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A PREDICTOR- CORRECTOR HYBRID METHOD FOR DIRECT SOLUTION OF MODELED REAL LIFE PROBLEMS OF SECOND ORDER ODES

[1]Friday Oghenerukevwe Obarhua, [1]Adelegan Momoh Lukman and [2]Adamu Bala

[1]Department of Mathematical Sciences, The Federal University of Technology, Akure, Ondo State, Nigeria.
[2]Department of Basic Sciences, Federal Polytechnic, Auchi, Edo State, Nigeria.

*Corresponding Auhtor's Email: adamubalahod@gmail.com

ABSTRACT

This paper develops a Chebyshevian hybrid linear multi-step method to solve general second-order ordinary differential equations (O.D.Es). The development of the method utilized the chebyshev polynomial of the first kind as the basis function for the approximate solution. The interpolation of the basis function was done at both grid and off grid points. While the differential systems are collocated at all grid points for step number k=2. The required continuous hybrid method is produced by the substitution of the unknown parameters into the basis function and the simplification of the resulting equations. The inherent demerit of predictor methods of lower order is circumvented by deriving predictors of same order with the methods. The methods were applied to solve real life second order initial value problems directly. The errors in the results obtained were compared to those from existing methods. The new results obtains has a better performance than the existing methods..

Keywords: Linear Multistep Method, Hybrid Point, Chebyshev Polynomial, Collocation and Interpolation, Predictor-Corrector Method

INTRODUCTION

Mathematics today is a diverse discipline that deals with data, measurement and observation from science with interference deduction and proof with mathematical models of natural phenomena of human behavior and social systems.

A mathematical model can be broadly defined as a formulation or an equation that expresses the essential features of physical system or process in mathematical terms. Models can be represented by a functional relationship between dependent variables, independent variables, parameters and forcing functions.

These models often yield equations that contain some derivatives of an unknown function of one or more several variables "Such equations are called a differential equation."

"In mathematics, a differential equation can be defined as an equations that contains some derivatives of an unknown function of one or more variables", while in the field of differential equations, an initial value problem (IVP is a differential equation with a set of additional constraints called the initial conditions. In applications, the functions usually represent physical quantities, the derivative represent their rates of change, and the equation defines a relationship between the two because such relationship are extremely common.

In this paper, we consider numerical methods for solving second order initial value problems of ordinary differential equations of the form

$$y''(x) = (x, y, y'),\ y(x_0) = y_o,\ y'(x_0) = y0' \qquad (1)$$

Equation (1) arises from many physical phenomena in a wide variety of applications especially in engineering problems such as resonance vibration of machine, cooling of a body and other area of application. Many authors have worked at solving (1) numerically by reducing it to a system of first order equations [James *et.al.* (2013), Singh and Ramos (2018), Akinfenwa *et.al.* (2020), Ogunniran *et.al.* (202)] and others. In spite of the success of this approach, it suffers some setbacks, according to [Awoyemi *et.al.* (2014), Kayode and Obarhua (2015), Areo and Rufai (2016)], the setback is, un-economical in term of cost implementation, increased computational burden and wastage of computer time, increased dimension of the resulting systems of equations to be solved. This led many authors to attempt at solving problem (1) directly without reduction to systems of first order equation. Badmus A.M. and Yahaya Y.A (2009), Develop an implicit collocation method for direct solution of second order ordinary differential equation directly. Awari and Abada (2014) presented a class of seven point zero stable continuous block method for solution of second order differential equation to solve (1). Kuboye et.al.(2018) Generalized hybrid block method for solving second order ordinary differential directly. Kayode et.a. (2021) develop two step continuous trigonometric ally fitted method for solving oscillatory second order (ODEs). Olanegan et. al. (2018) implicit hybrid points approach for solving general second order and the method is of order five. Like wise Adegboro (2022) develop a trigonometrically fitted predictor-corrector method for solving oscillatory second order ordinary differential equations without reduction to a system of first order. In this paper, efforts are made to develop a class of chebyshev polynomial to solve second order directly. The method is consistent, symmetric and of smaller error constant, and can effectively handle mildly stiff and Real life problems.

MATERIALS AND METHODS

This work consider numerical methods for solving second order initial value problems of ordinary differential equations of the form

$$Y''(x) = (x, y, y'),\ y(x_0) = y_o,\ y'(x_0) = y0', \qquad (2)$$

using Chybyshev polynomial of the form

$$y\,(x) = \sum_{i=0}^{(c+i)-1} a_i\,T_i\,(x) \qquad (3)$$

where c and i represent the number of collocation and interpolation points respectively, $a'is$ are parameters to be determined, $T_i(x)$ are continuous and differentiable terms of Chebyshev polynomial. The first and second derivatives of (2) yield

$$y'(x) = \sum_{i=0}^{(c+i)-1} a_i\,T'_i\,(x). \qquad (4)$$

$$y''(x) = \sum_{i=0}^{(c+i)-1} a_i\,T''_i\,(x). \qquad (5)$$

And Combining (1) and (4) gives

$$f(x, y, y') = \sum_{i=0}^{(c+i)-1} a_i\,T''_i\,(x). \qquad (6)$$

Interpolation (2) at points x_{n+i}, $i = 0, r, s, 1$ and Collocating (4) at all grid points x_{n+c}, $c = 0, 1, 2$ where r $\varepsilon(0, 1)$ giving rise to a system of $c + i$ equations

$$AX = B \tag{7}$$

Where x =

$$\begin{bmatrix} 1 & xn & -1+2r_n^2 & -3x_n+4x_n^3 & 1-8x_n^2+8x_n^4 & 5x_n-20x\,{}^{3}_{4}+16x\,{}^{5}_{n} & -1+18x_n^2-48x_n^4+32x_n^6 \\ 1 & xn+rh & -1+2(x_n+rh)^2 & -3(x_n+rh)+4(x_n+rh)^3 & 1-8(x_n+rh)^2+8(x_n+rh)^4 & 5(x_n+rh)-20\,(x_n+rh)^3+16\,(x_n+rh)^5 & -1+18(x_n+rh)^2-48(x_n+rh)^4+32(x_n+rh)^6 \\ 1 & xn+sh & -1+2(x_n+sh)^2 & -3(x_n+rh)+4(x_n+sh)^3 & 1-8(x_n+sh)^2+8(x_n+sh)^4 & 5(x_n+sh)-20\,(x_n+sh)^3+16\,(x_n+sh)^5 & -1+18(x_n+sh)^2-48(x_n+sh)^4+32(x_n+sh)^6 \\ 1 & xn+h & -1+2(x_n+h)^2 & -3(x_n+h)+4(x_n+h)^3 & 1-8(x_n+h)^2+8(x_n+h)^4 & 5(x_n+h)-20\,(x_n+h)^3+16\,(x_n+h)^5 & -1+18(x_n+h)^2-48(x_n+h)^4+32(x_n+h)^6 \\ 0 & 0 & 4 & 24x_n & -16+96x_n^2 & -120x_n+320x_n^3 & 36-57x_n^2+960x_n^4 \\ 0 & 0 & 4 & 24(x_n+h) & -16+96((x_n+h)^2 & -\,120\,(x_n+h)+320\,(x_n+h)^3 & 6-576(x_n+h)^2+960(x_n+h)^4 \\ 0 & 0 & 4 & 24(x_n+2h) & -16+96((x_n+2h)^2 & -\,120\,(x_n+2h)+320\,(x_n+2h)^3 & 36-576(x_n+2h)^2+960(x_n+2h)^4 \end{bmatrix}$$

$$\text{A}=\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \quad B = \begin{bmatrix} y_n \\ y_{n+r} \\ y_{n+s} \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \end{bmatrix}$$

The values of $ai's$ were determined, using Gaussian elimination method.

substituting the values of a_1, a_2, $a_3 \ldots a_6$ in to equation 2 to have the required continuous coefficients hybrid linear multi step method.

$$y_k(t) = \alpha_0(t)y_n+\alpha_r(t)y_{n+r}+\alpha_s(t)y_{n+s}+\alpha_1(t)y_{n+1}+h^2(\beta_0(t)f_n+\beta_1(t)f_{n+1}+\beta_2(t)f_{n+2}) \tag{8}$$

Applying the transformation $t = \left(\frac{x-xn+k-1}{h}\right), \frac{dt}{dx} = \frac{1}{h}$ t ε (0, 1) in Obarhua (2022), to (8),

its continuous coefficients $\alpha jl\ s$, and $\beta jl\ s$, as functions of t obtained to be

$\alpha_0(t) = \frac{A_o}{B_o}, \alpha_1(t) = \frac{A_1}{B_1}$,, $\alpha_2(t)\,\frac{A_2}{B_2}$, $\alpha_3(t) = \frac{A_3}{B_3}$, while $\beta_0(t)\,\frac{C_o}{B_4}, \beta_1(t) = \frac{C_1}{B_4}, \beta_2(t)\,\frac{C_2}{B_4}$.

Where

$$A_o = \left\{\begin{matrix} (-1+r-t)t(1-s+t)(11-22t-17t^2+16t^3+s^2(-47+61t+19t^2-24t^3)+2s^3(7-7t-3t^2+3t^3 \\ +s(22-55t-t^2+16t^3)+2r^3(7+3s^3+3s^2(-3+t)-7t-3t^2+3t^3+s(-1-6t+3t^2)+r^2 \\ (-47+6s^3(-3+t)+61t+19t^2-24t^3+s^2(49-36t+6t^2)+s(16+29t-30t^2+6t^3+r(22-55 \\ t-t^2+16t^3+2s^3(-1-6t+3t^2)+s(-36+28t+35t^2-24t^3)+s^2(16+29t-30t^2+6t^3)) \end{matrix}\right\} A_1 =$$

$$-\left\{\begin{matrix} \mathrm{t(1+t)(-(1+t)^2(11-33t+16t^2)-s(1+t)^2(11-33t+16t^2)+2s^4(7-7t-3t^2+3t^3)} \\ \mathrm{+s^3(-61+61t+39t^2-24t^3-6t^4)+3s^2(23-23t-27t^2+7t^3+8t^4))} \end{matrix}\right\}$$

$$A_2 = \left\{\begin{matrix} \mathrm{-\,t(1+t)(-(1+t)^2(11-33t+16t^2)-r(1+t)^2(11-33t+16t^2)+2r^4(7-7t-3t^2+3t^3)} \\ \mathrm{+r^3(-61+61t+39t^2-24t^3-6t^4)+3r^2(23-23t-27t^2+7t^3+8t^4)} \end{matrix}\right\}$$

$$A_3 = \left\{\begin{matrix} (-1+r-t)(1-s+t)(1+t)(2r^3(8+3s^3+3s^2)+2s^2(-4+t-6t^2+3t^3+s(8-9t+3t^2)) \\ \mathrm{+s(1+t)(20(3-5t+2t^2)+2s^2(8-9t+3t^2)-5s(13-17t+6t^2)+r^2(-65+6s^3(-4+t)} \\ \mathrm{+20t+55t^2-30t^3+s^2(91-48t+6t^2)+s(-49+83t-42t^2+6t^3))+r(2s^3(8-9t+3t^2)} \\ \mathrm{+20(3-2t-3t^2+2t^3)+s^2(-49+83t-42t^2+6t^3)-5s(1+16t-19t^2+6t^3)} \end{matrix}\right\}$$

$$C_0 = \left\{\begin{matrix} \mathrm{h^2(-1+r-t)(-1+s-t)t(1+t)(66-198t+96t^2+s(-91+207t-92t^2)+s^2(25-50t+21t^2)} \\ \mathrm{+r^2(25-50t+21t^2+s(-34+43t-15t^2)+3s^2(3-3t+t^2))+r(-91+207t-92t^2} \\ \mathrm{+s^2(-34+43t-15t^2)+s(125-193t+73t^2)/12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^2)} \\ \mathrm{+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3)} \end{matrix}\right\}$$

$$C_1 = \left\{\begin{matrix} \mathrm{h^2(-1+r-t)t(1+t)(1-s+t)(-s(1+t)(52-28t+s(-19+9t))+r^2(19+10t-9t^2+s(38)} \\ \mathrm{+s^3(-61+61t+39t^2-24t^3-6t^4)+3s^2(23-23t-27t^2+7t^3+8t^4))))} \end{matrix}\right\}$$

$$C_2 = \left\{\begin{matrix} \mathrm{h2(-1+r-t)(-1+s-t)t(1+t)(-s(1+t)(1-4t+s(-1+3t))+r^2(1-2t-3t^2} \\ \mathrm{s(2-5t-3t^2)+3s^3(-1+t+t^2)+r(-1+3t+4t^2+s^2(2-5t-3t^2)+s(-1+5t+t^2))))} \end{matrix}\right\}$$

$$B_0 = \{\mathrm{rs(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2+16s^3)-r2(65+49s-91s^2} \\ \mathrm{+\,24s^3))}\}$$

$$B_1 = \left\{\begin{matrix} \mathrm{(-1+r)r(s^2(-60+65s-16s^2)+rs^2(-60+65s-16s^2)+2r^4(8+8s-12s^2} \\ \mathrm{+3s^3)-r^3(65+65s-75s^2+6s^4)+3r^2(20+20s-25s^3+8s^4))} \end{matrix}\right\}$$

$$B_2 = \left\{\begin{matrix} \mathrm{(-1+s)s(s^2(60-65s+16s^2)+rs^2(60-65s+16s^2)-2r^4(8+8s-12s^2+3s^3)} \\ \mathrm{+\,(r^3(65+65s-75s^2+6s^4)-3r^2(20+20s-25s^3+8s^4))} \end{matrix}\right\}$$

$$B_3 = \left\{\begin{matrix} \mathrm{(-1+r)(-1+s)(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^2)} \\ \mathrm{+r(60-5s-49s^2+16s^3)-r^2(65+49s-91s^2+24s^3))} \end{matrix}\right\}$$

$$B_4 = \left\{\begin{matrix} \mathrm{(12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2} \\ \mathrm{16s^2)-r^2(65+49s-91s^2+24s^2))} \end{matrix}\right\}$$

$$B_5 = \left\{\begin{matrix} \mathrm{(12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2} \\ \mathrm{16s^2)-r^2(65+49s-91s^2+24s^2))} \end{matrix}\right\}$$

$$B_6 = \left\{\begin{matrix} \mathrm{(12(s(60-65s+16s^2)+2r^3(8+8s-12s^2+3s^3)+r(60-5s-49s^2} \\ \mathrm{16s^2)-r^2(65+49s-91s^2+24s^2))} \end{matrix}\right\}$$

$$B_4 = \{\mathrm{B_5 = B_6 = B_4}\}$$

Evaluating the coefficients value and the first derivative of the coefficients at t=1 yield the discrete scheme.

1
yn+2 = —
0 C
1

$\alpha_0 y_n -$
1 $\overline{C}$
1
$\alpha 1 yn+r +$
2 $\overline{C}$
1
$\alpha 2 yn+s -$
3 $\overline{C}$
$h2\ \alpha 3 yn+1 + 6C$ $\overline{}$
$[-\beta 0 f_n + 2\beta_1 f_{n+1} + \beta_2 f_{n+2}]$ 4 (9)

with first derivative

$$y'_{n+2} = \frac{1}{C_0}\alpha'_0 y_n - \frac{1}{C_1}\alpha'_1 y_{n+r} + \frac{1}{C_2}\alpha'_2 y_{n+s} - \frac{1}{C_3}\alpha'_3 y_{n+1} + \frac{h^2}{6C_4}[-\beta 0' f_n + 2\beta'_1 f_{n+1} + \beta'_2 f_{n+2}] \tag{10}$$

Where

$\alpha_0 = (-2 + r)(2 - s)(-12 - 18s + 9s^2 + r^2(9 + 21s + 19s^2 - 12s^3) + r(-18 + 3s + 21s^2 - 8s^3) + 2r^3(-4s - 6s^2 + 3s^3))$
$\alpha_1 = 2(24 + 24s - 36s^2 + 9s^3)$
$\alpha_2 = 2(24 + 24r - 36r^2 + 9r^3)$
$\alpha_3 = 2(-2 + r)(-2 + s)(2s(-10s + 4s^2) + r^2(-20 - 2s + 49s^2 - 18s^3) + 2r^3(4 + 2s - 9s^2 + 3s^3) + r(-20s - 2s^2 + 4s^3))$
$\beta_0 = h^2(-2 + r)(-2 + s)(-36 + 24s - 4s^2 + r(24 + 5s - 6s^2) + r^2(-4 - 6s + 3s^2))$
$\beta_1 = h^2(-2 + r)(2 - s)(-2(24 - 10s)s + r^2(20 + 30s - 15s^2) + r(-48 - 52s + 30s^2))$ $\beta_2 = h^2(-2 + r)(-2 + s)(-2s(-3 + 2s) + r(6 + 5s - 6s^2)$
$+ r^2(-4 - 6s + 3s^2))$
$C_0 = rs(s(60 - 65s + 16s^2) + 2r^3(8 + 8s - 12s^2 + 3s^3) + r(60 - 5s - 49s^2 + 16s^3) -$
$r^2(65 + 49s - 91s^2 + 24s^3))$
$C_1 = (-1 + r)r(s^2(-60 + 65s - 16s^2) + rs^2(-60 + 65s - 16s^2) + 2r^4(8 + 8s - 12s^2 + 3s^3) - r^3(65 + 65s - 75s^2 + 6s^4) + 3r^2(20 + 20s - 25s^3 +$
$8s^4))$
$C_2 = (-1 + s)s(s^2(60 - 65s + 16s^2) + rs^2(60 - 65s + 16s^2) - 2r^4(8 + 8s - 12s^2 + 3s^3) +$
$r^3(65 + 65s - 75s^2 + 6s^4) - 3r^2(20 + 20s - 25s^3 + 8s^4))$
$C_3 = (-1 + r)(-1 + s)(s(60 - 65s + 16s^2) + 2r^3(8 + 8s - 12s^2 + 3s^3) + r(60 - 5s - 49s^2 + 16s^3) - r^2(65 + 49s - 91s^2 + 24s^3))$
$C_4 = 6(s(60 - 65s + 16s^2) + 2r^3(8 + 8s - 12s^2 + 3s^3) + r(60 - 5s - 49s^2 + 16s^3) - r^2(65 + 49s - 91s^2 + 24s^3))$

and

$\alpha' 0 = 2r^3(3s^2 - 12s + 8) + r^2(6s^3 - 48s^2 + 107s - 49) + r(-24s^3 + 107s^2 - 114s - 5) +$
$16s^3 - 49s^2 - 5s + 602r^3(3s^3 - 12s^2 + 8s + 8) + r^2(-24s^3 + 91s^2 - 49s - 65) + r(16s^3 -$
$49s^2 - 5s + 60) + 16s^3 - 65s^2 + 60s$
$\alpha'_o = \mathrm{rs}(2r^3\,(3s^3 - 15s^2 + 20s) + r^2\,(-30s^3 + 145s^2 - 180s) + r\,(40s^3 - 180s^2 + 200s\alpha'_2$
$= 16s^4 - 65s^3 + 60s^2$
α'_3
$\beta' 0$
$\beta' 1\ \beta' 2$
$= 16r^4 - 65r^3 + 60r^2$
$= h^2 rsf_n\,(r^2\,(21s^2 - 92s + 96) + r\,(-92s^2 + 391s - 390) + 96s^2 - 390s + 360)$
$= h^2 rsf_{n+1}\,(r^2\,(28s - 9s^2) + r\,(28s^2 - 80s))$
$= h^2 rs\,(r^2\,(4s - 3s^2) + r\,(4s^2 - 5s))$
$C' 0 = s(2r^3(3s^3 - 12s^2 + 8s + 8) - r^2(24s^3 - 91s^2 + 49s + 65) + r(16s^3 - 49s^2 - 5s +$
$60) + s(16s^2 - 65s + 60))$
$C' 1 = (r - 1)(s - 1)(2r^3(3s^3 - 12s^2 + 8s + 8) - r^2(24s^3 - 91s^2 + 49s + 65) + r(16s^3 -$
$49s^2 - 5s + 60) + s(16s^2 - 65s + 60))$
$C' 2 = (r - 1)r(2r^4(3s^3 - 12s^2 + 8s + 8 - r^3(6s^4 - 75s^2 + 65s + 65) + 3r^2(8s^4 - 25s^3 + 20s + 20) + rs^2(-16s^2 + 65s - 60) + s^2(-16s^2 + 65s -$
$60))$
$C' 3 = (s - 1)s(-2r^4(3s^3 - 12s^2 + 8s + 8) + r^3(6s^4 - 75s^2 + 65s + 65) - 3r^2(8s^4 - 25s^3 + 20s + 20) + rs^2(16s^2 - 65s + 60) + s^2(16s^2 - 65s + 60))$
$C' 4 = (r - 1)r(2r^4(3s^3 - 12s^2 + 8s + 8) - r^3(6s^4 - 75s^2 + 65s + 65) + 3r^2(8s^4 - 25s^3 + 20s + 20) + rs^2(-16s^2 + 65s - 60) + s^2(-16s^2 + 65s -$
$60))$

in order to test the properties of the continuous methods (8), we choose the values of r and s at various points in the interval of (0,1) to obtain our derived discrete hybrid methods. Therefore, for the purpose of testing, the values of r and s are taken to $\frac{1}{5}, \frac{4}{5}$ respectively to have

$$y_{n+2} = \left[\frac{2927204}{369671} y_n - \frac{17828125}{369671}\, y_{n+\frac{2}{5}} + \frac{1850000}{369671}\, y_{n+\frac{3}{5}} - \frac{3229408}{369671}\, y_{n+1}\right] - \tag{11}$$

$$y'_{n+2} = \left[\frac{43812599}{2218026} y_n - \frac{266434375}{2218026}\, y_{n+\frac{2}{5}} + \frac{142287500}{1109013}\, y_{n+\frac{3}{5}} - \frac{30976612}{1109013}\, y_{n+1}\right] - \tag{12}$$

Analysis of Basic Properties of the Methods

Order and Error Constant of the Methods

let the linear difference operator l associated with the continuous method be defined as

$$L[y(x); h] = \sum_{j=0}^{k} [\alpha_r y\ (x + rh) - h^n \beta_r f\ (x\ +\ rh), \ldots, yv(x\ +\ rh)];\ r\ =\ 1, 2, \ldots, m \tag{13}$$

$y(x)$ represent an arbitrary test function that is continuously differentiable in the interval [a,b]. Taylor series expansion of $y(x + rh)$ and $y''(x + rh)$, r =1,2,...,m about x_n and collecting the like terms in h and y yields

$$L[y(x); h] = \bar{c}_0 y(x) + \bar{c}_1 h y^{I}(x) + \bar{c}_2 h^2 y^{II}(x) + \ldots + \bar{c}_p h^p y^p(x) \tag{14}$$

The difference operator L and the continuous multi step method are said to br of order

$c_0 = c_1 = c_2 = \ldots = c_p = 0,\ c_{p+2} \neq 0$

The term C_{p+2} is called the error constant and it implies that the local truncated error (LTE) is given by $(LTE) = c_{p+1} h^{p+1} y^{p+1}(x_n) + 0(h^{p+3})$. (lambert 1973)

$$y_{n+2}=\left[\frac{2927204}{197506}y_n-\frac{2687500}{98753}\,y_{n+\frac{1}{5}}+\frac{5953125}{197506}\,y_{n+\frac{4}{5}}-\frac{1646244}{98753}\,y_{n+1}\right]-\frac{h^2}{98753}\,[27756f_n-143640f_{n+1}-5994f_{n+2}]$$

its first derivative

$$y'_{n+2}=\left[\frac{1424436}{395012}y_n-\frac{1960000}{296259}\,y_{n+\frac{1}{5}}+\frac{9400375}{1185036}\,y_{n+\frac{4}{5}}-\frac{4861872}{98753}\,y_{n+1}\right]-\frac{h^2}{98753}\,[27756f_n-143640f_{n+1}-26973f_{n+2}]$$

$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0,\ C_7 = \frac{423}{493765} \approx 8.566\times 10^{-4}$

Hence the method is of order 5 with error constant C_7=8.566×10^{-4} and for the first derivative

$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0,\ C_7 = \frac{198328}{86408875} \approx 2.2952\times 10^{-3}$

Hence the first derivative of the method is of order 5 with error constant C_7=2.2952×10^{-3}

The main predictor

$$y_{n+2}=\frac{1}{4366}\left[284629y_n-2746875\,y_{n+\frac{4}{5}}-4250000\,y_{n+\frac{1}{5}}+\ 133878y_{n+1}\right]-\frac{h^2}{2183}\left[15093f_{n+1}+675435f_{n+\frac{1}{5}}-1890f_n\right]$$

with its first derivative

$$y'_{n+2}=\frac{1}{26196}\left[76055679y_n+72690652\,y_{n+\frac{4}{5}}-11327500\,y_{n+\frac{1}{5}}+\ 354571304y_{n+1}\right]-\frac{h^2}{4366}\left[120261f_{n+1}+675435f_{n+\frac{1}{5}}-19488f_n\right]$$ with its first derivative

$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0,\ C_7 = \frac{6921}{2728750} \approx 2.53363\times 10^{-3}$

Hence the main predictor of the method is of order 5 with error constant C_7=2.5363×10^{-3}

and

$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0,\ C_7 = \frac{495613}{38202500} \approx 1.2973\times 10^{-2}$

Hence the first derivative of the main predictor of the method is of order 5 with error constant C_7=1.2973×10^{-2}

Consistency

The linear multi-step method for [ivp] is said to be consistent if the following conditions are satisfied:

i. the order of the scheme must be greater than or equal to 1 ($p \geq 1$)
ii. $\sum_{j=0}^{k}\alpha_j = 0, \alpha'_j s$ are the coefficients of the first characteristic polynomial
iii. $\rho(r) = \rho^l(r) = 0$ for r=1

$\rho''(r) = 2!\sigma(r)$ for r=1. $\rho(r)$ and $\sigma(r)$ are first and second characteristic polynomial respectively.

consistency of the methods

$$y_{n+2}=\left[\frac{2927204}{197506}y_n-\frac{2687500}{98753}\,y_{n+\frac{1}{5}}+\frac{5953125}{197506}\,y_{n+\frac{4}{5}}-\frac{1646244}{98753}\,y_{n+1}\right]-\frac{h^2}{98753}\,[27756f_n-143640f_{n+1}-5994f_{n+2}]$$

condition (i) is satisfied since the scheme is of order 5

condition (ii)is satisfied since $\displaystyle\sum_{j=0}^{k}\alpha_j = 0$

$\alpha_0 = \frac{2911869}{197506}, \alpha_1 = -\frac{1646244}{98753},\ \alpha_{\frac{1}{5}} = -\frac{2687500}{98753}, \alpha_{\frac{4}{5}} = -\frac{5953125}{197506}, \alpha_2 = 1$

condition (iii) is satisfied since $\rho(r) = \rho'(r) = 0$ for r =1

$\rho'(\mathrm{r}) = \left[r^2-\left(\frac{5953124r^{4/5}}{197506}-\frac{1646244r^1}{98753}-\frac{2687500\sqrt[5]{r}}{98753}+\frac{2911869}{197506}\right), r \to 1\right] = 0$

$\rho'(\mathrm{r}) = \left[r^2-\left(\frac{537500}{98753\sqrt[5]{r}}+2r\frac{2381250}{98753\sqrt[5]{r}}+\frac{2911869}{197506}\right), r \to 1\right] = 0$

condition (iv) is satisfied since $\rho''(r) = 2!\sigma(r)$ for r=1. $\rho(r)$ and $\sigma(r)$ are first and second characteristic polynomial respectively.

$\rho''(r) = -\frac{430000}{98753r^{4/5}}+\frac{476250}{98753r^{6/5}}+\ 2$

and

$\sigma(r) = \frac{162r^2}{2669}+\frac{143640r^1}{98753}-\frac{27756}{98753}$

therefore, if r=1, then $\rho''(r) = 2!\sigma(r) = \frac{6588}{2669}$

hence, the four conditions of consistency are satisfied, the 2-step method is consistent.

Zero Stability

The linear multi step Method is Said to be Zero Stable if no Root of the First Characteristic Polynomial p(r) has a Modulus Greater than one and if Every Root of Modulus one has Multiplicity not more than one.

The first characteristic polynomial

$\rho(\mathrm{r}) = \left[r^2-\left(\frac{5953124r^{4/5}}{197506}-\frac{1646244r^1}{98753}-\frac{2687500\sqrt[5]{r}}{98753}+\frac{2911869}{197506}\right), r \to 0\right] = 1$

Equating to zero and solving for r gives r =1

The root r of 2-step method for which $|r|$ =1 is simple, hence the method is zero stable

Convergence

Theorem 1: convergence - The necessary and sufficient condition for a linear multi step method to be convergent is for it to be consistent and zero stable. From the theorem above, the new hybrid predictor - corrector method is convergent.

Region of Absolute Stability of the Methods

considering the stability polynomial of the linear multistep method defined by ρ and σ, $\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0$, where $\bar{h} = h\lambda$.

Definition: A linear multistep method is said to be absolutely stable for a given value $\bar{h}$, if for all roots of $|r_s|$ of $\pi(r, h) = \rho(r, h)$ and $\bar{h}\vartheta(r) = 0$ satisfies $|r_s| = 1$, $s = 1, 2,\ k$ and absolutely unstable for that $\bar{h}$ otherwise. The region of absolute stability is the region

where the stability of the method is guaranteed. We shall adopt the boundary locus method proposed by lambert (1973). The boundary locus method is given by

$$\bar{h}(\mathrm{r}) = \frac{p\,(r)}{\sigma\,(r)} \tag{15}$$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomial respectively, taking

$$r = e^{i\theta} = cos\theta + isin\theta \tag{16}$$

Applying equation (15) and (16) to our method we have

$$\bar{h}(\mathrm{r}) = \frac{r^2 - \left(\frac{5953125r^{4/5}}{197506} - \frac{1646244r^1}{98753} - \frac{2687500\sqrt[5]{r}}{98753} + \frac{2911869}{197506}\right)}{\frac{162r^2}{2669} + \frac{143640r^1}{98753} - \frac{27756}{98753}} \tag{17}$$

By setting

r = $\exp^{i\theta}$, $0 \le \theta \le \pi$, where $e^{i\theta} = \cos\theta + isin\theta$, then in equation (3.105), we have

$$\bar{h}(\mathrm{r}) = \frac{e^{2i\theta} - \left(\frac{5953125e^{40/5}}{197506} - \frac{1646244e^{i}\theta}{98753} - \frac{2687500e^{i\frac{\theta}{5}}}{98753} + \frac{2911869}{197506}\right)}{\frac{162e^{2i\theta}}{2669} + \frac{143640e^{i\theta}}{98753} - \frac{27756}{98753}} \tag{18}$$

$$\bar{h}(\theta) = \frac{\cos(2\theta) + isin(2\theta) - \left(\frac{5953125\left(cos\frac{40}{5} + isin\frac{40}{5}\right)}{197506} - \frac{1646244(\cos\theta + isin\theta)}{98753} - \frac{2687500\left(cos\left(\frac{\theta}{5}\right) + isin\left(\frac{\theta}{5}\right)\right)}{98753} + \frac{2911869}{197506}\right)}{\frac{162e^{2i\theta}}{2669} + \frac{143640e^{i\theta}}{98753} - \frac{27756}{98753}} \tag{19}$$

$$\bar{h}(\theta) = \frac{\left(\frac{2687500\left(cos\frac{\theta}{5} + isin\frac{\theta}{5}\right)}{98753} - \frac{1646244(\cos\theta + isin\theta)}{98753} + \cos 2\theta + isin2\theta - \frac{5953125\left(cos\frac{40}{5} + isin\frac{40}{5}\right)}{197506} - \frac{2911869}{197506}\right)}{\frac{143640(\cos\theta + isin\theta)}{98753} + \frac{162(\cos\theta + isin\theta)}{2669} - \frac{27756}{98753}} \tag{20}$$

$5375000\left(cos\frac{\theta}{5} + isin\,\frac{\theta}{5}\right)$ - $5953125\left(cos\frac{40}{5} + isin\,\frac{40}{5}\right)$

$$\bar{h}(\theta) = + \frac{3292488\ (\cos\theta + \mathrm{i}\sin\theta) + 197506\,(\cos 2\theta + \mathrm{i}\sin 2\theta) - 2911869}{108[2660(\cos\theta + \mathrm{i}\sin\theta) + 111\,(\cos 2\theta + \mathrm{i}\sin 2\theta) - 514]} \tag{21}$$

multiplying by conjugate gives $\bar{h}\,(\theta) = x\,(\theta) + iy(\theta)$ and considering the real part gives

$(5375000\cos\frac{\theta}{5}$ - $5953125\cos\frac{4\theta}{5}$ + $3292488\cos\theta$ $+197506\cos2\theta$ – $2911869)$

$$\bar{h}\,(\theta) = + \frac{i(5375000sin\frac{\theta}{5} - 5953125sin\frac{4\theta}{5} + 3292488\,sin\,\theta + 197506sin2\theta}{[108(2669\,cos(\theta) + 111\,cos(2\theta) - 514) + i(2660sin\theta + 111sin2\theta)]} \tag{22}$$

$$\bar{h}\,(\theta) = + \frac{5375000\cos\left(\frac{\theta}{5}\right) - 5953125\left(\frac{4\theta}{5}\right) + 3292488\cos(\theta) + 197506\cos(2\theta) - 2911869}{108(2660\cos(\theta) + 111\cos(2\theta) - 514)} \tag{23}$$

Evaluating at intervals of $(0^0, 180^0)$ gives (- 8.89600,0)

Implementation of the Methods

Implementation of the discrete method (10) obtained from (8) for problem (1) requires the generation of some starting values. This is obtained in predictor-corrector mode of the same order of accuracy. The following explicit Predictor scheme and its derivative, of the same order with the corrector scheme, are obtained using the same procedure in Section 2 for y_{n+2} and y'_{n+2}.

$$y_{n+2} = \frac{1}{4366}\left[284629y_n - 2746875\,y_{n+\frac{4}{5}} - 4250000\,y_{n+\frac{1}{5}} + 133878y_{n+1}\right] + \frac{h^2}{2183}\left[15093f_{n+1} + 74925f_{n+\frac{1}{5}} - 1890f_n\right] \tag{24}$$

$$y'_{n+2} = \frac{1}{26196}\left[76055679y_n + 72690652\,y_{n+\frac{4}{5}} - 11327500\,y_{n+\frac{1}{5}} + 354571304y_{n+1}\right] + \frac{h^2}{4366}\left[120261f_{n+1} + 675435f_{n+\frac{1}{5}} - 19488f_n\right] \tag{25}$$

Other explicit schemes were also generated to evaluate other starting values and Taylor's series expansion was used to evaluate the values for $y_{n+r,}$ that is

$$y_{n+r} = \mathrm{yn} + (\mathrm{rh})\,y'_n + \frac{(rh)^2}{2!}f_n + \frac{(rh)^3}{3!}\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n + \frac{\partial f_n}{\partial y'_n}\} + 0(\mathrm{h}^4) \tag{26}$$

$$y'_{n+r} = y'_n + (\mathrm{rh})\,f_n + \frac{(rh)^2}{2!}\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n + \frac{\partial f_n}{\partial y'_n}\} + 0(\mathrm{h}^4) \tag{27}$$

RESULTS AND DISCUSSION

The method is applied to solve the following stiff and real life second order initial value problems of ordinary differential equations directly without reduction to system of first order equations.

Problem 1

Dynamic Problem

Considered a problem in Engineering Dynamic Problem solved by Sabo et al. (2021).

A 10kg mass is attached to a spring having a spring constant of 140*N/m*. The mass is started in motion from the equilibrium position with an initial velocity of 1m/s in the upward direction and with an applied external force *F* (*t*) = 5*sint.* . Find the subsequent motion of the mass if the force due to air resistance is where is -90xN. Where m=10, k=140, a=90 and F(t)=5sint.

Upon simplification, the problem reduces to:
Solve $\left(\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x\,(t) = \frac{1}{2}(t), x\,(0) = 0, x'(0) = -1\right)$
To obtain the exact solution $x(t) = -\frac{9}{50}e^{-2t} + \frac{9}{500}e^{-7t} - \frac{9}{500}\cos(t) + \frac{13}{500}\sin(t)$

It follows that, the exponential terms, which come from the homogeneous solution represent an associated free over damped motion, quickly die out. These terms are the transient part of the solution. The terms coming from the particular solution, however, do not die out at t tends to infinity. They are the steady-state part of the solution.

Table 1: Comparison of Absolute Errors of Problem 1 with Exiting Methods

X-value	Exact-solution	Computed-solution	Error in New Method	Error in Ehigie *et al.*,(2010)	Error in Skwame *et al.*, (2017)
0.1	-0.064362051545524573	-0.064363716821136396	1.665276e-008	-	1.064677e-007
0.2	-0.084307205226447732	-0.084311263330738323	4.058104e-008	-	1.186950e-006
0.3	-0.084052253133900440	-0.084057834087608030	5.580954e-008	1.26e-05	2.263479e-006
0.4	-0.075293042133333835	-0.075299185053038023	6.142920e-008	1.66e-05	2.821855e-006
0.5	-0.063570639603558132	-0.063576670924472056	6.031321e-008	2.05e-05	2.953872e-006
0.6	-0.051421170693845074	-0.051426712465961047	5.541772e-008	2.41e-05	2.818687e-006
0.7	-0.039930529564386814	-0.039935416652912996	4.887089e-007	2.75e-05	2.546577e-006
0.8	-0.029498658628035464	-0.029502856255363242	4.197627e-007	3.07e-05	2.223454e-006
0.9	-0.020212691312590878	-0.020216235069677423	3.543757e-007	3.35e-05	1.899059e-006
1.0	-0.012026994254031254	-0.012029952070651674	2.957817e-007	3.60e-05	1.598776e-006

Source: Skwame et al. (2017)

Problem 2
(Real-life Problem) (www. mathinsite.bmth.ac.uk) Cooling of a Body
The temperature *y* degrees of a body, *t* minutes after being placed in a certain room,
satisfies the differential equation $3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$ By using the substitution $z = \frac{dy}{dt}$ or the
otherwise, find *y* in terms of *t* given that *y* = 60 when *t* = 0 and *y* = 35 when *t*6*In*4 Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulation of the Problem
$y" = \frac{(-y')}{3}, y(0) = 60, y'\,(0) = -\frac{80}{9}, h = \frac{1}{320}$

Analytical Solution
$y(x) = \frac{80}{3}\,e^{-}\left(\frac{1}{3}\right)x + \frac{100}{3}$

Table 2: Comparison of Absolute Errors of Problem 2 with Exiting Methods

X	Exact Solution	Computed Solution	Error in New Method	Error in Badmus et.al (2009)	Error in Olanegan et al. (2018)
0.1	59.125762679520165000	59.125761851677453000	8.278427e-007	3.83540e-05	7.476427e-06
0.2	58.280186267509812000	58.280183073229864000	3.194280e-006	7.50040e-04	2.939419e-05
0.3	57.462331147625598000	57.462324151610687000	6.996015e-006	1.05920e-4	6.480165e-05
0.4	56.671288507811937000	56.671276371414706000	1.213640e-005	1.35476e-04	1.127905e-05
0.5	55.906179330416393000	55.906160806831394000	1.852358e-005	1.55567e-04	1.724976e-04
0.6	55.166153415412850000	55.166127345107839000	2.607031e-005	1.86372e-04	2.431027e-04
0.7	54.450388435647504000	54.450353741982326000	3.469367e-005	1.96055e-04	3.238270e-04
0.8	53.758089023057281000	53.758044708102261000	4.431496e-005	2.21045e-04	4.139307e-04
0.9	53.088485884845788000	53.088431025373211000	5.485947e-005	2.05628e-04	5.127120e-04
1.0	52.440834948634347000	52.440768692268534000	6.625637e-005	2.77908e-04	6.195049e-04

Source: Olanegan et al. (2018)

Problem 3
Resonance Vibration of a Machine
A stamping machine applies hammering forces on metal sheets by a die attached to the plunger moves vertically up and down by a fly wheel spinning at constant set speed. The constant rotational speed of the fly wheel makes the impact force on the sheet metal, and therefore the supporting base, intermittent and cyclic. The bearing base on which the metal sheet is situated has a mass, *M* = 2000*kg* .The force acting on the base follows a function: *f* (*t*) = 2000*sin*(10*t*), in which *t* = *time* in seconds. The base is supported by an elastic pad with an equivalent spring constant *k* = 2 × 105*N/M*. Determine the

differential equation for the instantaneous position of the base $y(t)$ if the base is initially depressed down by an amount $0.1m$. [Kayode and Adegboro (2018)].

Solution

The mass spring system above is modeled as differential equation: The bearing base $mass = 2000kg$
Spring constant $k = 2 \times 10^5 \, N/M$
Force (ma) on the metal sheet = $m\frac{d^2y}{dt^2}$ = my''

i.e $ma = my'' = 2000sin(10t)$: where $a = y''$
Initial conditions on the system are
$y(t0) = y0; y(t0) = y(0) = 0.1, y(0) = 0$
Therefore, the governing equation for the instantaneous position of the base $y(t)$ is given by
$My'' + ky = F(t); y(t0) = y(0) = 0.1, y(0) = 0$

Theoretical Solution

$y(t) = \frac{1}{10}\, Cos\,(10t) + Sin(10t) \,-\, \frac{t}{20}\, Cos\,(10t)$

Table 3: Comparison of Absolute Errors of Problem 3 with Exiting Methods

x	Exact solution	Computed solution	Error in new method	Adegboro (2022)
0.1	0.099805157812128037	0.099805166345776630	8.533649e-009	5.001291880e-08
0.2	0.098788120701908358	0.098788150331618585	2.962971e-008	2.000545171e-07
0.3	0.096917121440198789	0.096917170760983273	4.932078e-008	4.501242281e-07
0.4	0.093137423045576082	0.093137466471499231	4.342592e-008	8.005089850e-07
0.5	0.089381288936266456	0.089381282088686342	6.847580e-009	1.251453858e-07
0.6	0.084879800636699518	0.084879686580073388	1.140566e-007	1.802820808e-06
0.7	0.079679648376031073	0.079679355525104301	2.928509e-007	2.454909110e-06
0.8	0.073833247429221927	0.073832690951938229	5.564773e-007	3.208057660e-06
0.9	0.065133042073048070	0.065131982336447347	1.059737e-006	4.061998320e-06
1.0	0.058010749973883932	0.058009186768479770	1.563205e-006	5.017051900e-06

Problem 4

Mass Spring Motion

A 128lb weight is attached to a spring having a spring constant of 64lb/ft. The weight is started in motion with no initial velocity by displacing it 6inches above the equilibrium position and by simultaneously applying to the weight an external force $F(t) = 8sin(4t)$. Assuming no air resistance, compute the subsequent motion of the weight at $t : 0.01t010$.

We model the problem into a mathematical equation of the motion on the weight attached to the spring.
m = 128, k= 64, b= 0 and $F(t) = 8sin4t$. Upon simplification, the problem boils down to: $\frac{d^2x}{dt^2} + 16x = 2sin(4t)$, $x(0) = -\frac{1}{2}$, $x'(0) = 0$
With the exact solution x(t) = $\frac{1}{2}\, Cos4t + \frac{1}{16}\, SinAt + \frac{1}{4}\boldsymbol{t}\, Cos4t$

Table 4: Comparison of Absolute Errors of Problem 4 with Exiting Methods

x	Exact-solution	Computed-solution	Error in New Method	Error in Kayode(2021)
0.2	-0.4983901330974940	-0.4983908975899810	7.044492e-07	4.98251761e-07
0.3	-0.4963686974027950	-0.4963697399280050	1.344253e-06	1.99750245e-06
0.4	-0.4935282660817960	-0.4935303512252360	2.108514e-06	4.49460223e-06
0.5	-0.4898678796894520	-0.4898704538772500	2.957419e-06	7.99469593e-06
0.6	-0.4853824289709970	-0.4853869341136930	3.850514e-06	1.24926327e-05
0.7	-0.4800796129056690	-0.4800770813407640	4.746844e-06	2.44963093e-05
0.8	-0.4739400736436170	-0.4739461243977930	5.605075e-06	3.20040361e-05
0.9	-0.4669895079202790	-0.4669933443085270	6.383639e-06	4.05095821e-05
0.1	-0.4592837545722390	-0.4592541631489060	7.040858e-06	5.00240831e-05

CONCLUSION

This paper presents an application of the linear multistep method (lMM) to directly solve general second-order ordinary differential equations without converting them into a system of first-order equations, utilizing chebyshev polynomial of the first kind as the basis function. In this paper, a new class of hybrid methods was developed using a collocation and interpolation approach. Collocation and interpolation points were selected to determine the hybrid points at different intervals. A class of 2-step methods, incorporating various hybrid points, was implemented in a predictor-corrector mode (PCM). This implementation was designed to compare the newly derived methods with some existing predictor-corrector and block methods.

The predictors (main components) were derived to have the same order of accuracy as the correctors. The results obtained from these methods were then compared with those achieved by existing methods. The comparison of errors demonstrated that the new methods outperformed any of the existing methods. Furthermore, the application of these new methods to practical problems, such as the cooling of a body, the resonance vibration of a machine, confirmed their effectiveness and usability. This shows that the new methods are highly applicable and beneficial in solving engineering problems that can be modeled as second-order ordinary differential equations. The superior performance of the new meth-ods in these diverse scenarios underscores their potential to be a valuable tool in various fields of engineering and applied mathematics.

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