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MATHEMATICAL GROUP REPLACEMENT MODEL WITH AN UNBOUNDED PLANNING HORIZON

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ABSTRACT

Traditional replacement models often rely on finite planning horizons, a framework misaligned with the perpetual operational needs of real-world systems such as infrastructure, industrial machinery, and fleet vehicles. This limitation necessitates robust strategies for optimising maintenance and replacement decisions over an unbounded planning horizon. This study develops and validates deterministic group replacement model to address this challenge, thereby providing a pragmatic framework for long-term asset management, and minimizing long-run average cost per unit time over an unbounded planning horizon. The model incorporates an exponential failure rate, determines the optimal periodic replacement interval by comparing the cost of individual failures with bulk replacement costs, with the optimal solution obtained numerically using the Newton-Raphson method. When applied to a case study in a bakery, the model demonstrated significant practical utility, it established an optimal replacement interval for baking pans at 4.3 years. Sensitivity analyses revealed that the optimal policy is highly responsive to changes in the maintenance cost growth rate and the group replacement cost, thereby providing managers with critical insight into the financial drivers of their decisions. This study contributes to knowledge by bridging the theory-practice gap in unbounded horizon replacement modelling. It provides computationally tractable and directly applicable unbounded-horizon replacement models, equipping industries, particularly those in resource-constrained environments, with a structured methodology to enhance capital budgeting, reduce lifecycle costs, and ensure operational sustainability over an indefinite planning horizon.

Keywords: Unbounded horizon, Group replacement model, Optimal replacement time, Asset lifecycle management, Preventive replacement

INTRODUCTION

Replacement theory constitutes an important area of operations research concerned with determining optimal policies for maintaining, repairing, or replacing deteriorating assets in order to minimize long-term operational costs (Abdelwali *et al.*, 2024). Many engineering and industrial systems consist of numerous identical components whose performance gradually declines due to wear, aging, and technological obsolescence. In such circumstances, replacement decisions must be carefully structured to ensure operational efficiency and cost effectiveness (Martyushev *et al.*, 2023; Hakimi *et al.*, 2014). Among the various strategies used in reliability and maintenance management, group replacement policies have received considerable attention because they allow multiple similar components to be replaced simultaneously at predetermined intervals rather than individually at failure. They have proven effective in systems involving multiple identical components, particularly in production environments where coordinated replacement reduces operational costs. Empirical evidence from Nigeria manufacturing systems supports this approach, demonstrating improved cost efficiency through structured replacement strategies (Abubakar *et al.*, 2020). According to Hakimi and Waziri, 2014, this approach is particularly beneficial in large-scale systems where the cumulative cost of individual replacements may become significantly higher than the cost of coordinated replacement actions.

In many practical applications, asset management decisions are not restricted to a predetermined planning period but instead extend indefinitely into the future (Liu *et al.*, 2021). This gives rise to replacement models with an unbounded (infinite) horizon, where decision-making is formulated over an infinite sequence of time periods. Unlike finite-horizon models, which optimise replacement decisions within a fixed timeframe, unbounded-horizon models focus on minimizing the long-run expected cost per unit time while accounting for continuous system operation, as emphasized by Liu *et al.*, 2021. Such models are particularly relevant for infrastructures, industrial machinery, transportation fleets, and manufacturing equipment that are expected to operate indefinitely with periodic maintenance and replacement interventions.

Mathematically, unbounded-horizon replacement models often employ analytical frameworks derived from stochastic processes, renewal theory, and Markov decision processes to capture the probabilistic nature of component deterioration and failure (Lefebvre and Yaghoubi, 2024; Vieten, and Stockbridge, 2025). These approaches enable the evaluation of long-term cost structures associated with preventive maintenance, corrective maintenance, and group replacement strategies. By incorporating cost convergence properties and steady-state behaviour, infinite-horizon models provide a rigorous basis for determining optimal replacement intervals and maintenance policies in complex operational environments.

In developing economies such as Nigeria, the application of systematic replacement policies is particularly relevant in sectors where operational continuity and cost efficiency are critical (Chen and Martinez-Vazquez, 2024; Wu *et al.*, 2024). Transportation fleets, manufacturing systems, and infrastructure networks often consist of numerous components or units operating under challenging conditions. In such settings, adopting mathematically grounded group replacement models can significantly reduce maintenance costs, minimize system downtime, and improve long-term operational planning. For example, transportation companies operating large fleets of vehicles must determine optimal schedules for replacing or overhauling components to maintain reliability while controlling maintenance expenditures.

Despite the practical importance of replacement modelling, the formulation of effective unbounded-horizon models presents several challenges (Ye *et al.*, 2024). These include accurately characterizing deterioration patterns, estimating replacement and maintenance costs, and developing analytical frameworks capable of capturing long-term system behaviour. Advances in computational modelling, data analytics, and reliability theory have, however, improved the feasibility of implementing such models in real-world settings. Modern analytical techniques enable more precise estimation of component lifetimes and allow decision-makers to evaluate replacement policies under varying operational conditions (Garcia *et al.*, 2025).

Against this background, this study develops a mathematical group replacement model with an unbounded horizon aimed at optimising maintenance and replacement decisions for systems consisting of multiple identical components. The study formulates the underlying mathematical framework for group replacement and investigates the determination of optimal replacement intervals that minimize long-run expected costs. By integrating theoretical modelling with practical considerations in asset management, the study contributes to the development of efficient replacement strategies applicable to large-scale operational systems.

MATERIALS AND METHODS

In cases where multiple identical assets (batteries, machine parts, light bulbs) fail over time, the group replacement strategies are used. The approaches used are either individual vs periodic group replacement or preventive vs corrective replacement.

Individual vs Periodic Group Replacement

Each item can be replaced individually upon failure, or all items can be replaced simultaneously at fixed intervals to reduce administrative costs. The decision is based on the cost of replacing items one-by-one, bulk replacement cost savings and failure distribution of the items.

A common approach is to determine an optimal group replacement period T by minimizing $A(T)$:

$$A(T) = \frac{total\ replacement\ cost\ over\ T}{T} \quad (1)$$

Preventive vs Corrective Replacement

Corrective replacement is carried out when an item fails. Whereas, preventive replacement is carried out to replace all items at a fixed interval to avoid high failure cost.

If preventive replacement is cheaper in the long run, an optimal replacement interval T^* is selected.

The expected number of failures in the interval $[0, T]$ for a single item is

$$M(T) = \int_0^T \lambda(t)dt \quad (2)$$

Where:

M(T) – expected number of failures in interval $[0, T]$

λ(t) – cumulative failure distribution over time t

T – fixed group replacement interval

Since there are N items, the total expected failures before the group replacement time t is

$$M_{total}(T) = N\int_0^T \lambda(t)dt \quad (3)$$

The cost of replacing individual items over one circle $[0, T]$ is

$$C_{ind}(T) = C_i N\int_0^T \lambda(t)dt \quad (4)$$

Where:

C_i – cost of replacing an item individually upon failure

N – number of identical items in operation

The cost of replacing a group of items at time T is given by:

$$C_{group}(T) = C_g \quad (5)$$

Where:

C_g – cost of replacing all N items at once

Thus, the total cost per cycle including individual and group replacement is given by:

$$C(T) = C_i N\int_0^T \lambda(t)dt + C_g \quad (6)$$

Since the replacement cycle repeats indefinitely, the long-run average cost per unit time is given by:

$$A(T) = \frac{C(T)}{T} = \frac{C_i N\int_0^T \lambda(t)dt + C_g}{T} \quad (7)$$

To minimize the cost per unit time, we differentiate A(T) with respect to T and set it to zero:

$$\frac{d}{dT}\left(\frac{C_i N\int_0^T \lambda(t)dt + C_g}{T}\right) = 0 \quad (8)$$

Using the quotient rule, we get

$$\frac{C_i N\lambda(T)T - \left(C_i N\int_0^T \lambda(t)dt + C_g\right)}{T^2} = 0 \quad (9)$$

Rearranging equation (9), gives

$$C_i N\lambda(T)T = C_i N\int_0^T \lambda(t)dt + C_g \quad (10)$$

Dividing through by T, we have:

$$C_i N\lambda(T) = \frac{C_i N\int_0^T \lambda(t)dt + C_g}{T} \quad (11)$$

If failure times follow an exponential distribution with failure rate α, then

$$\lambda(T) = 1 - e^{-\alpha T} \quad (12)$$

$$\int_0^T (T) = T - \frac{1-e^{-\alpha T}}{\alpha} \quad (13)$$

Substituting equation (13) into the cost function (7) gives

$$A(T) = \frac{C_i N\left(T - \frac{1-e^{-\alpha T}}{\alpha}\right) + C_g}{T} \quad (14)$$

Let $X(t) = C_i N\left(T - \frac{1-e^{-\alpha T}}{\alpha}\right) + C_g$ and differentiating equation (14) gives:

$$\frac{d}{dT}(A(T)) = \frac{d}{dT}\left(\frac{X(t)}{T}\right) \quad (15)$$

Using quotient rule, we have

$$\frac{d}{dT}\left(\frac{X(t)}{T}\right) = \frac{TX'(T) - X(T)}{T^2} \quad (16)$$

Differentiating X(T) gives

$$X'(T) = C_i N\left(1 - \frac{d}{dT}\left(\frac{1-e^{-\alpha t}}{\alpha}\right)\right) \quad (17)$$

Since $\frac{d}{dT}\left(\frac{1-e^{-\alpha t}}{\alpha}\right) = \frac{\alpha e^{-\alpha t}}{\alpha} = e^{-\alpha t}$, equation (3.40) becomes

$$X'(T) = C_i N(1 - e^{-\alpha t}) \quad (18)$$

Now applying the quotient rule

$$\frac{dA}{dT} = \frac{TC_i N(1-e^{-\alpha t}) - \left(C_i N\left(T - \frac{1-e^{-\alpha t}}{\alpha}\right) + C_g\right)}{T^2} \quad (19)$$

Setting $\frac{dA}{dT} = 0$ in equation (19) and simplifying gives

$$TC_i N(1 - e^{-\alpha t}) - \left(C_i N\left(T - \frac{1-e^{-\alpha t}}{\alpha}\right) + C_g\right) = 0 \quad (20)$$

Making C_g the subject of formula gives

$$C_g = TC_i N(1 - e^{-\alpha t}) - C_i N\left(T - \frac{1-e^{-\alpha t}}{\alpha}\right) \quad (21)$$

Equation (21) is non-linear in T and generally requires numerical methods such as Newton's method or bisection method to solve for T^*.

Numerical Method for Solving the Optimal Replacement Time

Equation (21) above gives a non-linear equation and generally requires numerical methods to solve for T^*. For this study, we employ the Newton-Raphson method to solve for T^*. We rearrange equation (21) into a function $f(T) = 0$ as thus:

$$f(T) = TC_i N(1 - e^{-\alpha T}) - C_i N\left(T - \frac{1-e^{-\alpha T}}{\alpha}\right) - C_g \quad (22)$$

We simplify equation (22) as thus:

$$f(T) = C_i N\left[T(1 - e^{-\alpha T}) - \left(T - \frac{1-e^{-\alpha T}}{\alpha}\right)\right] - C_g \quad (23)$$

Now, simplifying the expression inside the brackets, we have:

$$T(1-e^{-\alpha T})-\left(T-\frac{1-e^{-\alpha T}}{\alpha}\right)=T-Te^{-\alpha T}-T+\frac{1-e^{-\alpha T}}{\alpha}=\frac{1-e^{-\alpha T}}{\alpha}-Te^{-\alpha T} \quad (24)$$

Substituting back equation (3.47) into equation (3.46), we have:

$$f(T)=C_iN\left[\frac{1-e^{-\alpha T}}{\alpha}-Te^{-\alpha T}\right]-C_g \quad (25)$$

For Newton's method, we need the derivative of $f(T)$ with respect to T, denoted as $f'(T)$. We compute $f'(T)$ as thus:

Let $g(T)=\frac{1-e^{-\alpha T}}{\alpha}-Te^{-\alpha T}$, so

$$f(T)=C_iN[g(T)]-C_g \quad (26)$$

Thus:

$$f'(T)=C_iNg'(T) \quad (27)$$

Now, we find $g'(T)$ as thus:

$$g'(T)=\frac{d}{dT}\left[\frac{1-e^{-\alpha T}}{\alpha}-Te^{-\alpha T}\right] \quad (28)$$

The derivative in equation (28) is divided into two parts to make the derivation easy to follow. The first part of the derivation is:

$$\frac{d}{dT}\left[\frac{1-e^{-\alpha T}}{\alpha}\right]=\frac{1}{\alpha}\cdot\frac{d}{dT}(1-e^{-\alpha T})=\frac{1}{\alpha}\cdot\left(0-(-\alpha e^{-\alpha T})\right)=e^{-\alpha T} \quad (29)$$

Product rule is applied to the second part of the derivation as thus:

$$\frac{d}{dT}[-Te^{-\alpha T}]=u'v+uv'=(-1)(e^{-\alpha T})+(-T)(-\alpha e^{-\alpha T})=-e^{-\alpha T}+\alpha Te^{-\alpha T} \quad (30)$$

Substituting equations (29) and (30) back into equation (28), we have:

$$g'(T)=e^{-\alpha T}+(-e^{-\alpha T}+\alpha Te^{-\alpha T}\ ) \quad (31)$$

Simplifying equation (31), we have:

$$g'(T)=\alpha Te^{-\alpha T} \quad (32)$$

Recall that $f'(T)=C_iNg'(T)$. Now, substituting equation (32) into equation (27), we have:

$$f'(T)=C_iN\alpha Te^{-\alpha T} \quad (33)$$

Substituting equations (25) and (33) into the standard Newton iteration formula, we have:

$$T_{n+1}=T_n-\frac{f(T_n)}{f'(T_n)} \quad (34)$$

$$\therefore T_{n+1}=T_n-\frac{C_iN\left[\frac{1-e^{-\alpha T_n}}{\alpha}-Te^{-\alpha T_n}\right]-C_g}{C_iN\alpha T_ne^{-\alpha T_n}} \quad (35)$$

Where:

T_{n+1} is the future guess for replacement time
T_n is the current guess for replacement time
C_i is the individual replacement cost
N is the number of items
α is the failure rate
C_g is the group replacement cost

From the formula above, if $f(T_n)>0$, the average cost $A(T)$ is decreasing at T_n, suggesting T_n is too small. Similarly, if $f(T_n)<0$, the average cost $A(T)$ is increasing at T_n, suggesting T_n is too large. Furthermore, if $f(T_n)=0$, T_n is optimal.

The mathematical derivations for group replacement strategies show how total replacement costs are minimized using failure distribution functions, particularly under exponential failure assumptions. Overall, the model provides a structured approach to optimising replacement policies, ensuring cost-effective asset management in industries.

RESULTS AND DISCUSSION

To test the group replacement model in a bakery, we use the baking pan as our focus equipment since they are often deployed in multiples. For illustration purposes, we used a group-replacement dataset for 200 baking pans observed over a period of 12 years. Table 1 below presents an illustrative dataset for the baking pan's failure over the stated period of time.

Table 1: Illustrative Dataset for the Bakery Pan's Failure Rate Over a Period of 12 Years

Year	Observation Interval (years)	Number of failures
1	0-1	13
2	1-2	15
3	2-3	17
4	3-4	19
5	4-5	21
6	5-6	23
7	6-7	25
8	7-8	27
9	8-9	29
10	9-10	31
11	10-11	33
12	11-12	35
Total	-	**288**

Assumptions for Group Replacement Model

We made the following assumptions to implement the group replacement model:

i. Number of identical pans: $N=200$.
ii. Cost to replace one pan individually: $C_i=$ NGN5,000.
iii. Cost to replace whole group at once (bulk + labour): $C_g=$ NGN800,000.
iv. Observation period: 12 years.

The estimated failure rate α is given by:

$$\alpha=\frac{total\ number\ of\ pan\ failure\ over\ the\ observed\ period}{total\ number\ of\ pan\ \times observation\ period}$$

$$\therefore \alpha=\frac{288}{200\times12}=\frac{288}{2400}=0.12\text{ year}^{-1}$$

Newton's Iteration History

Table 2 presents the results obtained from applying the Newton–Raphson method to determine the optimal replacement interval T^* that minimizes the average cost function A(T). The iterative process begins with an initial guess of $T_1=10.00$ years. At each iteration, the value of the cost function $f(T_n)$ and its derivative $f'(T_n)$ are evaluated, and a new estimate T_{n+1} is computed using the Newton–Raphson update formula as given in equation (3.58). From the table, it is observed that the sequence of estimates converges rapidly. After the first iteration, the estimated replacement interval drops significantly from 10.00 years to approximately 4.43 years**,** indicating that the initial guess was far from the true optimum. Subsequent iterations show progressively smaller changes in T_n, with convergence achieved after the fourth iteration**.** At this point, the value of $f(T_n)$ becomes negligibly small (0.006), confirming that the optimal solution has been reached with high numerical accuracy. This result is consistent with the graphical analysis presented in Figure 1, where the minimum point on the average cost curve occurs at approximately 4 years. The close agreement between the analytical (Newton–Raphson) and graphical results validates the correctness of the model and the computational procedure used.

Table 2: Newton-Raphson Iteration History for Group Replacement

Iteration	T_n	$f(T_n)$	$f'(T_n)$	T_{n+1}
1	10.00	2011439.45	361433.05	4.4348
2	4.43	34289.03	312561.28	4.3251
3	4.32	201.15	308868.93	4.3245
4	4.32	0.006	308846.56	4.3245

Average Cost Function for the Group Replacement Model

Figure 1 presents the average cost function $A(T)$ for replacing the baking pans over an extended range of replacement intervals T (in years). The curve exhibits a distinct U-shape, which aligns with theoretical expectations in preventive maintenance optimisation. At very small replacement intervals, the average cost is high due to the frequent and unnecessary replacement of the pans, resulting in excessive planned replacement expenses. As the interval T increases, the average annual cost of replacing the pans decreases sharply, reaching a minimum value at approximately $T^* =$ 4 years. This point, indicated by the red marker, represents the optimal replacement interval where the total average cost of replacing the pans per year is minimized. The optimal average cost of replacing the entire group of pans is approximately NGN405,000. Beyond this optimal point, the average cost begins to increase gradually as the replacement interval extends further. This upward trend reflects the growing influence of unplanned failures and corrective maintenance costs that arise when components are kept in service beyond their economically efficient lifespan.

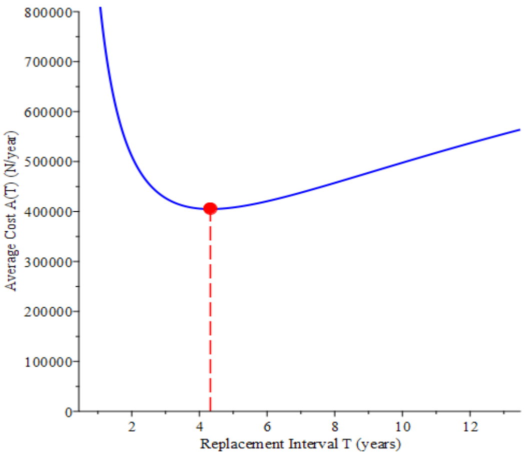

Figure 1: Plot Showing the Average Cost Function A(T) Over an Extended Replacement Intervals T

Sensitivity Analysis: Effect of Group Replacement Cost

Table 3 presents the sensitivity analysis results showing the effect of varying the group replacement cost (C_g) on the optimal replacement interval (T^*) and the corresponding minimum average annual cost ($A(T^*)$). This analysis was performed to assess how changes in the cost of performing a group replacement influence the economic decision on the timing of replacements. From the table, it is observed that as the group replacement cost C_g increases, both the optimal replacement interval T^* and the minimum average annual cost $A(T^*)$ also increase. This relationship follows a logical trend: when it becomes more expensive to carry out a full group replacement, it is economically preferable to extend the replacement interval, thereby postponing the costly maintenance operation. For instance, when $C_g =$ NGN800,000, the optimal replacement interval is approximately $T^* = 4.32$ years, with a minimum average annual cost of $A(T^*)$ = NGN404,846.56. However, increasing C_g to NGN1,400,000 shifts the optimal interval to $T^* = 6.12$ years, and the corresponding minimum average cost increases to $A(T^*)$ = NGN520,415.58. This demonstrates that higher group replacement costs push the system toward longer replacement cycles, but at the expense of higher annualized costs. The results of this sensitivity analysis align with theoretical expectations and findings in previous studies which indicate that the optimal replacement interval is positively correlated with the cost of preventive or group replacement.

Table 3: Sensitivity Analysis Table Showing the Effect of Group Replacement Cost on the **Optimal Replacement Interval and Minimum Average Annual Cost**

C_g(N)	T^*(years)	$A(T^*)$(N)
800,000	4.324	404,846.56
900,000	4.643	427,154.86
1,000,000	4.952	448,004.03
1,200,000	5.548	486,125.37
1,400,000	6.124	520,415.58

CONCLUSION

This study concludes that deterministic replacement models with unbounded horizons provides practical decision and support tools for long-term asset management. By moving beyond the limitations of finite-horizon assumptions, these models provide a sustainable framework for making cost-effective replacement decisions for assets intended for continued operations. The research successfully demonstrated that:

i. An optimal replacement point exists where the marginal cost of operation equals the long run average cost rate.
ii. Deterministic models, while simplifying real-world uncertainty, yield clear, actionable, and intuitive results that can be directly applied in resource-constrained settings, such as small and medium-sized enterprises.
iii. The models are adaptable, as shown by the sensitivity analyses, allowing decision-makers to understand how changes in key cost parameters affect their optimal strategy.

REFERENCES

Abdelwali, H. A., Aldaihani, F. S., Alrajhi, J. M., & Abdelati, M. H. (2024). On equipment replacement problem: A comparative study of heuristic and mathematical programming models. *International Journal of Traffic and Transportation Engineering*, 13(2), 41–48.

Abubakar, J., Abdullahi, I. L., Usman, S., Danjuma, N., & Agaie, B. G. (2020). LINEAR PROGRAMMING AS DECISION MAKING TOOL FOR OPTIMAL PRODUCTION: A CASE STUDY OF YOGHURT PRODUCTION BY ATS MULTI-CONCEPT WORLDWIDE LTD IN KATSINA STATE, NIGERIA. FUDMA JOURNAL OF SCIENCES, 4(1), 750-755. https://fjs.fudutsinma.edu.ng/index.php/fjs/article/view/110

Chen, C., & Martinez-Vazquez, J. (2024). Fiscal decentralization and public infrastructure maintenance expenditures: A cross-country panel analysis. Public *Finance and Management*, *23(1–2)*, 3–21. https://doi.org/10.1177/15239721241243141

Garcia, J., Rios-Colque, L., Peña, A., & Rojas, L. (2025). Condition monitoring and predictive maintenance in industrial equipment: An NLP-assisted review of signal processing, hybrid models, and implementation challenges. Applied Sciences, 15(10), 5465. https://doi.org/10.3390/app15105465

Hakimi, D., & Waziri, V. O. (2014). Mathematical Model for the Deterministic Extended Machine Replacement Policy. *International Journal of Computing and Optimization*, 3(1), 113 – 123.

Hakimi, D., Yusuf, A. U., Adeboye, K. R., & Waziri, V. O. (2014). Development of iterative method for replacement and maintenance process using inventory model. *International Journal of Computing and Optimization*, 1(3), 125–144.

Lefebvre, M., & Yaghoubi, R. (2024). Optimal inspection and maintenance policy: Integrating a continuous-time Markov chain into a homing problem. Machines, 12(11), 795. https://doi.org/10.3390/machines12110795

Liu, B., Pandey, M. D., Wang, X., & Zhao, X. (2021). A finite-horizon condition-based maintenance policy for a two-unit system with dependent degradation processes. *European Journal of Operational Research*, 295(2), 705–717. https://doi.org/10.1016/j.ejor.2021.03.010

Martyushev, N. V., Malozyomov, B. V., Sorokova, S. N., Efremenkov, E. A., Valuev, D. V., & Qi, M. (2023). Review models and methods for determining and predicting the reliability of technical systems and transport. *Mathematics*, 11(15), 3317. https://doi.org/10.3390/math11153317

Vieten, M. G., & Stockbridge, R. H. (2025). Finite element approximations for stochastic control problems with unbounded state space. *Journal of Applied Probability*, *62*(1), 367–394.

Wu, J., Qian, C., & Dohi, T. (2024). A Net Present Value Analysis of Opportunity-Based Age Replacement Models in Discrete Time. *Mathematics*, 12(10), 1472. https://doi.org/10.3390/math12101472

Ye, Z., Si, S., Zhang, Y., & Hu, C. (2024). Condition-based maintenance via Markov decision processes: A review. Frontiers of Engineering Management. https://doi.org/10.1007/s42524-024-4130-7