FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 4, February, 2026, pp 248 – 253
DOI: https://doi.org/10.33003/fjs-2026-1004-4861

HIGHER ORDER GAUSS-LEGENDRE QUADRATURE RUNGE-KUTTA TYPE METHOD FOR SOLVING STIFF AND OSCILLATORY PROBLEMS

[*1]Zainab Adamu, [1]Yakubu Muntaka Lawal and [2]Aliyu Jaafar Bello

[1]Department of Mathematics Sciences, Nigerian Defence Academy, Kaduna, Nigeria.
[2]Academic Programme Department, National Commission for Colleges of Education (NCCE), Abuja, Nigeria.

[*]Corresponding Auhtor's Email: adamuzainab006@gmail.com

ABSTRACT

In this Paper a four (4) stage super convergent implicit Runge-Kutta type methods of order eight (8) have been constructed. This is an improvement of existing lower order four and six Gauss Legendre method for solving Ordinary Differential Equations (ODEs). Legendre polynomials of fourth degree basic function was used to generate the special Gaussian points for the construction of an implicit four points Gauss Legendre Runge-Kutta type methods. Collocation and matrix inversion approach is used to obtain continuous formulas for the method. The continuous formula is evaluated at the special Gaussian points of fourth-degree Legendre polynomials to yield a block discrete scheme which is converted to Runge-Kutta evaluation functions for the integration of Ordinary Differential Equations of ODEs, especially highly stiff and oscillatory problems of first order ODEs which are found in science and engineering models. Experimental problems used, show that the developed methods is A-stable, consistent, more efficient with less error estimate than the lower stage Gauss Legendre Runge-Kutta methods of order 4 and six respectively.

Keywords: Implicit, More efficient, A-stable, Collocation methods, Perturbed Gaussian points, Order and Error constant

INTRODUCTION

Numerical methods for solving first order initial valued problems (IVPs) often fall into two large categories; one is the rule for discrete data and the other is for function of continuous data. A further division can be realized by dividing methods into those that are explicit and those that are implicit. The Gauss-Legendre quadrature is a well-known rule belonging to the latter categories (Chapra and Canale, 2006). According to Moshtaghi and Saadatmandi (2021), the Legendre–Gauss quadrature rule significantly improves on numerically solving of distributed order differential equations.

Recall that the first order Ordinary Differential Equations (ODEs)

$y' = f(t, y) \quad y(t_0) = y_0 \quad f: IR \times IR^m \rightarrow IR^m$ (1)

Over some interval $[a, b]$ where $a < \infty$, $b < \infty$, over a continuous range of the independent variable t (Abdul-Hassan et al., 2024)

The usual numerical methods for solving the IVP above are referred to as discrete variable methods because they discretize the interval $[a, b]$ into sub intervals and generate a sequence of approximate solution for $y(t)$ $i.e.$ $y_1, y_2, \dots, y_n$ at point $t_1, t_2, \dots, t_n$. No attempt is made to approximate the solutions $y(t)$, over a continuous range of the independent variable t. But, only a small class of differential equation possess analytical solution $y(t)$ expressible in terms of known tabulated transcendental functions that satisfy the differential equation, as well as the initial conditions. Hence, for such differential equations that are not solvable analytically, numerical integration is the only way to obtain information about the trajectory.

The Gauss–Legendre methods are family of numerical methods for ordinary differential equations. Gauss–Legendre methods are implicit Runge–Kutta methods. More specifically, they are collocation methods based on the points of Gauss–Legendre quadrature. The Gauss–Legendre method based on $\boldsymbol{r}$ points has order $\boldsymbol{2r}$. In fact, "all Gauss–Legendre methods are A-stable"(Iserles,1996; Butcher and Jackiewicz,1998; Babaei, M, 2024), and as also demonstrated in recent stability analyses of **(**Adeosun et al., 2026).

In the quest to tackle stability properties of numerical method, many researchers over decades have continued to study and developed implicit numerical schemes for solving ODEs of the form (1.0), most of which have generally been said to achieve great accuracy and stability Isserles (1996)**,** Agam and Yahaya (2014) developed order six (6) of such scheme titled "Highly efficient Implicit Runge-Kutta Method" for integrating first order stiff and oscillatory problems. The method is consistent and stable but of lower order and there for not efficient for highly stiff problems found in science and engineering. In this work, a higher order $p = 2r - stage$ direct Gauss Legendre methods of order 8 with only 4 stage-order to solve highly stiff and oscillatory problems was contructed.

MATERIALS AND METHODS

To find the approximate solution $\bar{y}(t)$ of the first order initial value problem of ordinary differential equation.

$y' = f(t, y), \; y(t_0) = y_0, \quad a \leq t \leq \; b$ (2)

By the general r-stage implicit R-K method through collocation method,

Assume an approximate solution to be polynomial of the form

$\bar{y}(t) = \sum_{i=0}^{k-1} q_i(t) y_{n+i} + h \sum_{i=0}^{m-1} p_i(t) f\left(c_i, y(c_i)\right)$ (3)

where k are the number of interpolation points t_{n+i} $(\, i = 0, 1, \dots, k-1)$ and m denote the number of collocation points c_i $(i = 0, 1, \dots, m-1)$ which are the Gaussian points. Here y_n and f are continuous differentiable real N-dimensional vector functions.

Using Onumanyi et al (1994), multi-step collocation process thereby representing the function $q_i(t)$ and $p_i(t)$ in (3) by polynomials of the form:

$$q_i(t) = \sum_{j=0}^{k+m-1} q_{i,j+1} t^j \,, (i = 0,1, \dots, k-1) \quad (4)$$

$$hp_i(t) = \sum_{j=0}^{k+m-1} hp_{i,j+1} t^j \,, (i = 0,1, \dots, m-1) \quad (5)$$

With constant coefficients $q_{i,j+1}$ *and* $hp_{i,j+1}$ to be determined. By putting (4) and (5) into (3) to have

$$\bar{y}(t) = \sum_{j=0}^{k+m-1} \left\{ \sum_{i=0}^{k-1} q_{i,j+1}\, y_{n+i} + \sum_{i=0}^{m-1} hp_{i,j+1}\, f_{n+i} \right\} t^j \quad (6)$$

And let $A = a_j = \left(\sum_{i=0}^{k-1} q_{i,j+1}\, y_{n+i} + \sum_{i=0}^{m-1} hp_{i,j+1}\, f_{n+i} \right) \; a_j \in \mathbb{R}^j, j = 0, 1, \dots, k+m-1$

Such that (6) reduces to a power series of a single variable t in the form

$$P(t) = \sum_{j=0}^{\infty} a_j\, t^j \quad (7)$$

And (7) is used as the basis or trial function to produce an approximate solution to (2) as

$$\bar{y}(t) = \sum_{j=0}^{k+m-1} a_j\, t^j \quad (8)$$

Assuming a power series solution of degree 4 of the form (8) as:

$$\bar{y}(t) = \sum_{j=0}^{4} a_j\, t^j \,,$$

$$\bar{y}'(t) = \sum_{j=1}^{4} j a_j\, t^{j-1} \quad (9)$$

Thus, the Legendre polynomial of degree r= 4, where r is the stage yield:

$$p_4(t) = \frac{1}{8}(3\,5t^4 - 30t^2 + 3)$$

To get the zeros we equate to zero

$$p_4(t) = \frac{1}{8}(3\,5t^4 - 30t^2 + 3) = 0,$$

And the approximate zeros of the fourth-degree r=4 are:

$$t_1 = -\frac{4\sqrt{511}}{105}$$

$$t_2 = -\frac{17}{50}$$

$$t_3 = \frac{17}{50}$$

$$t_4 = \frac{4\sqrt{511}}{105} \quad (10)$$

To get the Gaussian point c_i for the fourth degree, now transform (10) using the transformation formula

T(ti) = ½(1-ti) = C_i (Guo, B., & Wang, Z, 2018) (11)

To get approximate Gaussian points C_i as follows:

$$c_1 = \frac{1}{2} - \frac{2\sqrt{511}}{105}$$

$$c_2 = \frac{33}{100}$$

$$c_3 = \frac{67}{100}$$

$$c_4 = \frac{1}{2} + \frac{2\sqrt{511}}{105} \quad (12)$$

Interpolating at $t = t_n$ and collocating at $t = t_{n+c_i}$ $(i = 1, 2, 3, 4)$ in (9), yields the system of simultaneous equations.

$$a_0 + a_1 t_n + a_2 t_n^2 + a_3 t_n^3 + a_4 t_n^4 = \bar{y}_n$$

$$a_1 + 2a_2 t_{n+c_1} + 3a_3 t_{n+c_1}^2 + 4a_4 t_{n+c_1}^3 = f_{n+c_1}$$

$$a_1 + 2a_2 t_{n+c_2} + 3a_3 t_{n+c_2}^2 + 4a_4 t_{n+c_2}^3 = f_{n+c_2}$$

$$a_1 + 2a_2 t_{n+c_3} + 3a_3 t_{n+c_3}^2 + 4a_4 t_{n+c_3}^3 = f_{n+c_3}$$

$$a_1 + 2a_2 t_{n+c_4} + 3a_3 t_{n+c_4}^2 + 4a_4 t_{n+c_4}^3 = f_{n+c_4} \quad (13)$$

Where a_j's are to be determined. (13) can be written in matrix form as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 \\ 0 & 1 & 2t_{n+c_1} & 3t_{n+c_1}^2 & 4t_{n+c_1}^3 \\ 0 & 1 & 2t_{n+c_2} & 3t_{n+c_3}^2 & 4t_{n+c_2}^3 \\ 0 & 1 & 2t_{n+c_3} & 3t_{n+c_3}^2 & 4t_{n+c_3}^3 \\ 0 & 1 & 2t_{n+4} & 3t_{n+c_4}^2 & 4t_{n+c_4}^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \bar{y}_n \\ f_{n+c_1} \\ f_{n+c_2} \\ f_{n+c_3} \\ f_{n+c_4} \end{pmatrix} \quad (14)$$

That is

$BA = Y,$

where

$$B=\begin{pmatrix}1 & t_n & t_n^2 & t_n^3 & t_n^4\\ 0 & 1 & 2t_{n+c_1} & 3t_{n+c_1}^2 & 4t_{n+c_1}^3\\ 0 & 1 & 2t_{n+c_2} & 3t_{n+c_3}^2 & 4t_{n+c_2}^3\\ 0 & 1 & 2t_{n+c_3} & 3t_{n+c_3}^2 & 4t_{n+c_3}^3\\ 0 & 1 & 2t_{n+4} & 3t_{n+c_4}^2 & 4t_{n+c_4}^3\end{pmatrix} \quad (15)$$

$$A=(a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4)^T \quad (16)$$

$$Y=(\bar{y}_n \quad f_{n+c_1} \quad f_{n+c_2} \quad f_{n+c_3} \quad f_{n+c_4})^T \quad (17)$$

Using the maple software for the inversion of matrix B in (15) to obtain a continuous scheme which is evaluated at $c_1=\frac{1}{2}-\frac{2\sqrt{511}}{105},\ c_2=\frac{33}{100},\ c_3=\frac{67}{100},\ c_4=\frac{1}{2}+\frac{2\sqrt{511}}{105}$ and a block discrete scheme is obtained as:

$$y_{n+c_1}=y_n+\left(-\frac{2963}{10364096160}\sqrt{511}\right)hf_{n+c_1}+\left(\frac{30291425}{211186206}-\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_2}+\left(\frac{38573875}{211186206}-\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_3}+\left(-\frac{290479547\sqrt{511}}{72548673120}+\frac{34293}{394372}\right)hf_{n+c_4}$$

$$y_{n+c_2}=y_n+\left(\frac{43659}{448150}+\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_1}+\frac{19874217}{121896800}hf_{n+c_2}-\frac{3398769}{121896800}hf_{n+c_3}+\left(\frac{43659}{448150}-\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_4}$$

$$y_{n+c_3}=y_n+\left(\frac{188538}{2464825}+\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_1}+\frac{474626459}{1340864800}hf_{n+c_2}+\frac{218623613}{1340864800}hf_{n+c_3}+\left(\frac{188538}{2464825}-\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_4}$$

$$y_{n+c_4}=y_n+\left(\frac{290479547\sqrt{511}}{72548673120}+\frac{34293}{394372}\right)hf_{n+c_1}+\left(\frac{30291425}{211186206}+\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_2}+\left(\frac{38573875}{211186206}+\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_3}+\left(\frac{2963\sqrt{511}}{10364096160}+\frac{34293}{394372}\right)hf_{n+c_4} \quad (18)$$

The discrete schemes (18) must satisfy the first order IVP (2) in order to be converted into Runge-Kutta type formula, thus;

$$y'_{n+c_1}=f\left(x_{n+c_1},y_{n+c_1}\right)$$

$$y'_{n+c_1}=f\left\{x_{n+c_1},y_n+\left(-\frac{2963\sqrt{511}}{10364096160}+\frac{34293}{394372}\right)hf_{n+c_1}+\left(\frac{30291425}{211186206}-\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_2}+\left(\frac{38573875}{211186206}-\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_3}+\left(-\frac{290479547\sqrt{511}}{72548673120}+\frac{34293}{394372}\right)hf_{n+c_4}\right\}$$

$$y'_{n+c_2}=f\left(x_{n+c_2},y_{n+c_2}\right)$$

$$y'_{n+c_2}=f\left\{x_{n+c_2},\ y_n+\left(\frac{43659}{448150}+\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_1}+\frac{19874217}{121896800}hf_{n+c_2}-\frac{3398769}{121896800}hf_{n+c_3}+\left(\frac{43659}{448150}-\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_4}\right\}$$

$$y'_{n+c_3}=f\left(x_{n+c_3},y_{n+c_3}\right)$$

$$y'_{n+c_3}=f\left\{\begin{matrix}x_{n+c_3},\ y_n+\left(\frac{188538}{2464825}+\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_1}+\frac{474626459}{1340864800}hf_{n+c_2}\\ +\frac{218623613}{1340864800}hf_{n+c_3}+\left(\frac{188538}{2464825}-\frac{83993679\sqrt{511}}{20937568000}\right)hf_{n+c_4}\end{matrix}\right\}$$

$$y'_{n+c_4}=f\left(x_{n+c_4},y_{n+c_4}\right)$$

$$y_{n+c_4}=f\left\{x_{n+c_4},\ y_n+\left(\frac{290479547\sqrt{511}}{72548673120}+\frac{34293}{394372}\right)hf_{n+c_1}+\left(\frac{30291425}{211186206}+\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_2}+\left(\frac{38573875}{211186206}+\frac{46720\sqrt{511}}{6211359}\right)hf_{n+c_3}+\left(\frac{2963\sqrt{511}}{10364096160}+\frac{34293}{394372}\right)hf_{n+c_4}\right\} \quad (19)$$

Let:

$$y'_{n+c_1}=f\left(x_{n+c_1},y_{n+c_1}\right)=f_{n+c_1}=K_1 \quad (20)$$

$$y'_{n+c_2}=f\left(x_{n+c_2},y_{n+c_2}\right)=f_{n+c_2}=K_2 \quad (21)$$

$$y'_{n+c_3}=f\left(x_{n+c_3},y_{n+c_3}\right)=f_{n+c_3}=K_3 \quad (22)$$

$$y'_{n+c_4}=f\left(x_{n+c_4},y_{n+c_4}\right)=f_{n+c_4}=K_4 \quad (23)$$

The function evaluations are obtained as follows:

$$k_1=f\left\{x_n+\left(\frac{1}{2}-\frac{2\sqrt{511}}{105}\right)h,\quad y_n+\left(-\frac{2963\sqrt{511}}{10364096160}+\frac{34293}{394372}\right)hK_1+\left(\frac{30291425}{211186206}-\frac{46720\sqrt{511}}{6211359}\right)hK_2+\left(\frac{38573875}{211186206}-\frac{46720\sqrt{511}}{6211359}\right)hK_3+\left(-\frac{290479547\sqrt{511}}{72548673120}+\frac{34293}{394372}\right)hK_4\right\} \quad (24)$$

$$k_2=f\left\{x_n+\frac{33}{100}h,\ y_n+\left(\frac{43659}{448150}+\frac{83993679\sqrt{511}}{20937568000}\right)hK_1+\frac{19874217}{121896800}hK_2-\frac{3398769}{121896800}hK_3+\left(\frac{43659}{448150}-\frac{83993679\sqrt{511}}{20937568000}\right)hK_4\right\} \quad (25)$$

$$k_3=f\left\{x_n+\frac{67}{100}h,\ y_n+\left(\frac{188538}{2464825}+\frac{83993679\sqrt{511}}{20937568000}\right)hK_1+\frac{474626459}{1340864800}hK_2+\frac{218623613}{1340864800}hK_3+\left(\frac{188538}{2464825}-\frac{83993679\sqrt{511}}{20937568000}\right)hK_4\right\} \quad (26)$$

$k_4 = f\ \{x_n + \left(\frac{1}{2} + \frac{2\sqrt{511}}{105}\right)h,\ \ y_n + \left(\frac{290479547\sqrt{511}}{72548673120} + \frac{34293}{394372}\right)hK_1 + \left(\frac{30291425}{211186206} + \frac{46720\sqrt{511}}{6211359}\right)hK_2\ +$
$\left(\frac{38573875}{211186206} + \frac{46720\sqrt{511}}{6211359}\right)hK_3 + \left(\frac{2963\sqrt{511}}{10364096160} + \frac{34293}{394372}\right)hK_4\}$ (27)

And evaluating the continuous schemes at $t = t_n + h,$ the four (4) stage Gauss-Legendre Runge Kutta type (GLRKM4) is obtained as:

$y_{n+1} = y_n + \frac{34293}{197186}hf_{n+c_1} + \frac{32150}{98593}hf_{n+c_2} + \frac{32150}{98593}hf_{n+c_3} + \frac{34293}{197186}hf_{n+c_4}$ (28)

With weight $w = (w_1, w_2, w_3, w_4)$, where:

$w = \left(\frac{34293}{197186}, \frac{32150}{98593}, \frac{32150}{98593}, \frac{34293}{197186}\right)$

Where ki, i= 1,2,3,4 in equation (24) to (27).

Which is specifically written as

$y_{n+1} = y_n + h\sum_{i=1}^{4} w_i K_i\ = y_n + \frac{34293}{197186}h(K_1 + K_4) + \frac{32150}{98593}h(K_2 + K_3)$ (29)

Analysis of the Method

Consistency

The Runge Kutta type method is consistent since

Implicit Runge-Kutta Method that is developed based on Gaussian node points has order p= 2r where r is the stage.

Thus the order of constructed method is P = 2×4= 8.

$$\sum_{i=1}^{4} w_i = 1\,, \sum_{i=a_{ij}}^{4} a_{ij} = C_j\ \ (Agam\ and\ yahaya, 2014)$$

Stability

The stability of the method is investigated by considering the linear test equation.

$y' = \alpha y,\ \alpha \in \not\subset$

Putting $Z = \alpha h,\ h \in (0,1)$, the stability function is $R(Z)$

$R(Z) = I + Zw^T(I - ZA)$

Where I is the identity matrix, $w = (w_1, w_2, w_3, w_4)$ and A is the Runge Kutta matrix of coefficients in the butcher's table (16).

Order and Error Constant of the Scheme

$y_{n+1} = y_n + h\sum_{i=1}^{4} w_i K_i = y_n + h\sum_{i=1}^{4} w_i f\left(t_{n+c_i}, y_{n+c_i}\right)\ \ is\ the$ Approximate solution

And the exact solution is $y(t_n + h)$

The Error Constant $=\ y(t_n + h) -\ y_{n+1} =\ y(t_n + h) - y_n - h\sum_{i=1}^{4} w_i y'_{n+c_i}$

Where $C_1 = \frac{1}{2} - \frac{2\sqrt{511}}{105}, C_2 = \frac{33}{100}, C_3 = \frac{67}{100}, C_4 = \frac{1}{2} + \frac{2\sqrt{511}}{105}$) are the Gaussian point from the fourth (4th) degree Legendre polynomials and the $w = (w_1 = \frac{34293}{197186}, w_2 =, \frac{32150}{98593}, w_3 =, \frac{32150}{98593}, w_4 = \frac{34293}{197186}$) are the weight.

And the Tylor's series expansion for each term follows as:

$y(t_n + h) -\ h\sum_{i=1}^{4} w_i y'_{n+c_i} = [y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \frac{h^4}{4!}y_n^{(4)} + \frac{h^5}{5!}y_n^{(5)} + \frac{h^6}{6!}y_n^{(6)} + \frac{h^6}{7!}y_n^{(7)} + \frac{h^7}{8!}y_n^{(8)} +$
$\frac{h^9}{9!}y_n^{(9)}\ ] - y_n\ - w_1[hy'_n + c_1h^2y''_n + c_1^2\frac{h^3}{2!}y'''_n + c_1^3\frac{h^4}{3!}y_n^{(4)} + c_1^4\frac{h^5}{4!}y_n^{(5)} + c_1^5\frac{h^6}{5!}y_n^{(6)} + c_1^6\frac{h^7}{6!}y_n^{(7)}\ + c_1^7\frac{h^8}{7!}y_n^{(8)} +$
$c_1^8\frac{h^9}{8!}y_n^{(9)}$]- $w_2[hy'_n + c_2h^2y''_n + c_2^2\frac{h^3}{2!}y'''_n + c_2^3\frac{h^4}{3!}y_n^{(4)} + c_2^4\frac{h^5}{4!}y_n^{(5)} + c_2^5\frac{h^6}{5!}y_n^{(6)} + c_2^6\frac{h^7}{6!}y_n^{(7)} + c_2^7\frac{h^8}{7!}y_n^{(8)} + c_2^8\frac{h^9}{8!}y^{(9)}$]-
$w_3[hy'_n + c_3h^2y''_n + c_3^2\frac{h^3}{2!}y'''_n + c_3^3\frac{h^4}{3!}y_n^{(4)} + c_3^4\frac{h^5}{4!}y_n^{(5)} + c_3^5\frac{h^6}{5!}y_n^{(6)} + c_3^6\frac{h^7}{6!}y_n^{(7)} + c_3^7\frac{h^8}{7!}y_n^{(8)} + c_3^8\frac{h^9}{8!}y^{(9)}]$–$w_4\left[hy'_n +\right.$
$\left.c_4h^2y''_n + c_4^2\frac{h^3}{2!}y'''_n\ \ + c_4^3\frac{h^4}{3!}y_n^{(4)} + c_4^4\frac{h^5}{4!}y_n^{(5)} + c_4^5\frac{h^6}{5!}y_n^{(6)} + c_4^6\frac{h^7}{6!}y_n^{(7)}\ \ + c_4^7\frac{h^8}{7!}y_n{}^{(8)} + c_4^8\frac{h^9}{8!}y^{(9)}\right] +$
(30)

The R-K solution agrees with the Taylor's series expansion (30) up to term in h^8 i.e $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$, the truncation error at c_9 is $O(h^9)$ and the Error constant term is $\frac{1617363585350969}{3024568512000000000000000}$.

Thus, the four stage Gauss-Legendre Runge-Kutta method (GLRKM4) is of order 8.

RESULTS AND DISCUSSION

This paper used linear equation and stiff problems to compare our computational solutions with exact and similar existing method of lower order in order to show the efficiency of the new methods so derived.

Problem 1

$y'(t) = 10y(t) - 6y2\ (t),\ y(0) = 1,\ h = 0.2$

Analytical solution: $y(t) = 5/(3 + 2e\char`^(-10t))$

Problem 2

$$y' = -20t + 20\sin(t), \qquad y(0) = \frac{1}{10}, \qquad h = 0.$$

Analytical solution: $y(t) = -\frac{20}{401}\cos(t) + \frac{400}{401}\sin(t) + \frac{601}{4010}e^{-20t}$.

Problem 3

$y' = -8y + 8t + 1$

Analytical solution: $y(t) = t - 2e^{-8t}$

The approximate solution and error constant comparison are shown in Tables 1, 2 and 3 respectively

Discussion

Looking at the results from the numerical problems solved in the tables of Error (1- 3) it can be seen clearly that the error decreases continuously as against the newly method in solving the stiffs and oscillatory problems of ODEs. The Four (4) point Gauss-Legendre quadrature Runge-Kutta methods has advance better in solving stiff and Oscillatory Ordinary Differential Equations compared with the problems solved by the lower order Six (6) by Agam and Yahaya (2014). The new method has order eight (8). The new method developed converges better and faster to the exact solutions than the previous method of order six.

Table 1: Comparison of Numerical Solution of Problem 1

t	$y(t)$	GLRKM 4 (y_{n+1})	Abs Error GLRK3 (Butcher 1998)	Abs Error GLRKM3 (Agam andYahaya, 2014)	Abs Error of new method GLKM 4
0.2	1.52873849517394	1.52873976063222	$2.8010E-5$	$2.7905\text{E}-5$	$1.2655E-6$
0.4	1.64656145004854	1.64656148856071	$1.4641E-5$	$1.4611\text{E}-5$	$3.8512E-8$
0.6	1.66391704134132	1.66391699191142	$5.8562E-6$	$5.8423E-6$	$4.9430E-8$
0.8	1.66629401375419	1.66629399886806	$1.3404E-6$	$1.3371E-6$	$1.4886E-8$
1.0	1.66661622382700	1.66661622069037	$2.5573E-7$	$2.5509E-7$	$3.13663E-9$

Table 2: Comparison of Numerical Solution of Problem 2

t	$y(t)$	GLRKM 4 (y_{n+1})	Abs Error GLRK3 (Butcher 1998)	Abs Error GLRKM3 (Agam andYahaya, 2014)	Abs Error of new method GLKM 4
0.1	0.0702417303633198	0.0702421820987267	2.9997E-5	$2.9919E-5$	4.52E-7
0.2	0.152037832065867	0.152037954334925	8.1141E-6	$8.0931E-6$	1.22E-7
0.3	0.247507047101849	0.247507071919139	1.6472E-6	$1.6429E-6$	2.48E-8
0.4	0.342559297624288	0.342559302098593	2.9877E-7	$2.9800E-7$	4.47-E-9
0.5	0.434467064188660	0.434467064941735	5.2782E-8	$5.2647E-8$	7.53-10

Table 3: Comparison of Numerical Solution of Problem 3

t	$y(t)$	GLRKM 4 (y_{n+1})	Error GLRK3 (Butcher,1998)	Abs Error GLRKM3 (Agam and Yahaya,2014)	Abs Error of new GLRK4
0.1	0.99865792823444	0.99865793154319	$1.9166E-6$	$1.8866\text{E}-6$	$3.3087E-9$
0.2	0.60379303598931	0.603793038962742	$1.7223E-6$	$1.6954E-6$	$2.9734E-9$
0.3	0.481435906578825	0.481435908582898	1.1608E-6	$1.1427E-6$	$2.0040E-9$
0.4	0.48152440795673	0.481524409157384	$6.9548\text{E}-7$	$6.8460E-7$	$1.2006E-9$
0.5	0.5366127777746	0.536631278451828	$3.9062E-7$	$3.8451E-7$	$6.7436E-10$

CONCLUSION

This Paper was able to prove that chosen a higher degree Legendre polynomial to develop an implicit Gauss- Legendre Runge Kutta type method will give a better result, and a super convergent solution for the first order IVP. The work developed a four (4) points Gauss-Legendre Runge Kutta methods of order 8 (order is 2r). The new implicit method of order 8 is tested using experimental problems and found to be highly efficient, A-stable with lesser errors than the existing Gauss Legendre method of order 6, the new method is easy to implement. Furthermore, the Abs-Error of the fourth degree GLRKM method keep reducing respectively as shown in numerical example tables, when compared with lower degree method. Hence, the new fourth degree method converges better.

REFERENCES

Abdul-Hassan, N. Y., Kadum, Z. J., & Ali, A. H. (2024). An efficient third-order scheme based on Runge–Kutta and Taylor series expansion for solving initial value problems. *Algorithms*, *17*(3), 123.

Adeosun, A. T., Mohd Kasim, A. R., Akolade, M. T., & Tijani, Y. O. (2026). Gauss–Legendre weighted residual method on the stability of Eyring–Powell hybrid nanofluid flow past a shrinking surface with slip conditions. *International Journal of Numerical Methods for Heat & Fluid Flow*, *36*(2), 836-857.

Agam, S.A., and Yahaya, Y.A. (2014). A new three stage implicit Runge-Kutta type method with error estimation for first order ordinary differential equations. *Sci. Technol2015,1(1), 19-24.*

Agam, S.A (2014), Singly and multiply implicit Runge- kutta Type methods for solutions of first, second and third order Ordinary Differential Equations, Ph.D. thesis (unpublished).

Babaei, M. Optimized Gauss–Legendre–Hermite 2-point (O-GLH-2P) method for nonlinear time-history analysis of structures. *Meccanica* **59**, 305–332 (2024). https://doi.org/10.1007/s11012-023-01752-4

Brugnano, L., Iavernaro, F., & Susca, T. (2010). Numerical comparisons between Gauss-Legendre methods and Hamiltonian BVMs defined over Gauss points. *arXiv preprint arXiv:1002.2727*.

Butcher, J.C. (1964). Implicit Runge-Kutta processes. *Math Comp.*18:50-64.

Butcher, J.C. (1966). Convergence of Numerical Solution of Ordinary Differential Equation. *Mathematics of Computation*, 1-10.

Butcher, J.C and Jackiewicz (1997) Implementation of diagonally implicit multi stage Integration methods of ODEs. SIAM J. Numer. Anal 34:2119-2141.

Butcher, J.C. (2008). *Numerical methods for ordinary Differential Equations*, second Edition. John Wiley & Sons Ltd.

Chapra, S.C., & Canale, R.P. (2006), (5th edition). *Numerical Methods for Engineers*. New York, NY, USA: McGraw-Hill.

Guo, B., & Wang, Z. (2018). Legendre-Gauss collocation methods for ordinary differential equations. *Springer science-Journal of scientific computing*.

Hairer, E. and Wanner, G. (1996). *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problem.* New York: Springer-Verlag Berlin Heidelberg GmbH.

Iserles, A. *(1996). A First Course in the Numerical Analysis of Differential Equations. Cambridge:* Cambridge University Press. Retrieved from https://en,wikipedia.org/wiki/Gauss-Legendre_method. (08/05/2023)

Kuntzmann, J. (1961). Neure Entwickhengsweisen der methoden Von Runge und kutta. *Z. Angew Math Mech. 41: T28-T31.*

Lie, I., and Norsett, S.P. (1989). Super Convergence for multistep collocation methods. *Journal of mathematics computer, 52:65-79.*

Moshtaghi, N., & Saadatmandi, A. (2021). Polynomial–Sinc collocation method combined with the Legendre–Gauss quadrature rule for numerical solution of distributed order fractional differential equations. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 115(2), 47.

Onumanyi, P., Awoyemi, D.O., & Sirisena, U.W. (1994). New Linear Multi-Step with Continuous coefficient for first order initial value problems. *Journal of Mathematical Society of Nigeria*, 13:37-51.

Sasser, E. J. (2005). *History of ordinary differential equations. The first hundred years*. Cincinnati: Dover publications, Inc.

Teschl, G. (2012. Ordinary Differential Equations and Dynamical Systems. *American Mathematical Society. ISBN 978-0-8218-8328-0.*