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QUASI-EXACT SOLVABILITY OF TRI-CONFLUENT HEUN EQUATION

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ABSTRACT

The Tri-confluent Heun Equations (TCHE) are second order differential equations in the complex domain obtained by a limiting process which merges regular singularities of the canonical Heun differential equation with a regular singularity at $z = 0$ and the irregular singularity at $z = \infty$ of rank 3. In this paper, we present a new algebraisation of the TCHE by writing it as the linear combination of quadratic elements in the universal enveloping algebra of $sl(2,\mathbb{C})$. We also obtain a new exactly solvable potential from TCHE using a suitable guage transformation.

Keywords: Algebraization, Tri-Confluent Heun Equations, Gauge Potential, Exactly Solvable Potential

INTRODUCTION

The canonical Heun equation is a second order differential equation in the complex domain given (Ronveaux, 1995) by

$$\frac{d^2\psi}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-a}\right)\frac{d\psi}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}\psi = 0,$$

With regular singularities at $z = 0,1,a$, irregular singularity at ∞ and where, $\alpha, \beta, \gamma, \delta, \epsilon$ are complex numbers that obey the constraint equation $\gamma + \delta + \varepsilon = \alpha + \beta + 1$. The confluent forms of the Heun equation (1) arise when two or more of the regular singularities $z = 0,1,a$ merge to form an irregular singularity(∞). The Tri-confluent Heun Equation (TCHE) is given by $\frac{d^2\psi}{dz^2} + (\gamma + z)z\frac{d\psi}{dz} + (\alpha z - q)\psi = 0.$

This has irregular singularities at $z = 0$ and $z = \infty$ each of rank 2. The properties and connecting formulas of solutions of equation (2) have been discussed by (Whittaker & Whatson, 1902), (Bühring, 1994), (Ronveaux, 1995), (Wang & Guo, 1989), (Wolf, 1998), (Lay and Slavyanov, 2000). Schulze-Halberg (2002) considered a tri-confluent Heun operator (TCHO) having infinitely many Liouvillian eigenfunctions and transformed it into a radial Schrödinger operator with generalised power law potential. Three special cases were extracted namely with diverse singular fractional power potentials (SFPP) which yielded transformed Liouvillian eigenfunctions of the original TCHE seen as Schrödinger bound states. In (Schulze-Halberg's, 2002), algebraisation was not carried out. Osherov and Ushakov (2011) obtained the eigenfunction of the Stark problem in terms of Stokes multipliers for the TCHO (the quartic oscillator operator). Faleye et.al. (2014) applied extended Nikiforov–Uvarov (NU) method to solve problems in TCHE on two quantum dot models. Karayer *et. al.* (2015) reported some special eigenfunctions of BCHO and TCHO through extended NU method which was developed by changing the boundary conditions of NU method. Vieira and Bezerra (2016) obtained an exact solution of the Wheeler-DeWitt equation in a minisuperspace with a cosmological constant, which is given in terms of explicit eigenfunction of TCHO. The energy density spectrum were computed in all these papers but algebraisation of the TCHO was not carried out. Recent studies focus on extending the NU method through reduction algorithms (Aldossari, 2024) and obtained solutions in the form$\exp\left(\int r(x)dx\right) HeunT\left(q,\alpha,\gamma,\delta,\epsilon,f(x)\right)$.

In what follows, section 2 contains a methodology on Lie algebraisation of differential operators and section 3 contains the main result on quasi-exact solvability of TCHO. The final section gives the conclusion drawn from the result. In what follows, we shall denote the center of the universal enveloping algebra $U(\mathcal{G})$ by $\mathcal{Z}(U(\mathcal{G}))$.

MATERIALS AND METHODS

A k^{th} order differential operator given by

$\mathfrak{H}_k\psi = \lambda\psi,$

Where, $\mathfrak{H} = \sum_{j=0}^{k} a_j(z)\frac{d^j}{dz^j}$ is said to be quasi-exactly solvable if it can be written as quadratic combination of the generators of certain elements of the universal enveloping algebra of $sl(2,\mathbb{C})$ of the form

$\mathfrak{H}_k = \sum_{a,b=0,\pm} c_{ab}\, J_a J_b + \sum_{a=0,\pm} c_a\, J_a$, (3)

Where the number of free parameters $c_{ab} \neq 0$ is given by$par(\mathfrak{H}_k) = (k+1)^2$. When the number of free parameters $par(\mathfrak{H}_k) = \frac{(k+1)(k+2)}{2}$ then $\mathfrak{H}_k$ is said to be exactly solvable [see, H.L. Krall's classification (Krall, 1938)]. The technique of computing (3) is what we call algebraisation. In this work, the technique of solving quasi-exactly solvable problem wherein its operator act on monomials which form the basis of a block triangular matrix will be adapted since the Lie algebraic operator of $TCHE$ is quasi-exact. The standard quadratic expression for the Hamiltonian $-H$ as polynomials in terms of J_+, J_0 and J_- is given by the generators of the Lie algebra $\mathcal{G} = sl(2,\mathbb{C})$ have the form

$$J_+ := z^2\frac{d}{dz} - 2jz, \quad J_0 := z\frac{d}{dz} - j, \quad J_- := \frac{d}{dz} \quad (4)$$

Which obey the commutator relation

$$[J_0, J_+] = J_+, \quad [J_-, J_+] = 2J_0, \quad [J_-, J_0] = J_- \quad . (5)$$

The algebraisation requires that the Fuchsian operator $-H$ be written in the form

$$-H\psi = [c_{++}(J_+)^2 + c_{+0}[J_+J_0 + J_0J_+] + c_{00}(J_0)^2 + c_{+-}[J_+J_- + J_-J_+] + c_{0-}[J_0J_- + J_-J_0] + c_{--}(J_-)^2 + c_+J_+ + c_0J_0 + c_-J_- + c_*]\psi. \quad (6)$$

(see Gonzalo-Lopez et. al., 1994). The equation (6) in expanded form is given by

$$-H\psi = [c_{++}z^4 + 2c_{+0}z^3 + [c_{00} + 2c_{+-}]z^2 + 2c_{0-}z + c_{--}]\frac{d^2\psi}{dz^2} + [(2j-1)[2c_{++}z^3 + 3c_{+0}z^2 + (2c_{+-} + c_{00})z + c_{0-}] + c_+z^2 + c_0z + c_-]\frac{d\psi}{dz} + [2j(2j-1)c_{++}z^2 + 2j(2j-1)c_{+0}z + c_{00}j^2 - 2jc_{+-} - j[2c_+z + c_0] + c_*]\psi. \quad (7)$$

RESULTS AND DISCUSSION

In this section, we discuss the algebraisation of the TCHO, its quasi-exactly-solvability and corresponding eigenfunctions.

Theorem 1. *Consider the TCHE in equation (2) in which*
$H = D^2 + (z^2 + \gamma z)D + (\alpha z - q), D := \mathrm{d}/\mathrm{d}z.$
Then the Lie algebraization of H gives $\mathfrak{H} \in \mathcal{Z}(U(\mathcal{G}))$ such that
$\mathfrak{H} = J_-^2 + J_+ + \gamma J_0 - q,$
Provided the accessary parameter $q = \frac{n\gamma}{2}$. the equation $\mathfrak{H}\Psi(z) = 0$ has the eigenfunction $\mathcal{P}_n(z) = \sum_{k=0}^{n} a_k z^k$ and a_k satisfy the recurrent relations
$(k + n)a_{k+1} + (k\gamma + \Xi_{tch})a_k + k(k-1)a_{k-2} = 0.$
$a_1 = -\frac{\Xi_{tch}}{n} a_0 = \eta_1 a_0;$
$a_2 = \frac{(\gamma+\Xi_{tch})\Xi_{tch}}{n(n+1)} a_0 = \eta_2 a_0;$
$a_3 = -\left[\frac{(2\gamma+\Xi_{tch})(\gamma+\Xi_{tch})\Xi_{tch}}{n(n+1)(n+2)} + \frac{2}{(n+2)}\right] a_0 = \eta_3 a_0;$
$\vdots = \vdots$
$a_n = \Big[(-1)^k \frac{\prod_{k=0}^{n-1}(k\gamma+\Xi_{tch})}{(2n-1)_k} + \frac{2}{(2n-1)_{n-2}} +$
$(-1)^{n-2} \frac{(n-1)!}{(2n-1)_{n-3} n} \Xi_{tch}\Big] a_0 = \eta_n a_0.$
Where, the Pochammer symbol $(s)_n = s(s-1)\cdots(s-n+1)$ and $\Xi_{tch} = -(\frac{n\gamma}{2} + q)$ so that
$\Psi_n(z) = a_0(1 + \sum_{k=1}^{n} \eta_k z^k)\exp(\frac{z^3}{6} + \frac{z^2}{4}).$

Proof. Let equation (2) be re-written in the form
$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + (z^2 + \gamma z)\frac{\mathrm{d}\Psi}{\mathrm{d}z} + (\alpha z - q)\Psi = 0.$
Comparing equation (8) with (7), we have
$c_{++} = 0, \quad c_{+0} = 0, \quad c_{+-} = 0, \quad c_+ = 1$
$c_{0+} = 0, \quad c_{00} = 0, \quad c_{0-} = 0, \quad c_0 = \gamma$
$c_{-+} = 0, \quad c_{-0} = 0, \quad c_{--} = 1, \quad c_- = 0.$
The Casimir eigenvalue here $c_* = -q, \alpha = -n$. Now, we write the TCHE Hamiltonian in equation (8) in terms the quadratic polynomial of operators (4) as
$\mathfrak{H} = J_-^2 + J_+ + \gamma J_0 - q \in \mathcal{Z}(U(\mathcal{G})).$ (9)
Therefore, the structure metrics $[c_{ab}]_{a,b=\pm,0}$ and $[c_a]_{a,b=\pm,0}$ of the TCHE are respectively given by

$$[c_{ab}]_{a,b=0,\pm} = \begin{pmatrix} c_{++} & c_{+0} & c_{+-} \\ c_{0+} & c_{00} & c_{0-} \\ c_{-+} & c_{-0} & c_{--} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

And $[c_a] = (c_+, c_0, c_-) = (1, \gamma, 0)$. the norm of $[c_{ab}]_{a,b=\pm,0}$ is given as
$\| g \| = \det[c_{ab}]_{a,b=\pm,0} = 0.$ (11)

We observe here, that the free parameters for the TCHE Hamiltonian $\mathfrak{H}$ are six (6) in number which confirms the Hamiltonian equation (2) as an exactly solvable operator differential equation. Having considered the algebraisation of the confluent Heun equation, we now consider the evaluation of its ground state solution, its new eigenfunction and its new exactly solvable potential. Explicit form of $\mathfrak{H}$ is
$\mathfrak{H} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + (z^2 + \gamma z)\frac{\mathrm{d}}{\mathrm{d}z} + nz - (\frac{n\gamma}{2} + q).$
Thus, the Lie algebraic equation in the center of the universal enveloping algebra $\mathcal{Z}(U(\mathcal{G}))$ is
$\mathfrak{H}\Psi := \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (z^2 + \gamma z)\frac{\mathrm{d}}{\mathrm{d}z} + nz - (\frac{n\gamma}{2} + q)\right]\Psi = 0.$
Now, from the differential equation (8), let $\Xi_{tch} = -(\frac{n\gamma}{2} + q)$ the functions $p(z), q(z)$ and $r(z)$ are given as
$p(z) = 1, \quad q(z) = z^2 + \gamma z, \quad r(z) = nz + \Xi_{tch}.$
In what follows, the quasi-exact solvability is discussed. Following a technique similar to that of Panahi *et. al.* (2015) and Turbiner (2016), the gauge transformation of the Tri-confluent Heun equation is carried out to obtain an equivalent form, namely a Schrödinger operator $\mathcal{S}$ whose solution is obtained in terms of polynomial functions in the polynomial space$\mathcal{P}_{n+1}$. Thus,
$\mathcal{S} = \mu(z)^{-1} \cdot \mathfrak{H} \cdot \mu(z)$
where, $\mathcal{S} = -\frac{\mathrm{d}^2}{\mathrm{d}w^2} + U(w),$
Provided that
$U(w) = \frac{3(p\prime(z))^2 - 8p\prime(z)q(z) + 4q(z)^2}{16p(z)} - \frac{1}{4}p''(z) + \frac{1}{2}q'(z) - r(z),$
Where $w = \int_0^z \frac{\mathrm{d}u}{\sqrt{p(u)}} = z.$
For TCHE, the QES potential is
$U(w) = \frac{1}{4}(z^2 + \gamma z) + (2 - n)z + \gamma + \Xi_{tch}$
The guage function in this case is
$\mu(z) = \exp\left(\frac{1}{2}\int_0^z \frac{q(u)}{p(u)}\mathrm{d}u\right) = \exp(\frac{1}{2}\int_0^z (u^2 + \gamma u)\,\mathrm{d}u) = \exp(\frac{z^3}{6} + \frac{z^2}{4}).$
By applying $\mathfrak{H}$ to the space of monomials$\mathcal{P}_k$, one gets
$\mathfrak{H}z^k = (k + n)z^{k+1} + [k\gamma + \Xi_{tch}]z^k + k(k-1)z^{k-2}.$
The tridiagonal Jacobi matrix k-Th entry is given by
$\tau_{k,k+1} = k + n; \quad \tau_{k,k} = [k\gamma + \Xi_{tch}] \ \tau_{k,k-2} = k(k-1).$

Table 1: Table of Values for $\tau_{k,k+1}, \tau_{k,k}, \tau_{k,k-2}, k = 0, 1, 2, 3, \ldots, n$

k	$\tau_{k,k+1}$	$\tau_{k,k}$	$\tau_{k,k-2}$
0	$\tau_{0,1} = n$	$\tau_{0,0} = \Xi_{tch}$	$\tau_{0,-2} = 0$
1	$\tau_{1,2} = 1 + n$	$\tau_{1,1} = \gamma + \Xi_{tch}$	$\tau_{1,-1} = 0$
2	$\tau_{2,3} = 2 + n$	$\tau_{2,2} = 2\gamma + \Xi_{tch}$	$\tau_{2,0} = 2$
3	$\tau_{3,4} = 3 + n$	$\tau_{3,3} = 3\gamma + \Xi_{tch}$	$\tau_{3,1} = 6$
	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\tau_{n-1,n} = 2n - 1$	$\tau_{n,n} = (n-1)\gamma + \Xi_{tch}$	$\tau_{n-1,n-3} = (n-1)(n-2)$
n	$\tau_{n,n+1} = 2n$	$\tau_{n,n} = n\gamma + \Xi_{tch}$	$\tau_{n,n-2} = n(n-1)$

The next step will be to determine the nature of its eigenvalues and Eigen functions. Case by case treatment of each eigenvalue and Eigen function will be required to study the spectrum of the TCHO.

In this case, $\mathcal{S}$ possesses the invariant subspace $\mathcal{P}_1$ spanned by the basis{1}, thus, the function$\mathcal{P}_0(z) = a_0$. The matrix equation corresponding a_0 is given by the 1×1 matrix equation given by
$T_1 A_1 = [\tau_{0,0}][a_0] = 0 \Xi_{tch} a_0 = 0 q = -\frac{n\gamma}{2}$
Since $a_0 \neq 0$. The ground state solution is thus
$\Psi_0(z) = a_0 \exp(\frac{z^3}{6} + \frac{z^2}{4}).$
In this case, $\mathcal{S}$ possesses the invariant subspace $\mathcal{P}_1$ spanned by the basis$\{1, z\}$, thus, the function$\mathcal{P}_1(z) = a_0 + a_1 z$. The matrix equation corresponding a_0 is given by the 2×2 matrix equation given by

$$T_2 A_2 = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} \\ \tau_{1,0} & \tau_{1,1} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \Xi_{tch} & n \\ 0 & \gamma + \Xi_{tch} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0. \quad (12)$$

Since, $\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \neq 0$ it then follows that $\det(T_2)$ must be equal to zero. Hence, $\Xi_{tch}^2 + \gamma\Xi_{tch} = 0$ which gives $\Xi_{tch} = 0 \Rightarrow q = -\frac{n\gamma}{2}$ or $\Xi_{tch} + \gamma = 0 \Rightarrow q = -\frac{(n+2)\gamma}{2}$. By solving equation (12), one gets $a_1 = \frac{\Xi_{tch}}{n+\Xi_{tch}} a_0$. the eigenfunction for this case is thus obtained as

$\Psi_1(z) = a_0(1 + \eta_1 z)\exp(\frac{z^3}{6} + \frac{z^2}{4})$,

Where,

$\eta_1 = \frac{\Xi_{tch}}{n+\Xi_{tch}}$.

In this case, $\mathcal{S}$ possesses the invariant subspace $\mathcal{P}_1$ spanned by the basis $\{1, z, z^2\}$, thus, the function $\mathcal{P}_2(z) = a_0 + a_1 z + a_2 z^2$. The matrix equation corresponding a_0 is given by the 3×3 matrix equation given by

$$T_3A_3 = \begin{pmatrix} \tau_{0,0} & \tau_{0,1} & \tau_{0,2} \\ \tau_{1,0} & \tau_{1,1} & \tau_{1,2} \\ \tau_{2,0} & \tau_{2,1} & \tau_{2,2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \Xi_{tch} & n & 0 \\ 0 & \gamma + \Xi_{tch} & 1+n \\ 2 & 0 & 2\gamma + \Xi_{tch} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0. \quad (13)$$

$\det(T_3) = 0\Xi_{tch}^3 + 3\gamma\Xi_{tch}^2 + 2\gamma^2\Xi_{tch} - 2n(n+1) = 0.$

By standard formula for solving cubic polynomials (cf: Abramowitz and Stegun 1972, S 3.8.2, p.17) $\Xi_{tch}^3 + b_2\Xi_{tch}^2 + b_1\Xi_{tch} + b_0 = 0$ has the roots

$\Xi_{tch}^{(1)} = (s_+ + s_-) - \frac{b_2}{3}$,

$\Xi_{tch}^{(2)} = -\frac{1}{2}(s_+ + s_-) - \frac{b_2}{3} + i\frac{\sqrt{3}}{2}(s_+ - s_-)$,

$\Xi_{tch}^{(3)} = -\frac{1}{2}(s_+ + s_-) - \frac{b_2}{3} + i\frac{\sqrt{3}}{2}(s_+ - s_-)$.

Where

$s_\pm = [r \pm (t^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}}$

$t = \frac{1}{3}b_1 - \frac{1}{9}b_2^2$

$r = \frac{1}{6}(b_1 b_2 - b_0) - \frac{1}{27}b_2^3$

And

$b_2 = 3\gamma$

$b_1 = 2\gamma^2$

$b_0 = -2n(n+1)$.

Solving (13) we get

$a_2 = \frac{\Xi_{tch}(\gamma+\Xi_{tch}+n)}{n[\gamma+\Xi_{tch}+n+1]} a_0$.

Hence, the Eigenfunction for this case is

$\Psi_2 = a_0(1 + \eta_1 z + \eta_2 z^2)\exp(\frac{z^3}{6} + \frac{z^2}{4})$.

Here, $\eta_2 = \frac{\Xi_{tch}(\gamma+\Xi_{tch}+n)}{n[\gamma+\Xi_{tch}+n+1]}$.

In this case, the operator $\mathcal{S}$ has a finite-dimensional invariant subspace $\mathcal{P}_{n+1}$, which is spanned by the basis $\{z^k | k = 0,1,2,\ldots,n\}$. By the finite polynomial $\mathcal{P}_n(z) = \sum_{k=0}^n a_k z^k$ it is possible to obtain the tridiagonal Jacobi matrix equation $T_{n+1}A_n = 0$ which explicitly is given by

$$\begin{pmatrix} \Xi_{tch} & n & 0 & 0 & \ldots & 0 \\ 0 & \gamma + \Xi_{tch} & 1+n & 0 & \ldots & 0 \\ 2 & 0 & 2\gamma + \Xi_{tch} & 2+n & \ldots & 0 \\ 0 & 6 & 0 & \ddots & \ddots & 0 \\ \vdots & 0 & 12 & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & \vdots & (n-1)\gamma + \Xi_{tch} & 2n \\ 0 & \ldots & & 0 & 0 & n\gamma + \Xi_{tch} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = 0 \quad (14)$$

It is known (see: Agarwal and Elena 2017, Example 3.2, p.25) that $D_{n+1} = \det(T_{n+1})$,

$$D_{n+1} = (n\gamma + \Xi_{tch})D_n, \quad D_{-1} = 0, D_0 = 1, D_1 = \Xi_{tch}. \quad (15)$$

The auxiliary equation associated with (15) is

$$\lambda^{n+1} - (n\gamma + \Xi_{tch})\lambda^n = 0 \quad (16)$$

Whose roots are

$$\lambda = 0 \quad \text{or} \quad \lambda = n\gamma + \Xi_{tch}, \quad (n - \text{times}). \quad (17)$$

Thus,

$D_{n+1} = k_1[n\gamma + \Xi_{tch}]^{n+1}$

Following the initial conditions, when $n = -1, D_0 = k_1 = 1$. hence,

$D_{n+1} = [n\gamma + \Xi_{tch}]^{n+1}$.

Since, $A_{n+1} \neq 0, D_{n+1} = 0 n\gamma + \Xi_{tch} = 0 q = \frac{n\gamma}{2}$. we observe that the eigenfunction $\Psi_n(z) = \exp(\frac{z^3}{6} + \frac{z^2}{4})\mathcal{P}_n(z)$ where $\mathcal{P}_n(z) = \sum_{k=0}^n a_k z^k$ and a_k satisfy the recurrent relations

$(k+n)a_{k+1} + (k\gamma + \Xi_{tch})a_k + k(k-1)a_{k-2} = 0.$

$a_1 = -\frac{\Xi_{tch}}{n} a_0 = \eta_1 a_0;$

$a_2 = \frac{(\gamma+\Xi_{tch})\Xi_{tch}}{n(n+1)} a_0 = \eta_2 a_0;$

$a_3 = -\left[\frac{(2\gamma+\Xi_{tch})(\gamma+\Xi_{tch})\Xi_{tch}}{n(n+1)(n+2)} + \frac{2}{(n+2)}\right] a_0 = \eta_3 a_0;$

$\vdots = \vdots$

$a_n = \left[(-1)^k \frac{\prod_{k=0}^{n-1}(k\gamma+\Xi_{tch})}{(2n-1)_k} + \frac{2}{(2n-1)_{n-2}} + (-1)^{n-2}\frac{(n-1)!}{n(2n-1)_{n-3}}\Xi_{tch}\right] a_0 = \eta_n a_0.$

Here, the Pochammer symbol $(s)_n = s(s-1)\cdots(s-n+1)$.

$\Psi_n(z) = a_0(1 + \sum_{k=1}^n \eta_k z^k)\exp(\frac{z^3}{6} + \frac{z^2}{4})$.

CONCLUSION

In this work, we have presented a new algebraisation of the TCHE by writing it as the linear combination of quadratic elements in the universal enveloping algebra of $sl(2,\mathbb{C})$. We have seen that with Krall's classification of the Tri-confluent Heun Hamiltonian, it is exactly solvable when $\alpha = -n$. We also have obtained a new exactly solvable potential from TCHE using suitable guage transformations. New eigenfunctions and eigenvalues of TCHE have been obtained by letting its operator act on monomials which form the basis of a block triangular matrix.

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