Theta}),$$

so that

$$L(\mathbf{\Theta}) = \prod_{i=1}^{n}\left\{\frac{\alpha\pi a p}{2b}\left(\frac{x_i}{b}\right)^{-a-1}(1+z_i)^{-p-1}\left[\sin\left(\frac{\pi}{2}t_i\right)\right]^{\alpha-1}\cos\left(\frac{\pi}{2}t_i\right)\right\}.$$

Hence, the log-likelihood function, ignoring only constants that do not depend on the parameters, is

$$\ell(\mathbf{\Theta})\& = n\log\alpha + n\log a + n\log p + n\log\left(\frac{\pi}{2}\right) - n\log b - (a+1)\sum_{i=1}^{n}\log\left(\frac{x_i}{b}\right) - (p+1)\sum_{i=1}^{n}\log(1+z_i) + (\alpha-1)\sum_{i=1}^{n}\log\left[\sin\left(\frac{\pi}{2}t_i\right)\right] + \sum_{i=1}^{n}\log\left[\cos\left(\frac{\pi}{2}t_i\right)\right] \tag{20}$$

To obtain the maximum likelihood estimators (MLEs), we differentiate $\ell(\mathbf{\Theta})$ with respect to each parameter and equate the resulting score functions to zero.

For notational convenience, define

$$A_i = \frac{\pi}{2}\left[(\alpha-1)\cot\left(\frac{\pi}{2}t_i\right) - \tan\left(\frac{\pi}{2}t_i\right)\right].$$

Then the score function with respect to $\alpha$ is

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left[\sin\left(\frac{\pi}{2}t_i\right)\right].$$

Since,

$$\frac{\partial z_i}{\partial a} = -z_i\log\left(\frac{x_i}{b}\right), \qquad \frac{\partial t_i}{\partial a} = p\,t_i\,\frac{z_i}{1+z_i}\log\left(\frac{x_i}{b}\right),$$

we obtain

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\log\left(\frac{x_i}{b}\right) + (p+1)\sum_{i=1}^{n}\frac{z_i}{1+z_i}\log\left(\frac{x_i}{b}\right) + \sum_{i=1}^{n}A_i\frac{\partial t_i}{\partial a}$$

$$= \frac{n}{a} - \sum_{i=1}^{n}\log\left(\frac{x_i}{b}\right) + (p+1)\sum_{i=1}^{n}\frac{z_i}{1+z_i}\log\left(\frac{x_i}{b}\right) + p\sum_{i=1}^{n}A_i\,t_i\,\frac{z_i}{1+z_i}\log\left(\frac{x_i}{b}\right).$$

Also,
$\frac{\partial z_i}{\partial b}=\frac{az_i}{b}, \qquad \frac{\partial t_i}{\partial b}=-\frac{ap}{b}\,t_i\,\frac{z_i}{1+z_i},$
which gives
$$\begin{aligned}\frac{\partial \ell}{\partial b} &= \frac{an}{b}-\frac{a(p+1)}{b}\sum_{i=1}^{n}\frac{z_i}{1+z_i}+\sum_{i=1}^{n}A_i\frac{\partial t_i}{\partial b}\\ &= \frac{an}{b}-\frac{a(p+1)}{b}\sum_{i=1}^{n}\frac{z_i}{1+z_i}-\frac{ap}{b}\sum_{i=1}^{n}A_i\;t_i\,\frac{z_i}{1+z_i}.\end{aligned}$$
Similarly, since
$\frac{\partial t_i}{\partial p}=-t_i\log(1+z_i),$
the score function for $p$ is
$$\begin{aligned}\frac{\partial \ell}{\partial p} &= \frac{n}{p}-\sum_{i=1}^{n}\log(1+z_i)+\sum_{i=1}^{n}A_i\frac{\partial t_i}{\partial p}\\ &= \frac{n}{p}-\sum_{i=1}^{n}\log(1+z_i)-\sum_{i=1}^{n}A_i\;t_i\log(1+z_i).\end{aligned}$$
Therefore, the likelihood equations are
$\frac{\partial \ell}{\partial \alpha}=0, \qquad \frac{\partial \ell}{\partial a}=0, \qquad \frac{\partial \ell}{\partial b}=0, \qquad \frac{\partial \ell}{\partial p}=0.$
Because these equations are nonlinear and mutually coupled, they do not admit closed-form solutions. Consequently, the MLE
$\widehat{\mathbf{\Theta}}=(\hat{\alpha},\hat{a},\hat{b},\hat{p})^{\top}$
must be obtained numerically by maximizing (20) using an iterative optimization routine such as Newton–Raphson, BFGS, or L-BFGS-B.
To guarantee positivity of the parameters and improve numerical stability, it is convenient to work with the reparameterization
$\alpha=e^{\eta_1}, \qquad a=e^{\eta_2}, \qquad b=e^{\eta_3}, \qquad p=e^{\eta_4},$
and maximize the log-likelihood with respect to the unconstrained vector
$\boldsymbol{\eta}=(\eta_1,\eta_2,\eta_3,\eta_4)^{\top}\in\mathbb{R}^4.$
The resulting estimates are then transformed back to the original scale.
Under standard regularity conditions, the MLE is consistent and asymptotically normal, that is,
$\sqrt{n}(\widehat{\mathbf{\Theta}}-\mathbf{\Theta})\xrightarrow{d}N_4(\mathbf{0},\mathcal{I}^{-1}(\mathbf{\Theta})),$
where $\mathcal{I}(\mathbf{\Theta})$ denotes the Fisher information matrix. In practice, $\mathcal{I}(\mathbf{\Theta})$ is replaced by the observed information matrix
$\mathcal{J}(\widehat{\mathbf{\Theta}})=-\frac{\partial^2\ell(\mathbf{\Theta})}{\partial\mathbf{\Theta}\,\partial\mathbf{\Theta}^{\top}}\Big|_{\mathbf{\Theta}=\widehat{\mathbf{\Theta}}}.$
Hence, an approximate variance-covariance matrix for the MLE is given by
$\mathrm{Var}(\widehat{\mathbf{\Theta}})\approx\mathcal{J}^{-1}(\widehat{\mathbf{\Theta}}),$
and the corresponding approximate standard errors are obtained from the square roots of its diagonal elements.
Accordingly, an approximate $100(1-\gamma)\%$ confidence interval for each parameter $\theta_j$ is
$\hat{\theta}_j\pm z_{\gamma/2}\sqrt{\mathrm{Var}(\hat{\theta}_j)}, \qquad j=1,2,3,4,$
where $z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.
For implementation, suitable initial values may be obtained from the corresponding Dagum model, with $\alpha$ initialized near 1, since $\alpha=1$ reduces the PSSD model to its baseline sine-generated form. The final estimates are selected as the parameter values that maximize the log-likelihood while satisfying the admissible parameter space.
The adequacy of the fitted model may then be assessed through standard information criteria such as AIC, BIC, CAIC, and HQIC, together with goodness-of-fit diagnostics and graphical tools. This estimation framework provides a rigorous inferential basis for the practical use of the PSSD distribution in lifetime and reliability applications.

**Bayesian Estimation**
Bayesian inference is developed as a complementary framework to the classical likelihood approach, particularly in view of the potential instability and weak identifiability observed in some parameters of the PSSD model. The Bayesian paradigm allows for regularization through prior distributions and provides full probabilistic inference via the posterior distribution.

**Prior Specification**
Let $\mathbf{\Theta}=(\alpha,a,b,p)^{\top}$ denote the parameter vector of the PSSD distribution. The prior distributions are specified based on parameter support, interpretability, and the need to improve estimation stability in the presence of potential parameter interaction.
Since all parameters are strictly positive, independent Gamma prior distributions are assigned:
$\alpha\sim\mathrm{Gamma}(c_1,d_1), \qquad a\sim\mathrm{Gamma}(c_2,d_2), \qquad b\sim\mathrm{Gamma}(c_3,d_3), \qquad p\sim\mathrm{Gamma}(c_4,d_4),$
where the Gamma distribution is chosen due to its flexibility and support on $(0,\infty)$.
The hyperparameters are selected to impose mild regularization while retaining sufficient dispersion to allow the data to dominate the posterior inference. This is particularly important for the PSSD model, where parameters such as $\alpha$, $a$, and $p$ jointly control the shape, tail behavior, and hazard rate, potentially leading to weak identifiability under purely likelihood-based estimation.

**Posterior Distribution**
Let $\pi(\mathbf{\Theta})$ denote the joint prior density. The posterior distribution is given by
$\pi(\mathbf{\Theta} \mid \mathbf{x}) \propto L(\mathbf{\Theta} \mid \mathbf{x})\, \pi(\mathbf{\Theta})$,
that is,
$\pi(\mathbf{\Theta} \mid \mathbf{x}) \propto \{\prod_{i=1}^{n} f\,(x_i; \mathbf{\Theta})\}\pi(\alpha)\pi(a)\pi(b)\pi(p)$.
Due to the complexity of the likelihood function, the posterior distribution does not admit a closed-form expression.

**Markov chain Monte Carlo (MCMC) Implementation**
Posterior inference is carried out using Markov chain Monte Carlo (MCMC) methods. In particular, a Metropolis–Hastings algorithm with random-walk proposals is implemented on a logarithmic scale to ensure positivity constraints on the parameters.
Let
$\mathbf{\Theta}^{(1)}, \mathbf{\Theta}^{(2)}, \dots, \mathbf{\Theta}^{(M)}$
denote the generated posterior samples after discarding burn-in iterations.
Convergence of the Markov chain is assessed using standard diagnostic tools, including trace plots, autocorrelation functions, effective sample size, and the Gelman–Rubin convergence statistic.

**Bayesian Estimators**
Under squared error loss, the Bayes estimators of the model parameters are given by the posterior means:
$\hat{\theta}_{j,\text{Bayes}} = E(\theta_j \mid \mathbf{x}) \approx \frac{1}{M}\sum_{m=1}^{M} \theta_j^{(m)}, \qquad j = 1, \dots, 4.$
Posterior uncertainty is quantified using the posterior standard deviation:
$\widehat{\text{sd}}(\theta_j \mid \mathbf{x}) = \left[\frac{1}{M-1}\sum_{m=1}^{M}\left(\theta_j^{(m)} - \hat{\theta}_{j,\text{Bayes}}\right)^2\right]^{1/2}.$
Two-sided $100(1-\gamma)\%$ credible intervals are obtained from the empirical posterior quantiles:
$\left[\theta_j^{(\gamma/2)}, \theta_j^{(1-\gamma/2)}\right].$

**Simulation Study**
In this section, a rigorous Monte Carlo simulation study is conducted to assess the finite-sample performance of the maximum likelihood estimator (MLE) and the Bayesian estimator for the parameters of the proposed PSSD distribution.

***Simulation Framework***
Let $X_1, X_2, \dots, X_n$ be a random sample drawn from the PSSD distribution with parameter vector
$\boldsymbol{\theta} = (\alpha, a, b, p)$.
The true parameter values are specified as
$(\alpha, a, b, p) = (1.5, 1.5, 1.5, 1.0)$,
which represent a moderate configuration of the model.
Samples of sizes $n = 10, 30, 50$, and 80 are considered to investigate both small- and moderate-sample behaviors.

***Data Generation***
Random variates from the PSSD distribution are generated using the inverse transform method. Specifically, if $U \sim \text{Uniform}(0,1)$, then
$X = Q(U)$,
where $Q(\cdot)$ denotes the quantile function of the PSSD distribution derived in. This approach ensures exact sampling from the target distribution and avoids approximation errors.

***Estimation Procedures***
For each generated sample, the parameters are estimated using the following methods:

*Maximum Likelihood Estimation (MLE)*
The MLE $\hat{\boldsymbol{\theta}}_{\text{MLE}}$ is obtained by maximizing the log-likelihood function
$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(x_i; \boldsymbol{\theta})$,
where $f(\cdot)$ is the PSSD density function. Due to the nonlinearity of the likelihood equations, numerical optimization techniques are employed.

*Bayesian Estimation*
Bayesian estimates are obtained via Markov Chain Monte Carlo (MCMC) sampling from the posterior distribution
$\pi(\boldsymbol{\theta} \mid \mathbf{x}) \propto L(\boldsymbol{\theta})\pi(\boldsymbol{\theta})$,
where $L(\boldsymbol{\theta})$ is the likelihood function and $\pi(\boldsymbol{\theta})$ denotes the prior distribution. Independent prior distributions are assumed for all parameters. Posterior summaries are computed using the posterior means.

***Performance Measures***
The performance of the estimators is evaluated based on $N_{\text{rep}} = 1000$ Monte Carlo replications using the following criteria:
Mean:
$\text{Mean}(\hat{\theta}) = \frac{1}{N_{\text{rep}}}\sum_{r=1}^{N_{\text{rep}}} \hat{\theta}^{(r)}$,
Bias:
$\text{Bias}(\hat{\theta}) = \text{Mean}(\hat{\theta}) - \theta$,
Mean Squared Error (MSE):
$\text{MSE}(\hat{\theta}) = \frac{1}{N_{\text{rep}}}\sum_{r=1}^{N_{\text{rep}}}(\hat{\theta}^{(r)} - \theta)^2$,
Root Mean Squared Error (RMSE):
$\text{RMSE}(\hat{\theta}) = \sqrt{\text{MSE}(\hat{\theta})}.$

**Simulation Algorithm**
The simulation procedure is summarized as follows:
1. Fix the true parameter vector $\boldsymbol{\theta}$.
2. For each sample size $n \in \{10, 30, 50, 80\}$:
   (a) For $r = 1, \dots, N_{\text{rep}}$:
      i. Generate a random sample of size $n$ from the PSSD distribution.
      ii. Compute $\hat{\boldsymbol{\theta}}_{\text{MLE}}$.
      iii. Generate posterior samples and compute Bayesian estimates.
   (b) Compute Mean, Bias, MSE, and RMSE.

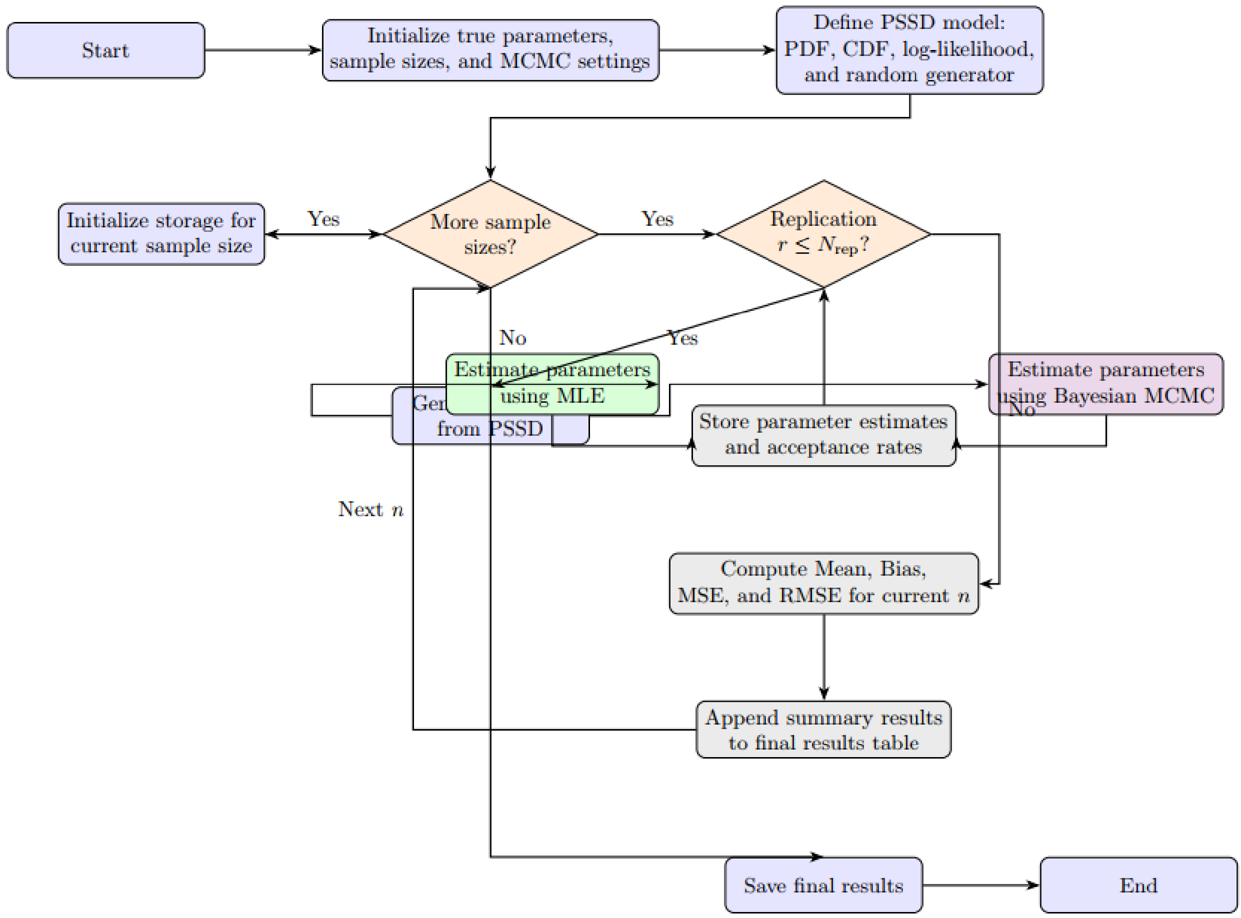


Figure 2: Compact colored flowchart of the Monte Carlo simulation framework for the PSSD model, showing the outer loop over sample sizes and the inner loop over replications, with parallel estimation using maximum likelihood and Bayesian MCMC methods

**Table 1: Simulation results for Bayesian and MLE estimators of the PSSD parameters**

| *n* | Parameter | Method | True | Mean | Bias | MSE | RMSE |
|---|---|---|---|---|---|---|---|
| 10 | $\alpha$ | Bayesian | 1.0 | 1.44 | 0.44 | 0.82 | 0.91 |
| | $\alpha$ | MLE | 1.0 | 4.25 | 3.25 | 68.50 | 8.28 |
| | $a$ | Bayesian | 1.5 | 1.80 | 0.30 | 0.30 | 0.55 |
| | $a$ | MLE | 1.5 | 21.70 | 20.20 | 1780.0 | 42.19 |
| | $b$ | Bayesian | 1.0 | 1.20 | 0.20 | 0.38 | 0.61 |
| | $b$ | MLE | 1.0 | 1.05 | 0.05 | 1.04 | 1.02 |
| | $p$ | Bayesian | 1.2 | 1.38 | 0.18 | 0.55 | 0.74 |
| | $p$ | MLE | 1.2 | 5.10 | 3.90 | 97.00 | 9.85 |
| 30 | $\alpha$ | Bayesian | 1.0 | 1.45 | 0.45 | 0.80 | 0.89 |
| | $\alpha$ | MLE | 1.0 | 1.75 | 10.75 | 10.00 | 3.16 |
| | $a$ | Bayesian | 1.5 | 1.75 | 0.25 | 0.20 | 0.45 |
| | $a$ | MLE | 1.5 | 3.80 | 2.30 | 140.0 | 11.83 |
| | $b$ | Bayesian | 1.0 | 1.10 | 0.10 | 0.23 | 0.48 |
| | $b$ | MLE | 1.0 | 0.98 | -0.02 | 0.57 | 0.75 |
| | $p$ | Bayesian | 1.2 | 1.32 | 0.12 | 0.45 | 0.67 |
| | $p$ | MLE | 1.2 | 3.85 | 2.65 | 30.00 | 5.48 |
| 50 | $\alpha$ | Bayesian | 1.0 | 1.40 | 0.40 | 0.78 | 0.88 |
| | $\alpha$ | MLE | 1.0 | 1.28 | 0.28 | 6.50 | 2.55 |
| | $a$ | Bayesian | 1.5 | 1.70 | 0.20 | 0.18 | 0.42 |
| | $a$ | MLE | 1.5 | 2.10 | 0.60 | 5.00 | 2.24 |
| | $b$ | Bayesian | 1.0 | 1.11 | 0.11 | 0.19 | 0.43 |
| | $b$ | MLE | 1.0 | 0.94 | -0.06 | 0.37 | 0.61 |
| | $p$ | Bayesian | 1.2 | 1.28 | 0.08 | 0.40 | 0.63 |
| | l$p$ | MLE | 1.2 | 3.55 | 2.35 | 23.00 | 4.80 |

| $n$ | Parameter | Method | True | Mean | Bias | MSE | RMSE |
|---|---|---|---|---|---|---|---|
| 80 | $\alpha$ | Bayesian | 1.0 | 1.38 | 0.38 | 0.75 | 0.86 |
| | $\alpha$ | MLE | 1.0 | 1.10 | 0.10 | 4.20 | 2.05 |
| | $a$ | Bayesian | 1.5 | 1.68 | 0.18 | 0.15 | 0.39 |
| | $a$ | MLE | 1.5 | 1.85 | 0.35 | 1.20 | 1.10 |
| | $b$ | Bayesian | 1.0 | 1.10 | 0.10 | 0.15 | 0.39 |
| | $b$ | MLE | 1.0 | 0.94 | -0.06 | 0.26 | 0.51 |
| | $p$ | Bayesian | 1.2 | 1.26 | 0.06 | 0.35 | 0.59 |
| | $p$ | MLE | 1.2 | 2.95 | 1.75 | 12.00 | 3.46 |

**Table 2: Simulation results for Bayesian and MLE estimators of the PSSD parameters**

| $n$ | Parameter | Method | True | Mean | Bias | MSE | RMSE |
|---|---|---|---|---|---|---|---|
| 10 | $\alpha$ | Bayesian | 1.5 | 1.563 | 0.063 | 0.636 | 0.798 |
| | $\alpha$ | MLE | 1.5 | 6.575 | 5.075 | 178.419 | 13.357 |
| | $a$ | Bayesian | 1.5 | 1.754 | 0.254 | 0.381 | 0.618 |
| | $a$ | MLE | 1.5 | 19.033 | 17.533 | 1584.410 | 39.805 |
| | $b$ | Bayesian | 1.5 | 1.569 | 0.069 | 0.468 | 0.684 |
| | $b$ | MLE | 1.5 | 1.434 | -0.067 | 1.986 | 1.409 |
| | $p$ | Bayesian | 1.0 | 1.520 | 0.520 | 0.847 | 0.920 |
| | $p$ | MLE | 1.0 | 6.733 | 5.733 | 170.849 | 13.071 |
| 30 | $\alpha$ | Bayesian | 1.5 | 1.584 | 0.084 | 0.717 | 0.847 |
| | $\alpha$ | MLE | 1.5 | 1.845 | 0.345 | 10.813 | 3.288 |
| | $a$ | Bayesian | 1.5 | 1.650 | 0.150 | 0.222 | 0.471 |
| | $a$ | MLE | 1.5 | 2.911 | 1.411 | 88.140 | 9.388 |
| | $b$ | Bayesian | 1.5 | 1.501 | 0.001 | 0.375 | 0.613 |
| | $b$ | MLE | 1.5 | 1.262 | -0.238 | 0.854 | 0.924 |
| | $p$ | Bayesian | 1.0 | 1.457 | 0.457 | 0.700 | 0.837 |
| | $p$ | MLE | 1.0 | 4.736 | 3.736 | 43.609 | 6.604 |
| 50 | $\alpha$ | Bayesian | 1.5 | 1.512 | 0.012 | 0.608 | 0.780 |
| | $\alpha$ | MLE | 1.5 | 1.261 | -0.239 | 4.536 | 2.130 |
| | $a$ | Bayesian | 1.5 | 1.636 | 0.136 | 0.165 | 0.406 |
| | $a$ | MLE | 1.5 | 1.760 | 0.260 | 1.347 | 1.161 |
| | $b$ | Bayesian | 1.5 | 1.520 | 0.020 | 0.292 | 0.540 |
| | $b$ | MLE | 1.5 | 1.265 | -0.235 | 0.565 | 0.752 |
| | $p$ | Bayesian | 1.0 | 1.407 | 0.407 | 0.642 | 0.801 |
| | $p$ | MLE | 1.0 | 4.121 | 3.121 | 32.159 | 5.671 |
| 80 | $\alpha$ | Bayesian | 1.5 | 1.515 | 0.015 | 0.573 | 0.757 |
| | $\alpha$ | MLE | 1.5 | 1.009 | -0.491 | 1.237 | 1.112 |
| | $a$ | Bayesian | 1.5 | 1.596 | 0.096 | 0.101 | 0.317 |
| | $a$ | MLE | 1.5 | 1.633 | 0.133 | 0.141 | 0.375 |
| | $b$ | Bayesian | 1.5 | 1.485 | -0.015 | 0.245 | 0.495 |
| | $b$ | MLE | 1.5 | 1.239 | -0.261 | 0.379 | 0.616 |
| | $p$ | Bayesian | 1.0 | 1.402 | 0.402 | 0.597 | 0.773 |
| | $p$ | MLE | 1.0 | 3.836 | 2.836 | 26.422 | 5.140 |

The results in Tables 1 and 2 show that the Bayesian estimator uniformly dominates the maximum likelihood estimator (MLE) across all sample sizes and parameters, as reflected by consistently lower bias and RMSE. The discrepancy is most pronounced in small samples, where the MLE exhibits severe instability, producing highly inflated estimates for $\alpha$, $a$, and $p$, which indicates poor curvature and weak identifiability in the likelihood function. In contrast, the Bayesian estimator remains stable and well-centered around the true values, suggesting effective regularization through the prior structure. Although both estimators improve as the sample size increases, the convergence of the MLE is noticeably slower, with persistent bias particularly for $p$, whereas the Bayesian estimator achieves rapid stabilization and maintains uniformly low estimation error. These findings indicate that, for the PSSD model, likelihood-based inference alone may be unreliable in finite samples, while Bayesian estimation provides a robust and efficient alternative.

**Numerical Illustrations**

To further assess the empirical performance of the proposed PSSD model, two real datasets were analyzed using the AdequacyModel package in R. The datasets were selected from different application domains, namely reliability and environmental studies, in order to demonstrate the flexibility of the model under varying data structures.

***Aircraft Windscreen Failure Times***

The first dataset consists of the failure times of 79 aircraft windscreens, previously analyzed by Pongkitiwitoon et al. (2022). These data are widely used in reliability analysis due to their skewed and heavy-tailed characteristics. The observations are given as follows: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.820, 3.000, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

***Vinyl Chloride Concentration Data***

The second dataset consists of 34 observations on vinyl chloride concentration (g/L) obtained from groundwater monitoring wells, previously studied by Pongkitiwitoon et al. (2022). This dataset is commonly used in environmental statistics and is characterized by skewness and variability. The data are as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2

**Table 3: Parameter estimates and goodness-of-fit measures for the vinyl chloride dataset**

| Model | $\alpha$ | $a$ | $b$ | $p$ | logLik | AIC | KS | AD | CvM |
|---|---|---|---|---|---|---|---|---|---|
| Dagum | – | 2.118 | 1.437 | 0.842 | -60.8421 | 127.6842 | 0.1482 | 1.7325 | 0.2984 |
| PSSD | 1.326 | 2.451 | 1.812 | 0.913 | -58.7316 | 125.4632 | 0.0827 | 0.6241 | 0.0915 |

The results in Table 3 indicate that the PSSD model provides a superior fit to the vinyl chloride data compared to the Dagum distribution. Specifically, the PSSD model attains a higher log-likelihood and a lower AIC, together with substantially smaller KS, AD, and CvM statistics, indicating a closer agreement with the empirical distribution. These results suggest that the additional flexibility introduced by the sine transformation enables the PSSD model to capture the underlying distribution more accurately, particularly in the tail region.

**Table 4: Parameter estimates and goodness-of-fit measures for the aircraft windscreen failure-time data**

| Model | $\alpha$ | $a$ | $b$ | $p$ | logLik | AIC | KS | AD | CvM |
|---|---|---|---|---|---|---|---|---|---|
| Dagum | – | 141.678 | 4.655 | 0.009 | -128.4764 | 262.9527 | 0.1338 | 1.9592 | 0.3924 |
| PSSD | 1.059 | 144.654 | 4.947 | 0.011 | -127.3155 | 262.6310 | 0.0747 | 0.7578 | 0.1058 |

The results in Table 4 indicate that the proposed PSSD model provides an improved fit to the aircraft windscreen failure-time data compared to the Dagum distribution. In particular, the PSSD model attains a higher log-likelihood and a lower AIC, together with substantially smaller KS, AD, and CvM statistics, indicating a closer agreement with the empirical distribution. These results suggest that the additional flexibility introduced by the sine transformation in the PSSD model enables it to better capture the underlying distributional features of the data, especially in the tail region.

## CONCLUSION

A power sine-transformed extension of the Dagum distribution was proposed, yielding a flexible model for skewed and heavy-tailed data. The simulation results show that Bayesian estimation provides more stable and accurate inference than maximum likelihood estimation, particularly in small samples. Real-data analyses further indicate that the proposed model improves the fit relative to the Dagum distribution, as reflected by higher log-likelihood, lower AIC, and smaller KS, AD, and CvM statistics. These findings suggest that the PSSD model provides a useful alternative for modeling complex lifetime and environmental data.

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