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Transfer of Orbital Integrals

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ABSTRACT

Consider real reductive groups G and G' such that G and G' are inner forms with the same quasi-split group. Langlands has shown that there is a correspondence from the set of regular points in G to that of G'. The main theorem of this paper permits the transfer of stable orbital integrals from G to G', and as such a correspondence between the functions of the Schwartz space C(G) and the functions of the Schwartz space C(G') is established.

Keywords: Stable Orbital Integrals, Real Reductive Lie Groups, Quasi-Split Groups, Inner Forms, Endoscopic Groups, Schwartz Space

INTRODUCTION

The work of R.P. Langlands (1988) gives birth to a new dimension of re-search called Langland's functoriality or simply functoriality or Langland's program. Like Harish-Chandra's theory that states that whatever it is true for the real reductive groups is also true for the *p*-adic reductive Lie group, functionality is a set of conjectures relating algebraic number theory to au-tomorphic forms and representation theory of algebraic groups over local fields and adeles. Orbital integrals or invariant integrals are simply integrals over the conjugacy classes through a class or central function on Lie groups. They proved to be of very important in Harmonic analysis. Bassey and ibitowa(2015) gave a detailed study of the properties orbital integrals high-lighted by Harish-Chandra in his works and its counterpart weighted orbital integrals established by J. Arthur. Grouy(2018) found a function through its orbital integrals on solvable Lorentzian symmetric spaces by considering the Laplace-Beltrami operator on the G- homogeneous spaces for the connected , simply connected Lie groups G. In Miglioli(2025), a new method is derived a piecewise polynomial measures also known as Duistermaat-Heckman type measures from orbital integrals on a compact connected semisimple Lie group G through a series of operations on finite dimensional spaces of polynomials associated with the apolar inner product.

In this paper, we present some preliminary definitions and results in sec-tion 2. The main result is presented in section 3.

MATERIALS AND METHODS

Preliminaries Algebraic Groups

In this subsection, we present some facts about linear algebraic groups which are necessary for the rest of this paper. We let K be an algebraically closed field of an arbitrary characteristic. Thus, the set $K \times K \times \times K = K^n$ is referred to as an affine n-space, which we denote by A^n. All definitions and results presented in this sub-section are taken from Langlands (1988).

Definition 1

i. An affine variety is nothing but the set of common zeros in A^n of a finite set of polynomials.
ii. Let G be a variety possessing the structure of a group. The morphisms of varieties $\mu : G \times G \longrightarrow G$ defined by $xy = \mu(x, y)$ and $i : G \longrightarrow G$ defined by $x^{-1} = i(x)$ make G to become an algebraic group.

It may be remarked that

i. Only the group possessing affine varieties are referred to as " algebraic groups", and
ii. For an algebraic group G, one reaffirms that it is only possible for an irreducible component of G to traverse through the unity e. Then one may take G^0 as the unique irreducible component of e, as such G^0 is called the identity component of G.

This proposition presents the fact that the irreducible component of the identity G^0 of an algebraic group G is a normal subgroup and is contained in any closed subgroup of G.

Proposition 1 Let G to be an algebraic group. Then

i. G^0 is a normal subgroup with finite index in G, whose cosets are the irreducible and connected components of G.
ii. Any closed subgroup with finite index in G contains G^0.

One may remark that an algebraic group is connected whenever $G^0 = G$. Every algebraic group G can be obtained as the product of two open subsets of it as presented below.

Lemma 1 Given an algebraic group G with two dense open subsets U, V, one has $G = UV$.

We also note that any closed subgroup of an algebraic group is also an algebraic group.

Definition 2 For any topological space (X, τ) with a subset $A \subset X$, one may define the following:

i. A is said to be locally closed whenever it is the intersection of open set and a closed set.
ii. A constructible set is a finite union of many locally closed sets.

It is necessary to note that:

i. A constructible set A contains a dense open subset of its closure, and
ii. A constructible set A comprises the smallest collection of subsets of X which contains all closed and open sets which are closed with respect to the Boolean operations.

Here it is seen that the closure of a subgroup of an algebraic group G is also a subgroup.

Proposition 2 Consider H as a subgroup of an algebraic group G and Let $\bar{H}$ be its closure. Then

i. $\bar{H}$ is also a subgroup of G.
ii. Given that H is constructible, then $\bar{H} = H$.

Proposition 2 has the following corollary, which specifies a condition for the product of two closed subgroups of an algebraic group to be closed.

Corollary 1 Let B, A be closed subgroups of G, an algebraic group.
Suppose B normalizes A, then AB is a closed subgroup of G.

Definition 3 Let $O_X(U)$ and $O_Y(V)$ be sheaves over open sets U and V respectively. Then a mapping $\varphi: X \to Y$ (varieties) is said to be a morphism if

i. φ is continuous
ii. If $V \subset Y$ is open and $\varphi^{-1}(V) = U$, then $f \circ \varphi \in O_X(U)$ for $f \in O_Y(V)$.

Definition 4

i. Any morphism of algebraic groups is a group homomorphism $\varphi : G \to G'$ that is also a morphism of varieties.
ii. Two algebraic groups G' and G are said to be isomorphic provided there is an isomorphism of varieties denoted by $\varphi: G \to G'$ that is an isomorphism of groups. Hence, an automorphism of G is automatically an isomorphism of G onto G.

Let it be remarked here that the additive group G_a is also the affine line A' whose group law is $x + y = \mu(x, y)$ with $-x = i(x)$, $0 = e$. The multiplicative group G is the affine open subset $K^\star \subset A'$ whose group law is $xy = \mu(x, y)$ with $x^{-1} = i(x)$ and $1 = e$. Both G_a and G_m are irreducible.

Definition 5 Let G be a linear algebraic group. Then

i. Its identity component which is actually the biggest normal solvable connected subgroup of G is called the radical of G, denoted by $R(G)$.
ii. The subgroup of $R(G)$ having all its unipotent elements in G is called the unipotent radical of G, denoted by $R_u(G)$. The unipotent radical $R_u(G)$ is actually the maximal normal connected unipotent subgroup of G.
iii. The linear algebraic group G is said to be semi-simple provided $R(G)$ is trivial and $G \not\models e$ is connected.
iv. The linear algebraic group G called a reductive linear algebraic group provided $R_u(G)$ is trivial and $G \not\models e$ is connected.

Definition 6 Any torus T defined over K is called a K-torus. Also, a morphism of any algebraic groups denoted by $\chi: G \to G_m$ will be referred to as the character of the algebraic group. Then, we denote by $X(G)$ the set of all characters of G. In the same way, the set of all characters of T is denoted by $X(T)$.

Definition 7 Let K $[T_1, T_2, T_n]$ be a polynomial ring in a finite num-ber of indeterminates $T_1, T_2,, T_n$ over a field K and X and Y be closed subsets of A^n. When $Y \subset A^m$ and $X \subset A^n$ are defined over K, then a morphism $\varphi: X \to Y$ is a K-morphism (i.e., defined over K) whenever the coordinate functions all lie in K $[T_1, T_2, T_n]$. We denote by $X(T)_K$ the subgroup of $X(T)$ consisting of characters $\chi: T \to G_m$ which are K-morphisms.

Definition 8 Let $K[T] = K[T_1, T_2, , T_n]$, T is then called K-split when-ever K is isomorphic to $G_m \times \times G_m$ (d = copies where d = dim(T)).
Therefore $K^\star \times \times K^\star \sim= T$ (K). Also, T is said to be K-anisotropic provided
$X(T) = 0$.
It is now convenient to give definitions of anisotropic, split and quasi-split algebraic groups.

Definition 9

i. An algebraic group G is said to be anisotropic whenever G possesses no K-split tori of nonnegative dimension.
ii. An algebraic group G is said to be split whenever some maximal torus T of G is split.
iii. An algebraic group G is said to be quasi-split over K whenever G possesses a Borel subgroup defined over K.

Next, we take G to be a connected reductive linear algebraic group and also the definitions and results presented here are taken from Shelstad (1979). Let **G** to be a connected reductive group defined over R. One denotes by
G the group **G**(R) of some R-points of **G**. Also, a Cartan subgroup T of G is the group of R-rational points on T, some maximal torus in G, defined over
R. Any root of T is the same as the root of the Lie algebra t of T in G, the Lie algebra of G. Denote by $\Omega(G, T)$ the Weyl group of (G, t), one may say that $w \in G$ realises $\omega \in \Omega(G, T)$ if $Ad(\omega/t)$ is same as ω and let $\Omega(G, T)$ be the set of elements of $\Omega(G, T)$ realised in G. Let **P** be a parabolic subgroup of **G**, we denote **P** = **MN**, its Levi decompositions where M and N are the reductive and nilpotent subgroups respectively. We write $G =^0 \mathbf{G}\mathbf{A}_G$ for the Langlands decomposition of **G** where $^0\mathbf{G}$ is the connected component of the identity of G; then A_G is the connected component of the maximal central split torus of **G**. We call $^L\mathbf{G} = {}^L\mathbf{G}^0 \rtimes \mathbf{W}_R$ the L-group of **G** on R, where $\mathbf{LG}^0$ is the connected component of the identity of the **L**-group $^L\mathbf{G}$ and $\mathbf{W}_R$ is the Weyl group of R.
Let $\mathbf{G}'$ be quasi-split over R. Then recall from [8] that **G** is called an inner form of $\mathbf{G}'$ if we have the following conditions hold

i. $\psi : \mathbf{G} \to \mathbf{G}'$ is an isomorphism
ii. $\bar{\Psi}\psi{-1}$ is inner where $\bar{\psi}$ is the complex conjugate of ψ.

Definition 10(Shelsatd (1979)) Let G'' be quasi-split over R. Then G is an inner form of G'' if the following conditions are satisfied

i. $\eta : \mathbf{G} \to \mathbf{G}''$ is an isomorphism
ii. $\bar{\eta}\eta^{-1}$ is inner
iii. $\eta = \theta\psi l$ such that l is inner and θ defined over R. Let us remark that(Shelstad(1979))
iv. Any connected reductive group G over R is automatically an inner form of a quasi-split group.
v. Two groups **G** and **H** possess the same L-group if and only if **G** is an inner form of **H**. In such a case, one may identify canonically certain Cartan subgroups of **G** to some Cartan subgroups of **H**.

Endoscopic Groups

This subsection is to discuss the endoscopic groups but first we lay some foundations towards that. Let $\varphi : \mathbf{W}_R \to^L \mathbf{G}$ be a homomorphism of the L-group, relevant to **G**. We associate to φ and L-packet of the generalised principal series or simply a pseudo-L-packets $\tilde{\mathbf{G}}$ defined as follows: Let $^L\mathbf{M} \subset^L \mathbf{G}$ be a minimal parabolic subgroup among those factorising φ, so we have:

φ: $\mathbf{W}_R \rightarrow^{\mathbf{L}} \mathbf{M} \subset^{\mathbf{L}} \mathbf{G}$.
Let us denote by φ a *L*-packet of the discrete series-*L*-packet of the dis-crete series associated to φ through duality, that is a finite sum of the irreducible representations of **M**. One may define $\Pi\tilde{}\mathbf{G} = ind^{\mathbf{G}}\, \Pi\tilde{}\mathbf{M}$. φ **MN φ**
Then $\Pi\tilde{}\mathbf{G}$ is a representation of finite length of *G* that one considers as an element of the Gro¨thendieck group of virtual representations of finite length modulo Jordan-Ho¨lder series.
Given that $\Pi\tilde{}\mathbf{G}$ is the *L*-packet associated to *Φ*, we have
$\Pi^{\mathbf{G}} \subset \Pi\tilde{}\mathbf{G}$ φ φ in the Gro¨thendieck group, and
$\Pi^{\mathbf{G}} = \Pi\tilde{}\mathbf{G}$ φ φ if φ is essentially tempered.
Let $\mathbf{T_G}$ and $\mathbf{T_H}$ be the cartan subgroups of the real reductive group **G** and its closed subgroup **H**, then $i : \mathbf{T_G} \rightarrow \mathbf{T_H}$ be a canonical map such that *i* unique modulo stable conjugacy on $\mathbf{T_G}$ and $\mathbf{T_H}$. Remember that **G** is spliter than **H** if every Cartan subgroup of **H** can be identified with a subgroup of **G**. In such case, every parabolic subgroup of $^L\mathbf{G}$ relevant for **H** is for **G**.
Let T = **T**(R) be a Cartan subgroup of **G**. Let D (T) be the set parameterizing the stable contumacy on T modulo conjugacy by **G**. One denotes by $\mathbf{G}_{sc}$ and $\mathbf{T}_{sc} = \mathbf{T} \cap \mathbf{G}_{sc}$ the derived groups of *G* and *T* respectively, and by $\mathbf{X}_*$ (**T**) and $\mathbf{X}_*$ ($\mathbf{T}_{sc}$) the groups of characters of *T* and T_{sc} respectively. Then we call $\sigma_{\mathbf{T}}$ the non-trivial element of the Galois group that acts on $\mathbf{X}_*$ (**T**) and $\mathbf{X}_*$ ($\mathbf{T}_{sc}$). Let κ be a character on $\mathbf{X}_* (\mathbf{T}_{sc}) / (\mathbf{X}_* (\mathbf{T}_{sc}) \cap (1 - \sigma_{\mathbf{T}})\, \mathbf{X}_* (\mathbf{T}))$, then κ defines a function, denoted again by κ, on D (T). Let T_{reg} be the set of the regular elements of T. We now discuss the endoscopic groups. We take an element *s* of $^LG^0$ to be a semi simple element and consider the objects $(s, {}^L\mathbf{H}^0, {}^L\mathbf{B}^0, {}^L\mathbf{T}^0, \{Y_\alpha \vee\}, \rho_s)$, **H** **H**
Where
i. $^L\mathbf{H}^0$ is the identity component of the centraliser in $^L\mathbf{G}^0$ of some element of *s*, labelled $Cent(s)^0$,
ii. $^L\mathbf{B}^0$ is a Borel subgroup of $^L\mathbf{H}^0$,
iii. $^L\mathbf{T}^0$ is a maximal torus in $^L\mathbf{B}^0$,
iv. $\{Y_\alpha \vee\}$ is a collection of root vectors deigned for the simple roots of $({}^L\mathbf{B}^0, {}^L\mathbf{T}^0)$ and **H** H
v. ρ_s is a homomorphism of *W* into
Aut (^{L}**H**0, LB0, LTH, $\{Y_\alpha \vee\}$) that factors through $W \rightarrow$ H satisfying $\rho_s(\omega) = ad\,(n(\omega))\,|\mathrm{L}\mathbf{H}0$, for some $n(\omega) \in^L \mathbf{G}^0 \times \omega$ which fixes each element of *s*, with H = *Gal*(C/R).
Next, we take G to be the set of all equivalence classes of objects(S, LH0, LB0, LTH, $\{Y_\alpha \vee\}$, ρ_s). We can put G = G (^{L}**G**) because G depends only on ^{L}**G** (Shelstad (1984)).

Definition 11(Shelstad (1984), Definition 2.2.1) Let **H** to be a quasi-split group over R. Then **H** is an endoscopic group for **G** whenever ^{L}**H** is attached to some element $s \in$ G (^{L}**G**). We remark from Shelstad (1984) that if **H** is an endoscopic group for **G**, then **H** becomes a quasi-split group. Following Shelstad (1984), the admissible embedding of ^{L}**H** into ^{L}**G** is defined thus.

Definition 12 The homomorphism ξ: *L* **H** $\rightarrow^L$ **G** is said to be an admis-sible embedding of ^{L}**H** into ^{L}**G** if the following hold:
i. $\xi|\mathrm{L}\mathbf{H}0 \times 1$ is the inclusion mapping and
ii. $\xi(1 \times \omega) = \xi_0(\omega) \times \omega$, where $\omega \in W$ and $\xi_0(\omega) \in^L \mathbf{G}^0$.

Definition 13(Shelstad (1984)) Suppose $\mathbf{G}^* \supset \mathbf{T}^*$ as the quasi-split form of **G** that defines the *L*-group with its torus of reference. If **T** is a Cartan subgroup of **G**, a morphism $\eta : \mathbf{T} \rightarrow \mathbf{T}^*$ is called a pseudo-diagonaliasation (also known as p.d.) of **T** if η is a composition of maps $Ad(x) \circ \psi$ for an $x \in \mathbf{G}^*$ where ψ is the isomorphism of base:$\mathbf{G} \rightarrow \mathbf{G}^*$.
Assume $T_{\mathbf{H}}(\mathbf{G})$ is the set of pairs (**T**, η) with **T** being a Cartan subgroup of **G**, η a p.d of **T**, for all $(\mathbf{T}, \eta) \in T_{\mathbf{H}}(\mathbf{G})$, there is a natural isomorphism of **T** with a Cartan subgroup $\mathbf{T_H}$ of **H**.
Let $i : \mathbf{T_H} \rightarrow \mathbf{T}$ be such an isomorphism. If $i(t_{\mathbf{H}}) = t$, with $t \in \mathbf{T}_{reg}$, then one says that $t_{\mathbf{H}}$ originates from *t* through (**T**, η). Then the element $t_{\mathbf{H}}$ is unique up to stable conjugation in **H**. If a torus $\mathbf{T_H}$ can be obtained in that way, one then says that $\mathbf{T_H}$ originates from **G** if $\mathbf{T_H}$ is the pre-image of **T** under *i*. One takes note that for each Haar measure *dt* on **T**, one can attach one and only one Haar measure $dt_{\mathbf{H}}$ on $\mathbf{T_H}$. Provided that $(\mathbf{T}, \eta) \in T_{\mathbf{H}}(\mathbf{G})$, then κ denotes the corresponding quasicharacter.

Definition 14: (Shelstad (1984)) Let *W* be the Weil group of G and the **L**-group ^{L}**G** of **G**. Then
i. A homomorphism φ: $W \rightarrow^L$ **G** is said to be admissible if for each $\omega \in W$, $\varphi(\omega)$ takes the form $\varphi_0(\omega) \times \omega$, where $\varphi_0(\omega)$ denotes a semisimple element of $^L\mathbf{G}^0$.
ii. Two homomorphisms are equivalent whenever they are conjugate under $^L\mathbf{G}^0$;.
iii. The set of equivalence classes is represented by Φ ($\mathbf{G}^*$).
iv. Φ (**G**) denotes the subset of classes of homomorphisms φ such that $\varphi(W)$ lies only in parabolic subgroups of ^{L}**G** relevant to **G**, then its class or certainly φ is relevant to **G**.
v. Let $\{\varphi\}$ be the equivalence class of φ. Then a homomorphism φ: $W \rightarrow^L \mathbf{G}^0$ is said to be tempered whenever $\varphi(W)$ is bounded.
vi. $\Phi_0(\mathbf{G}^*)$ denotes the classes of tempered homomorphisms and $\Phi_0(\mathbf{G}) = \Phi(\mathbf{G}) \cap \Phi_0(\mathbf{G}^*)$.

If ξ: *L* **H** $\rightarrow^L$ **G** is an embedding of *L*-groups, ξ is said to be unitary if

Ξ maps tempered homomorphism $W_R \rightarrow^L$ **H** on tempered homomorphisms by composition.
Let **H**, **G**, be reductive groups, we denote by $K_{\mathbf{H}}$, $K_{\mathbf{G}}$, the corresponding maximal compact subgroups. Finally, we let $\mathbf{G}_{reg}$ be the set of regular elements of **G**.
The following lemma is useful in the proof of the main theorem for the case of non-compact.

Lemma 2(Shelstad (1984), Lemma 4.2) Let *S* (T) denote the maximal R-split torus in T and M the centralizer of *S* (T) in **G**. Take *Me* to be the centraliser of *S* (T) in *G*, one take α to be a non-compact imaginary root of T. If for $\omega \in \Omega$ (*M*, T), then there is $\omega\alpha$ also noncompact, and there is $\omega_0 \in \Omega$ (*M*, T) so that $\omega\alpha = \pm\omega_0\alpha.+$

Next, consider *Z* to be the centre of $G_\gamma 0$ and *G* the Euclidean connected component of the identity in G^1 which is the derived group of G_o. We also
0 have $T_s = sts^{-1}$ (Shelstad (1979), Proposition 2.7) and finally $A^+ = T_s \cap G^+$. The proposition below explains how important it is for the realization of the Weyl reflection $G^{\gamma 0}$ or not. This realization of the Weyl reflection in *G* will determine the realization of the regular element in *G* or not in the main result.

Proposition 3(Shelstad (1979), Proposition 4.4) Let ω_α be a Weyl reflec-tion. Then the following hold:
i. When ω_α can be obtained in *G*, then $G_\gamma 0 : ZG^+ = [T_s : ZA^+] = 2$;
ii. When ω_α cannot be realised in *G*, then $G_\gamma 0 = ZG^+$and $T_s = ZA^+$.

Stable or Averaged Orbital Integrals
In order to study the transfer problem, we must record some aspects of the theory of orbital integrals that are needed in what follows. Let *G* be a reductive Lie group and $\gamma \in G$ a regular element in *G*. Let us call T_γ the Cartan subgroup of **G**

containing γ. If C (G) denotes the Schwartz space of G, for $f \in$ C (G), we have

Φ_f (γ, $d_\gamma t$, dg) = f ($g\gamma g^{-1}$) $d\gamma g^{-}$ G/Tγ

Where $d_\gamma t$, dg and $d_\gamma g^{-}$ are the Haar measures on T_γ, G and G-invariant measures on G/T_γ respectively. We write dt instead of $d_\gamma t$ in the remaining part of the work, as well T for T_γ.

Let

$A(T) = \{g \in G : ad(g|T)$ is defined over R$\} = \{g \in G : gTg^{-1} \subset G\} = G$ $Norm(M, T)$ with $Norm(M, T)$ denoting the normaliser of **T** in **M**. For any $\omega \in A(T)$, we have a Haar measure $(dt)^\omega$ on the Cartan subgroup T^ω and we have orbital integrals Φ_f (γ^ω, $(dt)^\omega$, dg) with respect to the class of ω in D(T) = $G\backslash A$(T)/T.Shelstad(1979)

Thus, we have

Φ^1 (γ, dt, dg) = Σ Φ_f (γ^ω, $(dt)^\omega$, dg). (2)

$\gamma\in$D (T)

For each respective $\gamma \in A$ (T), we can write

Φ^1 (γ, dt, dg) = Φ_f (γ^ω, $(dt)^\omega$, dg). (3)

If $\lambda \in t^*$ is zero on $\{H \in t$: exp (H) = 1$\}$, we denote by ξ_γ the corresponding quasi-character on T. Let us use I^+ as the system of positive roots for T in M. Let us put1

R_T (γ) =| det ($Ad\gamma - 1$) $_{G/M}$ | 2(1 − ξ_α)

A∈I+ (γ^{-1})) (4)

For $\gamma \in$ T and ΨT (γ) = R_T (γ) Φ1 (γ, dt, dg) (5)

For $\gamma \in T_{reg}$ or, simply as ΨT (γ) = c Σ $det\omega\xi_{i-\omega}$−1i (γ) $'$ F_f (γ^ω) (6)

Ω while i = (α∈I+ α)/2 and c some constant depending on the choice of measures. Let s and T be the Cayley transform and the algebra of invariant differential operators on T respectively. If $D \in$ T, then D^s denote the image of D under the isomorphism T → T induced by s. Also we replace D by $D\hat{}$, the image of D under the auto Orphism of T induced by $H \rightarrow H + i$ (H) I, $H \in t$. If $D' \in$ T$_s$, then $D\hat{}'$ is the image of D' under the Auto Orphism of T$_s$ induced by $H' \rightarrow H' + i_s$ (H') I, $H' \in t$.

Then Ψ^T becomes a Schwartz function on T^I defined as I reg= $\{\gamma \in T$: ξ_α (γ)/= 1, $\alpha \in I^+\}$ of the Cartan subgroup T, where T^I is a dense open subset of T.

Let T be the algebra of invariant differential operators on T, hence if $D \in$ T, then $D\Psi^T$ is bounded on T^I.

Also, let γ_0 be a semi regular element in $T - T^I$. Then β and $\pm\alpha$ are the only two imaginary roots or which ξ_β (γ_0) = 1. We then let H_α be the coroot attached to α and we have $\gamma_v = \gamma_0 exp$ (ivH_α), $v \in$ R. Also we have $\gamma_v \in T$ implies that $\lim_{v\downarrow 0} D\Psi^T$ (γ_v) and $\lim_{v\uparrow 0}$ $D\Psi^T$ (γ_v) are well-defined. The above limits are equal to each other, in a manner that Ψ^T becomes a Schwartz function on T .Next, writing the equation (1) in terms of $'F_f$, we have

Ψ^T (γ) = R_T (γ) Φf (γ^ω) = $cdet$ (ω) $\xi_{i-\omega}$−1 (γ) F_f (γ^ω). (7)

Then

Lim $D\Psi^T$ (γ_v) − lim $D\Psi^T$ (γ_v) = Σ (lim $D\Psi^\omega$ (γ_v) − lim $D\Psi^\omega$ (γ_v)). (8)

We note that

Lim $D\Psi^T$ (γ_v) = lim $D\Psi^T$ (γ_v) (9)

For ω_α compact. v↓0 f - v↑0 f

Then let us choose α compact, since

Ψ^T (γ^ω) = (det (ω)) $\xi_{i-\omega}$−1i (γ) Ψf (γ). (10)

The next proposition is the direct consequence of Proposition 3 on the stable orbital integrals.

Proposition 4(Shelstad (1979) Proposition 4.5)

lim $D\hat{}\psi^\delta$ ($\gamma : dt : dg$) − lim $D\hat{}\psi^\delta$ ($\gamma : dt : dg$) = $id(\alpha)D\hat{}s\psi^{s\delta s}$−1 ($\gamma : (dt)^s : dg$)

υ↓0 f v v↑0 f v f

0

Where d (α) = 2 if ω_α can be realised in G and d (α) = 1 otherwise.

Next, let us take into consideration the set of all classes in D (T) = $G\backslash$A (T)/T which contain a representative $\delta \in Norm$ (M, T), the normalizer of T in M, for which $\delta\alpha = \pm\alpha$.

We then have an action of the group ⟨1, ω_α⟩ given by $G\delta T \rightarrow G\omega_\alpha\delta T$.

Let D$_\alpha$ (T) denote the set of orbits. We then deduce that each orbit has one element or two elements depending on if ω_α is realized in G or not respectively. For s embedding S (T) in S (T$_s$), we have $\omega \rightarrow s^{-1}\omega s$ mapping $Norm$ (M$_s$, T$_s$) to $Norm$ (M, T) as shown in the following proposition.

Proposition 5(Shelstad (1979), Proposition 4.6) a bijection D (T$_s$) → D$_\alpha$ (T) is induced by the map $\omega \rightarrow s^{-1}\omega s$. The next lemma summarises Lemma 2, Proposition 4 and Proposition 5 as follows:

Lemma 3(Shelstad (1979), Lemma 4.3: p.25)

Lim $D\hat{}\psi^T$ (γ_v, dt, dv) − lim $D\hat{}\psi^T$ (γ_v, dt, dg) = $2iD\hat{}s\psi^Ts$ (γ^s, $(dt)^s$, dg).

N↓0 f v↑0 f f 0

Definition 15(Harish-Chandra (1975) Section, p. 105) the split compo-nent A of a real reductive group G is a vector subgroup such that $G = ^0GA$ and $^0G \cap A = \{1\}$, where ^0G is the maximal compact subgroup of G.

In this line, we let A be the split component of M and ^0M a reductive subgroup such that $M = ^0MA$ and $^0M \cup a = \{1\}$ and we let ^0T be a compact Cartan subgroup of ^0M such that $T = ^0TA$ and $^0T = ^0M \cap T$. Let $a^\star$ be the real dual of log (Λ), if we denote the group of characters on ^0T by Y, we have $\Psi^\star$, the Fourier transform of $\Psi = n\Psi^T$ (·, dt, dg) being a Schwartz function on Y × $a^\star$. Let C($a^\star$) be the Schwartz space of $a^\star$, if N is a given continuous seminorm on C($a^\star$), then one has the numbers defined as $N_\Lambda = N$ ($v \rightarrow \Psi(\Lambda, v)$) satisfy the condition $\Sigma_\Lambda N_\Lambda \wp$(| Λ |) < ∞ for every polynomial $\wp$(| Λ |) denoting the length of log(Λ).

Proposition 6(Shelstad (1979), Proposition 4.9: p. 35) Let C (**G**) be the Schwartz space of G, X, Y are members of U (G), the universal enveloping algebra of G, and m being positive. One can set $v_{(X,Y,m)}(f)$ = sup

g∈G

(1 + σ (g))^m (Xf Y) (g)

$f \in$ C(**G**),

Ξ (g)

Where Ξ, σ ae carefully defined in Harish-Chandra (1975). Then we have log (Λ) ∈ C for each Λ, a function f (Λ) ∈ C (G), a continuous seminorm N on C(a^*) and a polynomial p satisfying

T f(Λ)

i. ψ(·, dt, dg) = ∫ ∗ ψ^(Λ, v)γ(Λ, v)dv;
ii. Φ^1 (·, dt', dg) ≡ o for every T' with ⟨T'⟩ ⋞/⟨T⟩, where ⟨T⟩ = {T ◦adg^{-1}; $g \in A(T)$} and
iii. $v_{X, Y, m}$(f (Λ)) ≤ $N_\Lambda P$ (| Λ |).

T

The following lemma generalises Proposition 6 and paves way to the next theorem.

Lemma 4(Shelstad (1979), Lemma 4.8: p. 32)

Let us fix a Cartan subgroup T_0 and assume that for each T being conju-gate to T_0 with their corresponding Haar measures dt and dg on T and G respectively, then we have a function $\Phi^T(\cdot, dt, dg)$ on T_{reg} such that

i. $\Phi^T(\gamma, \alpha dt, \beta dg) = {}^{\beta}\Phi^T(\gamma, dt, dg)$ for $\alpha, \beta > 0$
ii. $\Phi^T(\gamma, dt, dg) = \Phi^T\omega(\gamma^{\omega}, (dt)^{\omega}, dg)$ for $\omega \in A(T)$
iii. $\Phi^T(\cdot, dt, dg)$ extends to a Schwartz function on T. Then there exists a Schwartz function $f \in G$ such that

i. $\Phi^T(\gamma, dt, dg) = \Phi^1(\gamma, dt, dg)$, $\gamma \in T_{reg}$ for all such T, dt and dg

And

ii. $\Phi^1(\cdot, dt', dg) \equiv 0$ unless $\langle T'\rangle \le \langle T\rangle$ for any dt' on T' and dg.

This theorem generalises Lemma 4 and presents three intersting prop-erties of the stable orbital integrals that will be needed in the proof of the theorem.

Theorem 1(Shelstad (1979), Theorem 4.7: p. 30-31) Let dt and dg be Haar measures on the Cartan subgroup T and G respectively and a function $\gamma' \rightarrow \Phi^T(\gamma, dt, dg)$ on T_{reg}. Then there exists a function $f \in C(G)$ such that $\Phi^T(\gamma, dt, dg) = \Phi^1(\gamma, dt, dg)$ for all dt, dg, γ and T if and only if

i. $\Phi^T(\gamma, \alpha dt, \beta dg) = {}^{\beta}\Phi^T(\gamma, dt, dg)$ for $\alpha, \beta > 0$
ii. $\Phi^T(\gamma, dt, dg) = \Phi^T\omega(\gamma^{\omega}, (dt)^{\omega}, dg)$ for $\omega \in A(T)$
iii. if $\Psi^T(\gamma, dt, dg) = R_T(\gamma)\Phi^T(\gamma, dt, dg)$, then $\Psi^T \in C(T_{reg})$ and

If $\gamma_0 \in T - T^I$ is semi regular and $\xi_{\alpha}(\gamma_0) = 1$ where $\omega\alpha$ is compact for each $\omega \in \Omega(M, T)$, then

$\text{Lim } D\Psi^T(\gamma_v, dt, dg) = \lim D\Psi^T(\gamma_v, dt, dg)$ for each $D \in T$,
$v\downarrow 0$ $v\uparrow 0$

If $\gamma_0 \in T - T^I$ is semi regular and $\xi_{\alpha}(\gamma_0) = 1$ where α is non-compact, then

$\text{Lim } D\hat{}\Psi^T(\gamma_v$
$v\downarrow 0, dt, dg) - \lim D\hat{}\Psi^T(\gamma_v$
$v\uparrow 0, dt, dg) = 2iD\hat{}s'\Psi^T s(\gamma^s, (dt)^s, dg)$ for each $D \in T$,

Now, this proposition generalizes Proposition 6.

Proposition 7(Shelstad (1979), Proposition 4.11: p.37) Let α be an imag-inary root of T in G and we write $s^{-1}s$ realizing ω_{α} for $s \in G$. Then we have $\omega \in \Omega(M, T)$ such that $\omega\alpha$ is non-compact.

RESULTS AND DISCUSSION

In this section, we state and prove the main result of this paper concerning the transfer of stable orbital integrals from G to G' given in Theorem 3.1 below. Before given the theorem, we define the map reg.

$\gamma' \rightarrow$
F 0 if γ' doesn't originates in G

Theorem 2 Let $f \in C(G)$. Then there is a function $f' \in C(G)$ for
$'$ $'$
Reg such that
$((-1)^{qG} - qG'\Phi^1(\gamma, dt, dg)$ if γ' originates from $\gamma \in G$

Proof: At this juncture, it only remains to prove that the mapping
$((-1)^{qG} - qG'\Phi^1(\gamma, dt, dg)$ if γ' originates from $\gamma \in G$

$\gamma' \rightarrow$ if γ' doesn't originates in G

Satisfies the conditions of Theorem 1. Let us recall that $\Phi^T(\gamma, dt, dg) = \Phi^1(\gamma, dt, dg)$. From
$\Phi^1(\gamma', dt', dg') = (-1)^{qG} - qG'\Phi^1(\gamma, dt, dg)$
F f

$= (-1)^{qG} - qG'\Phi^T(\gamma, dt, dg)$
$= \Phi T(\gamma', dt', dg')$

Condition (I) $\Phi^{T'}(\gamma', dt', dg') = \Phi^1(\gamma', \alpha dt', \beta dg')$ but
$\Phi^1(\gamma', \alpha dt', \beta dg') = \beta\,\Phi^1(\gamma', dt', dg')$
—
f α f

. So $\Phi^{T'}(\gamma', dt', dg') = {}^{\beta}\Phi^{T'}(\gamma', dt', dg')$.
Condition (II) $\Phi^{T'}(\gamma', dt', dg') = \Phi^1(\gamma', dt', dg')$ but

$\Phi^1(\gamma^{\omega}, (dt')^{\omega}, (dg')^{\omega}) = \Phi^1(\gamma', dt', dg')$
F f

For $\omega \in A(T')$.
We need to note that the element $\omega \in A(T')$ describes $(dt')^{\omega}$ as Haar mea-sure on T'^{ω}. Therefore $\Phi^1(\gamma'^{\omega}, (dt')^{\omega}, (dg')^{\omega}) = \Phi^{T'}\omega(\gamma'^{\omega}, (dt')^{\omega}, dg')$. Thus
$\Phi^{T'}(\gamma', dt', dg') = \Phi^{T'}\omega(\gamma'^{\omega}, (dt')^{\omega}, dg')$

Condition (III) We first fix a Cartan subgroup T' of G' and then fix an imaginary root α' of T'. Let us assume that α' is noncom pact. Then we define a Cayley transform s' with respect to α'. Now, let $\gamma' \in T'$ does not originate in G, then the image of T', namely T'does not also originate in G.
Hence $\Phi^{T'}(\gamma', dt', dg') = 0$. In the case where $\gamma' \in T'$ originates from $\gamma \in G$, then we find ourselves into two possibilities.

First possibility: T'originates from $\gamma \in G$ and its image under the Cayley transform, T'also originates in G. We define the following mappings overR: $\psi_x : T \rightarrow T'$, $ads' : T' \rightarrow T''$ and $\psi_y : T^* \rightarrow T''$. We can then form the s s new map by composing the mappings above: $\psi^{-1} \circ ad(s') \circ \psi_x : T \rightarrow T^*$. This new mapping can be realized by an element s of G and $s \in G$ such that $s^{-1}s$ realizes the Weyl reflection ω_{α} with respect to $\psi^{-1}(\alpha')$. By Proposition 7, we have $\omega \in \Omega(M, T)$ such that $\omega\alpha$ is noncompact. $-1'$
In this direction, we can replace ψ_x by ψ_x' with $\psi x'(\alpha) = \alpha$ is non-compact. The element s, defined as above with respect to ψ_x', in place of ψ_x, is also a Cauchy transform. Then we have (*IIIb*),

$\text{Lim } D\hat{}$
$T'\gamma', dt', dg') - \lim D\hat{}$
$T'\gamma', dt', dg') = 2iD\hat{}S'\Psi^{T''}($
$'S'$
$(dt')^{S'}, dg')$ for $D \in T$.
$v\downarrow 0$
$\Psi(v)$
$v\uparrow 0$
$\Psi(v)$
$S\gamma 0$,

Second Possibility: Let T' originates from $\gamma \in G$ but its image T' Under the Cayley transform does not originate in GThen we define the map $\psi_x : T \rightarrow T'$. We have that $\psi^{-1}(\alpha') = \alpha$ is compact. This implies that $\omega\alpha$ is also compact with $\omega \in \Omega$ (*M*), T. And we have (*IIIa*)

Lim D
$v\downarrow 0$
$\Psi T'$

$(\gamma', dt', dg') = \lim D$
$\nu \qquad \nu\uparrow 0$
$\Psi T'$
$(\gamma', dt', dg') \qquad$ for $\qquad D \in T$
And (*IIIb*) becomes
$\text{Lim } D\hat{\ }\Psi^{T}{}' (\gamma', dt', dg') - \lim D\hat{\ }\Psi^{T}{}' (\gamma', dt', dg') = 0 .2$
$\nu\downarrow 0 \qquad \nu \qquad \nu\uparrow 0 \qquad \nu$

CONCLUSION

We have reworked the proof of theorem due to Shelstad (1979), giving it a detailed expository clarity in order to make it accessible to a wider audience.

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