FUDMA Journal of Sciences (FJS)
ISSN online: 2616-1370
ISSN print: 2645 - 2944
Vol. 10 No. 4, February, 2026, pp 139 – 147
DOI: https://doi.org/10.33003/fjs-2026-1004-4691

A HYBRID EXPONENTIAL-GENERALIZED GAMMA DISTRIBUTION WITH MEAN BASES MIXING PROPORTION: THEORY AND APPLICATIONS

*** Uchechukwu Kalu, Samuel Adewale Aderoju, Bello Ishola Sanni, Toheeb Akorede Yussuf, Adediran Dauda Adeshola and Saheed Kunle Ajibade**

Department of Mathematics and Statistics, Kwara State University, Malete, P.M.B. 1530, Ilorin, Nigeria.

* Corresponding authors' email: kaluuchechukwu45@gmail.com

ABSTRACT

In our study, we propose the Exponential-Generalized Gamma Distribution (EGGD) with Mean-Based Mixing Proportion. A new hybrid survival distribution developed to overcome the limitations of existing parametric models in modeling complex hazard functions and structures. The EGGD combines the simplicity of the exponential distribution with the flexibility of the generalized gamma distribution. The analytical calculations of the distribution's important statistical properties, namely moments, skewness, kurtosis, survival, and hazard functions, have been derived to provide further insights into the distribution's behavior. The EGGD parameter estimation is conducted using maximum likelihood estimation (MLE). The performance of the maximum likelihood estimates was rigorously examined through a Monte Carlo simulation study. The performance measures used in the study were bias and MSE. The practicality of the Model was examined through its application to real-world lifetime data. Its performance was compared with that of other existing three-parameter and two-parameter lifetime distributions. The model adequacy is assessed using information criteria, including AIC, AICc, HQIC, and BIC. Across three datasets, the EGGD consistently exhibits superior goodness-of-fit compared to the other considered models, highlighting its flexibility and robustness as a tool for survival and reliability analysis.

Keywords: Survival Analysis, Exponential Distribution, Generalized Gamma Distribution, Parametric Survival Models, Likelihood Estimation.

INTRODUCTION

Accurate modeling of lifetime data remains one of the biggest challenges facing survival analysis and reliability studies. For instance, there are cases where the risk of failure is time-dependent, there is heavy-tailed behavior, and there are underlying differences between individuals. Such cases can be observed in biomedical studies, engineering, and environmental studies. For such cases, the hazard rates are not monotonic, as assumed by most lifetime models. This has, therefore, created a need for more flexible parametric models, which can accommodate non-monotonic hazard rates, skewness, and heavy tails, yet they are simple enough for statistical inference (Mahmoudi & Jafari, 2012).

One of the most used distributions is the exponential distribution, which is considered simple. However, the exponential distribution has the limitation of constant hazard rates, which is not realistic. For instance, Balakrishnan & Basu (1996) indicate that the hazard rates of most systems are not constant over time. Contrary to the exponential distribution, the generalized gamma distribution is very flexible. It can accommodate increasing, decreasing, bathtub, and unimodal hazard rates. Moreover, the generalized gamma distribution can accommodate several special cases, including the exponential, Weibull, lognormal, and chi-square distributions (Stacy, 1962; Mahmoudi & Jafari, 2012). However, the generalized gamma distribution can be difficult to work with because of its complex form and challenges in parameter estimation (Al-Hussaini & Al-Nadif, 2019).

To overcome these issues, flexible lifetime distributions using approaches such as exponentiation, transmutation, compounding, and mixture modeling have been developed (Mudholkar & Srivastava, 1993; Gómez et al., 2008; Hosseini et al., 2022). A good example is the Samade distribution developed by Aderoju (2021), which outperformed several established distributions in real-life data analysis. Among the aforementioned modeling strategies, mixture modeling is an attractive option because it accommodates population variation and failure mechanisms (Uma Maheswari & Alexander, 2017). Nevertheless, the established mixture models often assume fixed mixing proportions or have a large number of parameters, which may result in identifiability and estimation difficulties

Inspired by the aforementioned limitations, the current study introduces a novel mixture distribution, termed the Exponential-Generalized Gamma Mixture with Mean-Based Mixing Proportion (EGGD). Unlike the conventional mixture distributions with fixed weighting factors, the proposed EGGD model uses the first moment matching method to obtain the mixing proportion. The first moment matching method relates the mixture weight to the expectation of the component distributions. Consequently, the proposed model inherits the simplicity of the exponential distribution and the flexibility of the generalized gamma distribution (Stacy, 1962; Mahmoudi & Jafari, 2012).

This paper derives important mathematical properties of the EGGD, comprising its moments, Rényi entropy, order statistics, survival function, and hazard rate function. Model Parameters are estimated using the Maximum Likelihood Estimation (MLE) method, and a Monte Carlo simulation is applied to study the performance of the estimators. The practical usefulness of the new model is displayed using real datasets from epidemiology, engineering, and hydrology. The outcome of the EGGD is compared with the Burr–Weibull, Generalized Gamma, and Exponentiated Weibull Extension models using standard information criteria.

The rest of the paper is structured as follows: Section 2 illustrates the derivation of the PDF and CDF of the EGGD. In Section 3, the statistical and mathematical properties are obtained. The parameter estimation technique is presented in Section 4. Section 5 contains the simulation study. Application to real-life datasets is presented in Section 6, while Section 7 presents the concluding remarks of the study.

MATERIALS AND METHODS

In this section, the probability density function (PDF) of an Exponential-Generalized Gamma with Mean-Based Mixing (EGGD) proportion distribution is derived using the exponential distribution (Balakrishnan and Basu, 1996; Shanker & Shukla, 2019) and generalized gamma distribution (Stacy, 1962), which are defined respectively as;

$$g(x) = \theta e^{-\theta x}, \;\; x > 0, \theta > 0, \qquad (1)$$

$$E(x) = \frac{1}{\theta}$$

and

$$h(x) = \frac{1}{\Gamma(\alpha)} \beta \theta^{\alpha} x^{\beta\alpha - 1} e^{-\theta x^{\beta}}, \; x > 0, \; \alpha > 0, \theta > 0, \beta > 0 \qquad (2)$$

$$E(x) = \frac{\theta^{-1/\beta}\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\Gamma(\alpha)}$$

A component of the two distributions, the exponential distribution and the generalized gamma distribution, was utilized to derive the mixing proportion through the proportion of the first moment.

$$P_1 = \frac{\theta^{\frac{1}{\beta}}\Gamma(\alpha)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha) + \theta\Gamma\left(\alpha + \frac{1}{\beta}\right)}$$

$$P_2 = \frac{\theta\Gamma\left(\alpha + \frac{1}{\beta}\right)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha) + \theta\Gamma\left(\alpha + \frac{1}{\beta}\right)}, such\ that \;\; p_1 + p_2 = 1$$

$$f(x; \alpha, \beta, \theta) = p_1 g(x) + \; p_2 h(x) \qquad (3)$$

Thus, substituting (1) and (2) into (3) leads to:

$$f(x; \alpha, \beta, \theta) = \frac{e^{-x^{\beta}\theta} x^{\alpha\beta} \beta \theta^{\alpha + \frac{1}{\beta}} + e^{-x\theta} x \theta^2 \Gamma\left(\alpha + \frac{1}{\beta}\right)}{x\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha) + \theta\Gamma\left(\alpha + \frac{1}{\beta}\right)\right)}, \;\; \alpha > 0, \beta > 0, \theta > 0 \qquad (4)$$

Note: $\int_0^{\infty} f(x; \alpha, \beta, \theta) dx = 1$

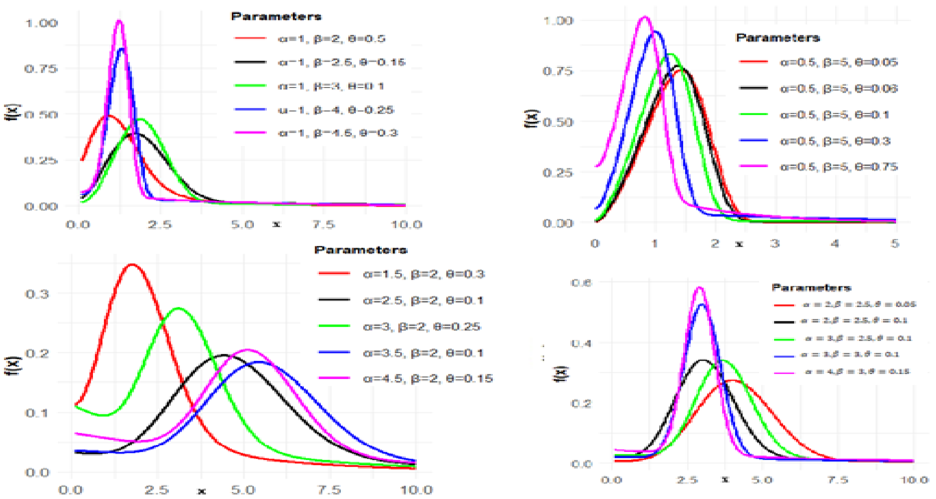

Figure 1: Probability Density Function Plots of the EGGD Parameters' Values

Figure 1 shows the probability density functions of the EGGD for various values of α, β, and θ. The plots illustrate that the parameters strongly influence the shape, location, and spread of the distribution. Increasing θ concentrates the density near the origin and produces faster tail decay, while smaller θ results in flatter curves with heavier tails. Lower values of β generate more right-skewed and dispersed densities, whereas higher values of β produce sharper peaks. The parameter α mainly controls peakedness and overall form, with larger α yielding more pronounced modes. These patterns confirm the flexibility of the EGGD for modelling diverse lifetime data.

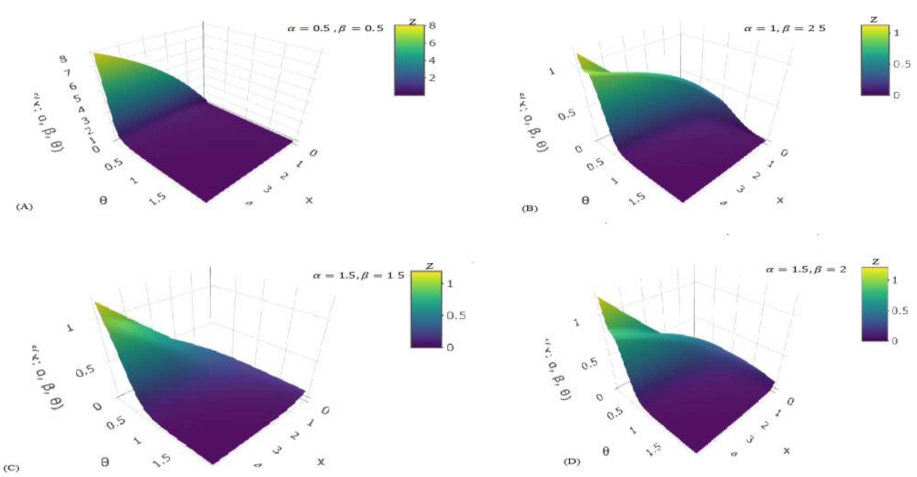

Figure 2: The 3D Density Surface Profiles of the EGGD for Different Parameter Combinations

Figure 2 illustrates three-dimensional (3D) surface plots of the probability density function of the EGGD for selected combinations of the shape parameters α and β, highlighting the joint effect of x and θ on the density. Clear variations in surface height and curvature are observed across parameter settings. Smaller values of β yield smoother surfaces with wider peaks, whereas larger β values produce lower and more rapidly decaying surfaces along the θ direction. Changes in α primarily influence the overall magnitude of the density, with smaller α leading to higher surface levels. In all cases, increasing θ attenuates the density, confirming its role in regulating the spread of the distribution.

However, the corresponding Cumulative distribution is given as follows;

$$F(x;\alpha,\beta,\theta) = \frac{(1-e^{-x\theta})\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta^{\frac{1}{\beta}}\Gamma(\alpha,0,x^{\beta}\theta)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)} \qquad (6)$$

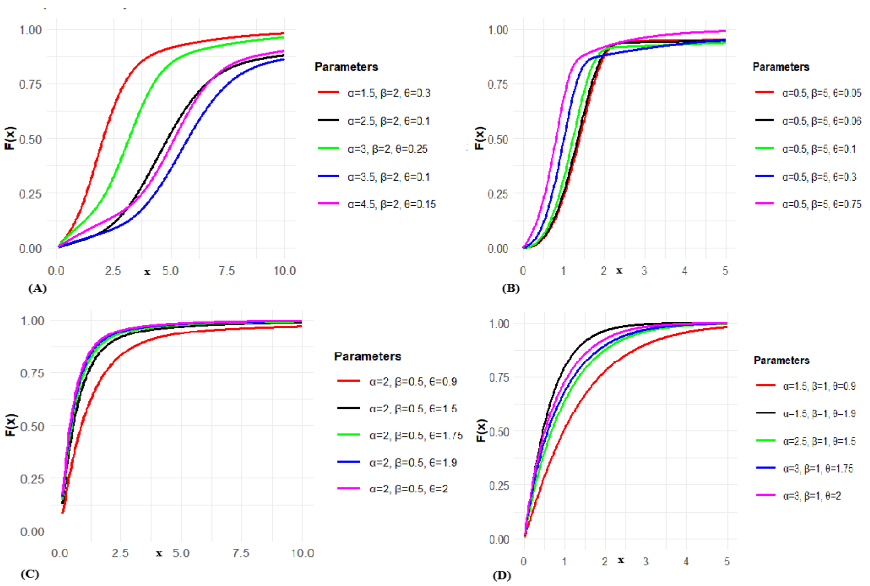

Figure 3: The CDF Plots of the EGGD for Different Combinations of Parameters α, β, and θ

Figure 3 displays the cumulative distribution functions of the EGGD for different values of α, β, and θ. The plots show that the parameters significantly affect the growth rate and location of the CDF curves. Larger values of θ produce steeper curves that reach one more quickly, indicating shorter lifetimes, while smaller θ leads to slower accumulation and heavier tails. Variations in β mainly control the skewness and spread, with lower β yielding more gradual increases and higher β giving sharper transitions. The parameter α influences the overall position of the curves, with larger α shifting the distribution to the right. These behaviours demonstrate the flexibility of the EGGD model in capturing diverse lifetime patterns.

Mathematical Properties

The r^{th} moment of a random variable X with the newly obtained distribution is given by

$$\mathrm{E}(x^r) = \mu^r = \int_0^{\infty} x^r f(x;\alpha,\beta,\theta)dx$$

$$\textit{when } r \textit{ is a positive whole number}$$

$$M_r = \frac{\theta^{-r\left(1+\frac{1}{\beta}\right)}\left(\theta^{\frac{r+\beta}{\beta}}\Gamma(1+r)\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta^{r+\frac{1}{\beta}}\Gamma\left(\alpha+\frac{r}{\beta}\right)\right)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)}$$

From equation (4), the first four moments can be expressed as (r =1,2,3,4)

$$M_1 = \frac{2\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta^{\frac{1}{\beta}}\Gamma(\alpha)}$$

$$M_2 = \frac{\theta^{-\frac{1+\beta}{\beta}}\left(2\theta^{\frac{1}{\beta}}\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta\Gamma\left(\alpha+\frac{2}{\beta}\right)\right)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)}$$

$$M_3 = \frac{6\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta^{2-\frac{2}{\beta}}\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\theta^{2+\frac{1}{\beta}}\Gamma(\alpha)+\theta^3\Gamma\left(\alpha+\frac{1}{\beta}\right)}$$

$$M_4 = \frac{24\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta^{3-\frac{3}{\beta}}\Gamma\left(\alpha+\frac{4}{\beta}\right)}{\theta^{3-\frac{1}{\beta}}\Gamma(\alpha)+\theta^4\Gamma\left(\alpha+\frac{1}{\beta}\right)}$$

$$var(x) = \frac{2\theta^{-\frac{1}{\beta}}\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{\beta}\right)+\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\beta}\right)-2\left(\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2+\theta^{1-\frac{1}{\beta}}\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2}$$

$$CV = \frac{\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right).\sqrt{2\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{\beta}\right)+\theta\Gamma(\alpha)\Gamma\left(\alpha+\frac{2}{\beta}\right)+\theta^2\Gamma\left(\alpha+\frac{1}{\beta}\right)\Gamma\left(\alpha+\frac{2}{\beta}\right)-2\theta^{\frac{1}{\beta}}\left(\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2}}{2\Gamma\left(\alpha+\frac{1}{\beta}\right)\theta^{\frac{1}{2\beta}}}$$

$$Skew = \frac{\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2+\left(6\Gamma\left(\alpha+\frac{1}{\beta}\right)\theta^{\frac{2}{\beta}}+\Gamma\left(\alpha+\frac{3}{\beta}\right)\theta^2\left(\theta^{\frac{3}{2}+\frac{3}{2\beta}}\right)\right)}{\left(-4\left(\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2\theta^{1+\frac{1}{\beta}}+\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\left(2\Gamma\left(\alpha+\frac{1}{\beta}\right)\theta^{\frac{1}{\beta}}+\Gamma\left(\alpha+\frac{2}{\beta}\right)\theta\right)\right)^{\frac{3}{2}}}$$

$$kur = \frac{\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2\left(24\Gamma\left(\alpha+\frac{1}{\beta}\right)\theta^{2+\frac{5}{\beta}}+\Gamma\left(\alpha+\frac{4}{\beta}\right)\theta^{5+\frac{2}{\beta}}\right)}{\left(-4\left(\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^2\theta^{1+\frac{1}{\beta}}+\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)\left(2\Gamma\left(\alpha+\frac{1}{\beta}\right)\theta^{\frac{1}{\beta}}+\Gamma\left(\alpha+\frac{2}{\beta}\right)\theta\right)\right)^2}$$

Rényi's Entropy

Rényi entropy is a useful measure for quantifying the uncertainty, diversity, or complexity of a probability distribution. For a continuous random variable X with probability density function $f(x)$, the Rényi entropy of order $\tau > 0(\tau \neq 1)$ is defined as:

$$H_\tau(X) = \frac{1}{1-\tau}log\left[\int_0^{\infty} f(x)^{\tau}dx\right] \qquad (6)$$

Substituting (4) into (6), the Rényi entropy can be expressed as:

$$H_\tau(X) = \frac{1}{1-\tau} log\left[\int_0^\infty \left(= \frac{e^{-x^\beta \theta} x^{\alpha\beta} \beta \theta^{\alpha+\frac{1}{\beta}} + e^{-x\theta} x\theta^2 \Gamma\left(\alpha+\frac{1}{\beta}\right)}{x\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha) + \theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)}\right)^\tau dx\right]$$

We use the binomial series expansion to obtain an approximate expression for the Rényi entropy:

$$H_\tau(X) =$$

$$\frac{1}{1-\tau} log\left[\sum_{k=0}^{\infty} \binom{\tau}{k} \int_0^\infty \frac{(e^{-x^\beta\theta}\, x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}})^{\tau-k}\, (e^{-x\theta} x\theta^2\, \Gamma(\alpha+\frac{1}{\beta}))^k}{x^\tau\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)^\tau} dx\right] \tag{7}$$

This formulation allows numerical computation of Rényi entropy for given parameter values $\alpha, \beta, and\ \theta$, capturing the degree of randomness and complexity inherent in the EGGD

Order Statistics

Order statistics provide a fundamental framework for drawing inferences from lifetime and reliability data. Let $X_1, X_2, X_3, \ldots, X_n$ be independent and identically distributed random variables following the EGGD. The smallest and largest observations in a sample are referred to as the minimum and maximum order statistics, denoted $X_{(1)} = \min(X_1, X_2, X_3, \ldots, X_n)$ and $\max(X_1, X_2, X_3, \ldots, X_n)$, respectively. The complete set of ordered values satisfies:

$X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$

The PDF of the k^th^ order statistic, $X_{k,}$ is given by

$$f_{X(k)}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}, \tag{8}$$

Therefore, substituting (4) and (5) into (8), the functions of the first and n*th* order statistics, respectively, are:

$$f_{X(1)}(x) = n\left(\frac{e^{-x^\beta\theta} x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}} + e^{-x\theta} x\theta^2\Gamma\left(\alpha+\frac{1}{\beta}\right)}{x\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)}\right)\left(1 - \frac{(1-e^{-x\theta})\theta\Gamma\left(\alpha+\frac{1}{\beta}\right) + \theta^{\frac{1}{\beta}}\Gamma(\alpha, 0, x^\beta\theta)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)}\right)^{n-1},$$

and

$$f_{X(n)}(x) = n\left(\frac{e^{-x^\beta\theta}\ x^{\alpha\beta}\beta\theta^{\alpha} +\ e^{-x\theta} x\theta\ \Gamma(1\ +\ \alpha)}{x\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)}\right)\left(\frac{(1-e^{-x\theta})\theta\Gamma\left(\alpha+\frac{1}{\beta}\right) + \theta^{\frac{1}{\beta}}\Gamma(\alpha, 0, x^\beta\theta)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)}\right)^{n-1}$$

Reliability Analysis

The reliability analysis of any PDF is typically evaluated using the survival function S(t) and the hazard rate function h(t). These functions provide essential insights into the likelihood of survival over time and the instantaneous failure rate at any given moment. Their derivations are outlined below.

Survival Function

The survival function is typically defined as the probability that a subject remains functional beyond a given time. It is represented as:

$$S(x) = 1 - F(x; \alpha, \beta, \theta)$$

$$S(x) = 1 - \frac{(\theta - e^{-x\theta}\theta)\Gamma\left(\alpha+\frac{1}{\beta}\right) + \theta^{\frac{1}{\beta}}\Gamma(\alpha, 0, x^\beta\theta)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)} \tag{9}$$

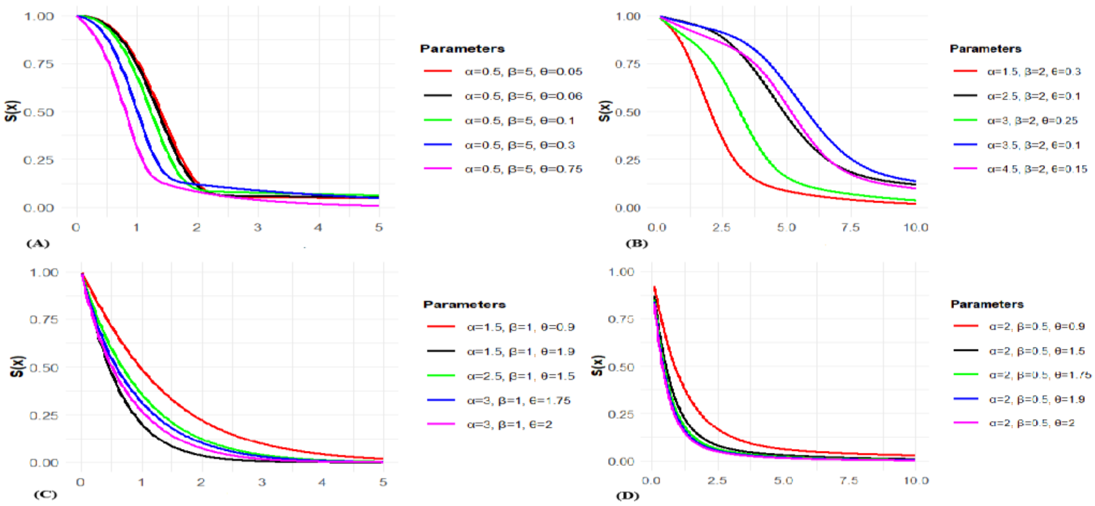

Figure 4: Survival Function Plots of the EGGD for Different Combinations of Parameters

Figure 4 displays the survival function of the EGGD for various combinations of the parameters α, β, and θ, illustrating a monotone decreasing pattern across all settings as expected. The plots show that increasing θ accelerates the decline of the survival curves, indicating shorter lifetimes, while larger values of α generally result in slower decay and heavier survival tails. Variations in β further modulate the curvature of the survival function, controlling the rate at which survival probabilities diminish over time, thereby highlighting the flexibility of the EGGD in modelling different survival behaviours and failure-time characteristics.

Hazard Function

The hazard rate function represents the likelihood of failure occurring at a specific time, provided that the subject has survived up to that point. It is defined as:

$$h(x;\alpha,\beta,\theta)=\frac{e^{(x-x^{\beta})\theta}x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}}+x\theta^{2}\Gamma\left(\alpha+\frac{1}{\beta}\right)}{x\left(e^{x\theta}\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)-e^{x\theta}\theta^{\frac{1}{\beta}}\Gamma(\alpha,0,x^{\beta}\theta)\right)} \quad (10)$$

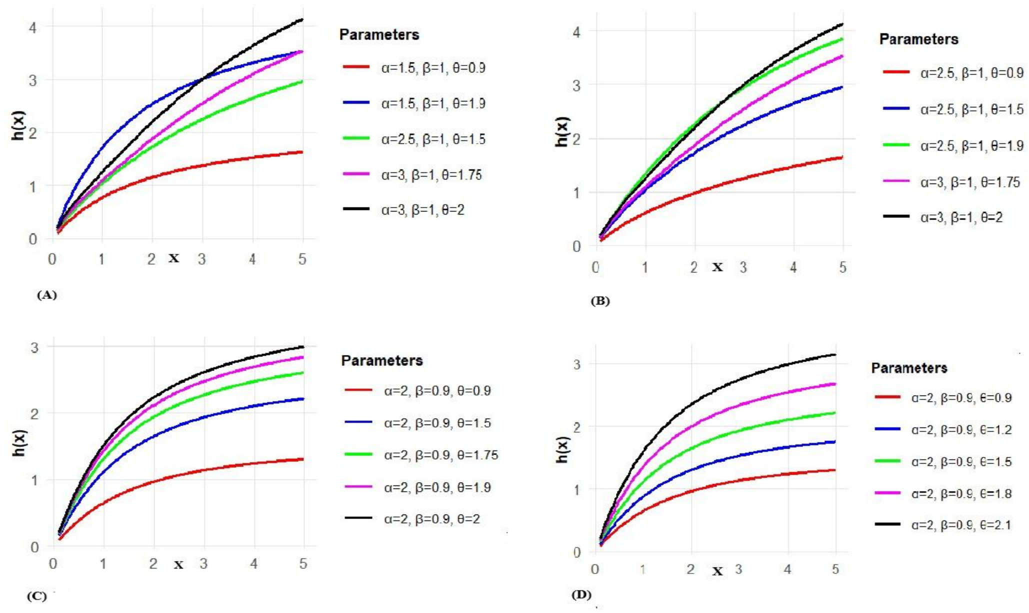

Figure 5: Hazard Function Plots of the EGGD for Different Combinations of Parameters

Figure 5 illustrates the hazard rate function of the EGGD for different parameter combinations, revealing an overall increasing hazard pattern across all cases, which is characteristic of aging and wear-out failure mechanisms. The plots show that larger values of the scale parameter θ significantly elevate the hazard level and steepen its growth over time, while increases in the shape parameter α also lead to higher hazard rates, indicating a stronger tendency toward failure as x increases. Variations in β modify the curvature of the hazard function, allowing the model to represent different rates of risk escalation, thereby demonstrating the flexibility of the EGGD in capturing a wide range of lifetime and reliability behaviours.

Maximum Likelihood Estimation

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n drawn from the exponential–generalized gamma mixture distribution with probability density function defined in (4).

For a sample size $x_1, x_2, x_3, \dots, x_n$, the likelihood function is given as

$$L(\alpha,\beta,\theta)=\prod_{i=1}^{n} f(x;\alpha,\beta,\theta) \quad (11)$$

Substituting the PDF in (4) into (10), we have:

$$L(\alpha,\beta,\theta)=\prod_{i=1}^{n} f\left(\frac{e^{-x^{\beta}\theta}x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}}+e^{-x\theta}x\theta^{2}\Gamma\left(\alpha+\frac{1}{\beta}\right)}{x\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)}\right)$$

Taking the natural logarithm, we obtain the log-likelihood function

$$\ell\big((\alpha,\beta,\theta)\big)=\sum_{i=1}^{n} \ln f(x;\alpha,\beta,\theta)$$

$$\ell\big((\alpha,\beta,\theta)\big)=\sum_{i=1}^{n} in\left(\frac{e^{-x^{\beta}\theta}x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}}+e^{-x\theta}x\theta^{2}\Gamma\left(\alpha+\frac{1}{\beta}\right)}{x\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)}\right)$$

$$\ell\big((\alpha,\beta,\theta)\big)=\sum_{i=1}^{n}\left(in\left(e^{-x^{\beta}\theta}x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}}+e^{-x\theta}x\theta^{2}\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)-in\mathrm{x}-in\left(\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)\right)\right)$$

Derivative with Respect to θ

$$\frac{\delta L}{\delta\theta}=\sum_{I=1}^{n}\frac{1}{f(X_i)}\;\frac{\delta f(X_i)}{\delta\theta}$$

$$\frac{\delta L}{\delta\theta}=\sum_{i-1}^{n}\left[\frac{e^{-\theta x^{\beta}}x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}-1}\left(\left(\alpha+\frac{1}{\beta}\right)-\theta x^{\beta}\right)+e^{-\theta x}x\Gamma\left(\alpha+\frac{1}{\beta}\right)(2\theta-\theta^{2}x)}{e^{-x^{\beta}\theta}x^{\alpha\beta}\beta\theta^{\alpha+\frac{1}{\beta}}+e^{-x\theta}x\theta^{2}\Gamma\left(\alpha+\frac{1}{\beta}\right)}-\frac{\frac{1}{\beta}\theta^{\frac{1}{\beta}-1}\Gamma(\alpha)+\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\theta^{\frac{1}{\beta}}\Gamma(\alpha)+\theta\Gamma\left(\alpha+\frac{1}{\beta}\right)}\right]=0 \quad (12)$$

Derivative with Respect to α

$$\frac{\delta L}{\delta\alpha}=\sum_{I=1}^{n}\frac{1}{f(X_i)}\frac{\delta f(X_i)}{\delta\alpha}$$

$$\frac{\delta L}{\delta \alpha} = \sum_{i=1}^{n} \left[\frac{e^{-\theta x_i^{\beta}} x_i^{\alpha\beta} \beta \theta^{\alpha + \frac{1}{\beta}} (in\, \theta + \beta \ln x_i) + e^{-\theta x_i}\, \theta^2 x_i\, \Gamma\left(\alpha + \frac{1}{\beta}\right) \psi\left(\alpha + \frac{1}{\beta}\right)}{e^{-\theta x_i^{\beta}} x_i^{\alpha\beta} \beta \theta^{\alpha + \frac{1}{\beta}} + e^{-\theta x_i} x_i \theta^2\, \Gamma\left(\alpha + \frac{1}{\beta}\right)} - \frac{\theta^{\frac{1}{\beta}} \Gamma(\alpha) \psi(\alpha) + \theta\, \Gamma\left(\alpha + \frac{1}{\beta}\right) \psi\, \Gamma\left(\alpha + \frac{1}{\beta}\right)}{\theta^{\frac{1}{\beta}}\, \Gamma(\alpha) + \theta \Gamma\left(\alpha + \frac{1}{\beta}\right)} \right] = 0 \tag{13}$$

Where $\psi(.)$ denotes the digamma function

Derivative with Respect to β

$$\frac{\delta L}{\delta \beta} = \sum_{i=1}^{n} \frac{1}{f(X_i)} \frac{\delta f(x_i)}{\delta \beta}$$

Derivative with Respect to β

$$\frac{\delta L}{\delta \beta} = \sum_{i=1}^{n} \left[\frac{e^{-\theta x_i^{\beta}} x_i^{\alpha\beta} \beta \theta^{\alpha + \frac{1}{\beta}} \left(\frac{1}{\beta} \alpha \ln x_i - \theta x_i^{\beta} \ln x_i \right)}{e^{-\theta x_i^{\beta}} x_i^{\alpha\beta} \beta \theta^{\alpha + \frac{1}{\beta}} + e^{-\theta x_i} x_i \theta^2\, \Gamma\left(\alpha + \frac{1}{\beta}\right)} - \frac{\frac{1}{\beta^2} \theta^{\frac{1}{\beta}} \Gamma(\alpha) \ln \theta + \frac{1}{\beta^2} \theta\, \Gamma\left(\alpha + \frac{1}{\beta}\right) \psi\left(\alpha + \frac{1}{\beta}\right)}{\theta^{\frac{1}{\beta}}\, \Gamma(\alpha) + \theta \Gamma\left(\alpha + \frac{1}{\beta}\right)} \right] = 0 \tag{14}$$

The maximum likelihood estimates (MLEs) of the parameters $\alpha, \beta, and\ \theta$ are obtained by solving the system of equations (12)– (14) simultaneously. These estimates provide the parameter values that maximize the likelihood function for the given sample and form the basis for subsequent inference and model assessment for the exponential–generalized gamma mixture distribution.

Simulation Study

A simulation study was conducted to evaluate the finite-sample performance of the maximum likelihood estimators for the EGGD. Random samples were generated using the acceptance-rejection method for three parameter combinations:
$(\alpha, \beta, \theta) = (1.5, 1.8, 1.2), (1.8, 2.2, 1.0), (1.9, 1.0, 1.5)$ with sample sizes n = 50, 100, 300, 500, 800, and 1000. The estimators' performance was assessed in terms of bias and mean squared error (MSE). Results are summarized in Table 1, illustrating that the MLEs are generally consistent and efficient across different sample sizes.

Table 1: The Numerical Illustration of the Simulation Study of the EGGD

Parameters	N	Parameters	MLEs	Biases	MSEs
	50	α	1.7220	0.2220	0.7972
		β	2.2602	0.4602	0.5470
		θ	1.2849	0.0849	0.0907
α = 1.5	100	α	1.9267	0.4267	0.5564
		β	2.1025	0.3025	0.1884
β= 1.8		θ	1.2490	0.0490	0.0309
	300	α	1.5544	0.0544	0.0505
$\theta = 1.2$		β	1.8526	0.0526	0.0257
		θ	1.2335	0.0335	0.0117
	500	α	1.5424	0.0424	0.0490
		β	1.8003	0.0003	0.0138
		θ	1.2450	0.0450	0.0080
	800	α	1.5311	0.0311	0.0229
		β	1.7893	−0.0107	0.0204
		θ	1.2091	0.0091	0.0035
	1000	α	1.4996	−0.0004	0.0256
		β	1.7758	−0.0242	0.0175
		θ	1.1964	−0.0036	0.0049
	50	α	2.0302	0.2302	0.5472
		β	2.1769	−0.0231	0.3241
		θ	1.1127	0.1127	0.1382
α = 1.8	100	α	2.0563	0.2563	0.4931
		β	2.3185	0.1185	0.1641
β= 2.2		θ	1.0044	0.0044	0.0106
	300	α	1.7956	−0.0044	0.0345
$\theta = 1.0$		β	2.1450	−0.0550	0.0714
		θ	0.9988	−0.0012	0.0056
	500	α	1.7110	-0.0890	0.0237
		β	2.1778	-0.0222	0.0170
		θ	0.9895	-0.0105	0.0033
	800	α	1.8486	0.0486	0.0191
		β	2.2232	0.0232	0.0094
		θ	0.9974	-0.0026	0.0010
	1000	α	1.7065	−0.0935	0.0179
		β	2.2086	0.0086	0.0145
		θ	0.9756	−0.0244	0.0023
	50	α	3.6978	1.7978	7.8645
		β	1.5154	0.5154	0.4985
		θ	1.6822	0.1822	0.0848

Parameters	N	Parameters	MLEs	Biases	MSEs
α = 1.9	100	α	1.9870	0.0870	0.6616
		β	1.2285	0.2285	0.1734
β= 1.0		θ	1.4512	−0.0488	0.0677
	300	α	1.9745	0.0745	0.3575
$\theta = 1.5$		β	1.1025	0.1025	0.0525
		θ	1.4516	−0.0484	0.0414
	500	α	2.0460	0.1460	0.2000
		β	1.0364	0.0364	0.0100
		θ	1.5119	0.0119	0.0128
	800	α	1.6936	−0.2064	0.4922
		β	1.1109	0.1109	0.0262
		θ	1.3684	−0.1316	0.0746
	1000	α	1.7964	−0.1036	0.1866
		β	1.0469	0.0469	0.0090
		θ	1.4409	−0.0591	0.0264

Table 1 reports the simulation results for the EGGD based on different sample sizes N, showing the maximum likelihood estimates (MLEs), biases, and mean squared errors (MSEs) of the parameters α, β, and θ. Across all parameter settings, the MLEs move closer to the true values as the sample size increases. Correspondingly, the absolute biases and MSEs decrease steadily with larger N, indicating improved estimation accuracy and consistency. For small samples (N=50 and 100), noticeable biases and larger MSEs are observed, particularly for α, reflecting higher variability in estimation. As N increases to 300 and above, the biases become small, and the MSEs markedly reduce for all parameters. At large sample sizes (N≥800), the estimators exhibit minimal bias and low MSEs, confirming the consistency and reliability of the MLEs for the EGGD.

Application to Real World Datasets

To evaluate their practical applicability, five well-known life time distributions are considered: the Exponential–Generaliz ed Gamma with Mean-Based Mixing Proportion (EGGD), th e Log-normal 3-parameter (LN3), the Gamma 3-parameter (GA3), the Half-Logistic Inverse Rayleigh (HLIRD), and the Exponentiated Inverse Rayleigh (EIRD) distributions. The E GGD integrates the simplicity of the exponential with the fle xibility of the generalized gamma, allowing it to model diver se hazard patterns, including increasing, decreasing, and bath tub-shaped rates. The LN3 distribution is suitable for positive ly skewed data and includes location, scale, and threshold pa rameters, making it appropriate for shifted lognormal dataset s. The GA3 distribution also models skewed positive data an d incorporates a threshold parameter that shifts the distributio n along the x-axis, a feature the standard gamma distribution cannot accommodate. The HLIRD distribution is a two-para meter model commonly used for reliability and lifetime data with non-monotone hazard rates, particularly when the failur e rate rises and then falls. Similarly, the EIRD distribution is a two-parameter model that extends the Inverse Rayleigh dist ribution with an exponentiation parameter, enabling it to capt ure increasing, decreasing, or bathtub-shaped hazard rates.

Dataset 1 comprises failure-time data (in units of 1000 hours) for 63 aircraft windshields, originally documented by Murthy et al. (2004). Each windshield is a multi-layered structure with a strong outer skin and inner layers. Failures occur due to heating system malfunctions or delamination of the outer ply, necessitating replacement. The data exhibit an increasing hazard pattern typical of fatigue-related failures in engineering systems, making them appropriate for assessing the performance of lifetime distributions designed for reliability analysis.

Dataset 2 contains 73 annual exceedances of maximum flood-peak levels recorded at the Wheaton River in the Yukon Territory, Canada, as reported by Hameldarbandi and Yilmaz (2019). This dataset is characterized by high variability and heavy-tailed behaviour, providing a challenging environment for testing the adaptability of flexible parametric distributions in hydrological applications

Dataset 3 comprises survival-time data (in years) for 46 patients undergoing chemotherapy, originally documented by Sapkota et al. (2023). The data reflect the time until an event (death or censoring) and are commonly used in reliability and survival analysis studies to evaluate lifetime distributions for biomedical applications.

Table 2: Results for Failure-time Data for 63 Aircraft Windshields

Models	Parameters (MLEs (S.E.s))	AIC	AICC	HQIC	BIC
EGGD	$\alpha = 4.3977\ (0.347)$ $\beta = 1.4478\ (0.043)$ $\theta = 1.000\ (0.001)$	222.0113	222.4181	224.5401	228.4407
LN3	$\alpha = 0.4506\ (0.121)$ $\beta = 0.925\ (0.101)$ $\theta = 0.0020\ (0.026)$	231.7390	232.1457	234.2677	238.1684
GA3	$\alpha = 1.3584\ (0.059)$ β =1.0000 (0.005) $\theta = 4.8667\ (0.346)$	810.8313	811.2381	813.3600	817.2607
HLIRD	$\alpha = 0.4646\ (0.057)$ $\beta = 0.3674\ (0.0467)$	315.4338	315.6338	317.1196	319.7201
EIRD	$\alpha = 0.3254\ (0.047)$ $\beta = 0.3827\ (0.047)$	331.0424	331.2424	332.7282	335.3286

For Dataset one, the EGGD model provided a relatively good fit, with parameter estimates α = 4.3977 (0.347), β = 1.4478 (0.043), and θ = 1.000 (0.001), and AIC, AICc, HQIC, and BIC values of 222.0113, 222.4181, 224.5401, and 228.4407, respectively. The LN3 model showed slightly higher information criteria values, indicating a somewhat poorer fit. Although the GA3 model had very large AIC and BIC values (810.8313 and 817.2607, respectively), the parameter estimates suggest over-parameterization and poor numerical stability. The HLIRD and EIRD models had substantially higher information criteria (315.4338 and 331.0424 for AIC, respectively), indicating that they were unable to capture the variability in the dataset effectively. Overall, the EGGD model appears to offer the most flexible and reliable fit among the tested models, making it the most appropriate choice for accurately describing the underlying distribution and supporting reliable predictions or inference for this dataset.

Table 3: Results for Hydrological Flood-peak Exceedance Data from the Wheaton River

Models	Parameters (MLEs (S.E.s))	AIC	AICC	HQIC	BIC
EGGD	$\alpha = 2.6883\ (0.789)$ $\beta = 1.7427\ (0.236)$ $\theta = 0.5323\ (0.093)$	204.8926	205.2994	207.4213	211.3220
LN3	$\alpha = 0.4506\ (0.097)$ $\beta = 0.9250\ (0.109)$ $\theta = 0.1000\ (0.001)$	231.739	232.1457	234.2677	238.1684
GA3	$\alpha = 9.9121\ (0.001)$ $\beta = 0.0017\ (0.002)$ $\theta = 5.5455\ (0.005)$	315.6058	315.1990	313.0771	309.1764
HLIRD	$\alpha = 0.4646\ (0.057)$ $\beta = 0.3674\ (0.018)$	315.4338	315.6338	317.1196	319.7201
EIRD	$\alpha = 0.3254\ (0.047)$ $\beta = 0.3827\ (0.032)$	331.0424	331.2424	332.7282	335.3286

For Dataset 2, the EGGD model provided the best overall fit, with parameter estimates α = 2.6883 (0.789), β = 1.7427 (0.2 36), and θ = 0.5323 (0.093), and the lowest AIC (204.8926), AICc (205.2994), HQIC (207.4213), and BIC (211.3220) val ues among the competing models. The LN3 model showed a poorer fit with higher information criteria values, while GA3 produced substantially larger AIC and BIC values together w ith a very small estimate of β, suggesting possible boundary b ehavior and reduced model adequacy. The HLIRD and EIRD models had much higher AIC and BIC values, indicating the y were unable to capture the variability in the dataset effectiv ely. Taken together, these results indicate that EGGD is the m ost flexible and reliable model for modeling the data in this c ase.

Table 4: Table 4: Results for Survival-time Data for 46 Patients Undergoing Chemotherapy

Models	**Parameters (MLEs (S.E.s))**	**AIC**	**AICC**	**HQIC**	**BIC**
EGGD	$\alpha = 9.976\ (1.185)$ $\beta = 1.722\ (0.109)$ $\theta = 1.173\ (0.026)$	118.8806	119.4520	120.9356	124.3665
LN3	$\alpha = 0.0021\ (0.181)$ $\beta = 1.1344\ (0.174)$ $\theta = 0.001\ (0.041)$	126.7688	127.3402	128.8239	132.2547
GA3	$\alpha = 1.247\ (0.122)$ $\beta = 1.00001\ (0.021)$ $\theta = 1.0793\ (0.006)$	1441.7163	1442.2877	1447.2022	1443.7713
HLIRD	$\alpha = 0.3798\ (0.052)$ $\beta = 0.1259\ (0.021)$	141.2919	141.5710	142.6619	144.9492
EIRD	$\alpha = 0.2746\ (0.046)$ $\beta = 0.1362\ (0.024))$	150.0223	150.3014	153.6796	151.3923

For Dataset 3, which comprises survival-time data (in years) for 46 patients undergoing chemotherapy, the EGGD model provided the best fit, with parameter estimates α = 9.976 (1.185), β = 1.722 (0.109), and θ = 1.173 (0.026), and AIC, AICc, HQIC, and BIC values of 118.8806, 119.4520, 120.9356, and 124.3665, respectively. The LN3 model had slightly higher information criteria (AIC = 126.7688, BIC = 132.2547), indicating a somewhat poorer fit. The GA3 model displayed extremely large AIC and BIC values (AIC = 1441.7163, BIC = 1443.7713), reflecting numerical instability or over-parameterization. The HLIRD and EIRD models also had considerably higher information criteria (AIC = 141.2919 and 150.0223, respectively), suggesting they were unable to adequately capture the variability in the survival times. However, the EGGD model provides the most flexible and reliable fit among the tested models, making it the preferred choice for modeling patient survival and supporting accurate inference in biomedical reliability analysis.

CONCLUSION

This paper introduces the Exponential–Generalized Gamma distribution incorporating a mean-based mixing weight, presenting a novel and adaptable framework for analyzing lifetime and survival data. Key theoretical characteristics of the model were systematically established, encompassing its probability density function, cumulative distribution function, moment expressions, survival and hazard rate functions, order statistics, and Rényi entropy, all contributing to a thorough understanding of the distribution's underlying behavior. Parameter estimation was accomplished through maximum likelihood techniques, while an extensive Monte Carlo simulation study verified the consistency and precision of the resulting estimators across diverse sample configurations. The empirical utility of the EGGD was illustrated through its application to authentic engineering and hydrological datasets, with its performance benchmarked against several well-established lifetime models. Model fit was evaluated using conventional information-theoretic criteria, including AIC, AICc, BIC, and HQIC. Across all examined scenarios, the EGGD demonstrated either superior or comparable performance, showing enhanced capacity to accommodate skewness, tail characteristics, and intricate hazard patterns relative to alternative models. These results establish the EGGD as a resilient and versatile instrument for survival and reliability investigations, with promising extensions to engineering disciplines, environmental research, and risk assessment contexts. Subsequent investigations could explore Bayesian estimation frameworks, regression-based adaptations, or multivariate extensions to broaden the proposed model's practical applicability.

REFERENCES

Aderoju, S. A. (2021). Samade probability distribution: Its properties and application to real lifetime data. *Asian Journal of Probability and Statistics, 14*(1), 1–11. https://doi.org/10.9734/AJPAS/2021/v14i130317

Aderoju, S. A., Dauda, K. A., & Olaifa, J. B. (2025). Development of a new flexible lifetime model: Its associated inferences and applications to cancer and COVID-19 data. *Journal of Science and Technology, 30*(8), Article 3024. https://doi.org/10.20428/jst.v30i8.3024

Aderoju, S. A., Aleshinloye, N. I., Taiwo, B. L., & Sanni, B. I. (2025). A new lifetime distribution and its application to cancer data. *Journal of Biostatistics and Epidemiology, 9*(4), 451–460. https://doi.org/10.18502/jbe.v9i4.16670

Al-Hussaini, A. H., & Al-Nadif, A. A. (2019). On generalized gamma distribution and its application to survival data. *International Journal of Statistics and Probability, 8*(3), 16–28. https://doi.org/10.5539/ijsp.v8n3p16

Alghamdi, A. S., & AbdEl-Raouf, M. M. (2023). A new alpha power cosine–Weibull model with applications to hydrological and engineering data. *Mathematics, 11*, 673. https://doi.org/10.3390/math11030673

Gómez, H. W., Bolfarine, H., & Louzada-Neto, F. (2008). A new lifetime model: The complementary exponential power distribution. *Computational Statistics & Data Analysis, 52*(7), 3853–3864. https://doi.org/10.1016/j.csda.2007.12.013

Hameldarbandi, M., & Yilmaz, M. (2019). A new perspective on transmuted distribution. *Communications Faculty of Sciences, University of Ankara Series A1, 68*, 1144–1162.

Hosseini, B., Afshari, M., Alizadeh, M., & Afify, A. Z. (2022). A new weighted-Lindley distribution: Properties, classical and Bayesian estimation with an application. *Pakistan Journal of Statistics and Operations Research, 18*(4), 1049–1066. https://doi.org/10.18187/pjsor.v18i4.4106

Mahmoudi, E., & Jafari, A. A. (2012). Generalized gamma distribution and its application in survival analysis. *Journal of Data Science, 10*(4), 371–386.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability, 42*(2), 299–302. https://doi.org/10.1109/24.229504

Murthy, D. N. P., Raghavan, V. R., & Rachamadugu, R. (2004). *Product reliability: Specification and performance*. Springer.

Sanni, B. I., Aderoju, S. A., Lamidi, R. K., Adeshola, A. D., Jimoh, A. K., & Adeniyi, E. J. (2025). *The new two-parameter Exponential-Gamma-based distribution: Properties and application to non-communicable disease data.* Journal of Science and Technology**, 30**(12). https://doi.org/10.20428/jst.v30i12.3314

Sapkota, L. P., Kumar, V., Gemeay, A. M., Bakr, M. E., Balogun, O. S., & Muse, A. H. (2023). New Lomax-G family of distributions: Statistical properties and applications. *AIP Advances, 13*(9), 095128. https://doi.org/10.1063/5.0171949

Shanker, R., & Shukla, D. (2019). A study on exponential distribution and its applications. *Journal of Statistics Applications & Probability, 8*(3), 13–25.

Shanker, R., & Shukla, K. K. (2019). A new generalization of the Lindley distribution: Its properties and applications. *Biostatistics & Epidemiology, 3*(1), 57–68. https://doi.org/10.1080/24709360.2018.1562303

Stacy, E. W. (1962). A generalization of the gamma distribution. *The Annals of Mathematical Statistics, 33*(3), 1187–1192. https://doi.org/10.1214/aoms/1177704472

Uma Maheswari, R., & Alexander, L. (2017). A mixture of identical distributions of exponential, gamma, lognormal, Weibull, and Gompertz approaches to heterogeneous survival data. *International Journal of Current Research, 9*(9), 57521–57532.

Umar, M. A., & Yahya, W. B. (n.d.). A new exponential–gamma distribution with applications. *Journal of Modern Applied Statistical Methods*.

Wang, X., & Wang, L. (2019). Generalized lognormal distribution and its application in survival analysis. *Statistical Papers, 60*(3), 1045–1067.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics, 18*(3), 293–297.