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ON THE PROPERTIES AND MLEs OF GENERALIZED ODD GENERALIZED EXPONENTIAL- EXPONENTIAL DISTRIBUTION

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ABSTRACT

For proper actualization of the phenomenon contained in some lifetime data sets, a generalization, extension or modification of classical distributions is required. In this paper, we introduce a new generalization of exponential distribution, called the generalized odd generalized exponential-exponential distribution. The proposed distribution can model lifetime data with different failure rates, including the increasing, decreasing, unimodal, bathtub, and decreasing-increasing-decreasing failure rates. Various properties of the model such as quantile function, moment, mean deviations, Renyi entropy, and order statistics. We provide an approximation for the values of the mean, variance, skewness, kurtosis, and mean deviations using Monte Carlo simulation experiments. Estimating of the distribution parameters is performed using the maximum likelihood method, and Monte Carlo simulation experiments is used to assess the estimation method. The method of maximum likelihood is shown to provide a promising parameter estimates, and hence can be adopted in practice for estimating the parameters of the distribution. An application to real and simulated datasets indicated that the new model is superior to the fits than the other compared distributions.

Keywords – *flexible generalized exponential model, non-monotone failure rates, moments, and method of maximum likelihood.*

INTRODUCTION

Many classical probability distributions have been used to make inferences about a population based on a set of data from the population. It is well-known that some of these distributions lack the ability to properly explain the hidden information contained in some of the available real data. The exponential distribution (ED) is one of the most important distributions with a wide range of applications in statistical practice. It is used in reliability and biological studies for modelling lifetime data with constant failure rate (FR). However, the data sets from these fields, might not adequately be describe by the ED due to its constant failure rates assumption. Moreover, if we look around the physical world we live in, we can find undeniable evidence that the failure rates of most, if not all, real-world objects are not constant. Thus, the major weakness of the exponential distribution is its inability to accommodate non-constant (monotone and non-monotone) failure rates. Many generalizations of the distribution have been proposed to cope with the situation. Few among the modifications of the ED are: the generalized exponential (GE) distribution (Gupta & Kundu, 1999), which introduced a shape parameter to the ED. The authors derived many distributional properties and reliability characteristics of the GE distribution. The distribution fits better than other distributions when subjected to real data set applications. The Exponential-Weibull distribution (Cordeiro et al., 2014). The distribution has monotone and non-monotone FR, which allows it to provides a good fit when fitted to a real-life data set. (Abba & Singh, 2018), also proposed a three-parameters new odd generalized exponential-exponential (NOGE-E) distribution and studied some of the distributional properties, including the quantile function, moments, moment generating function, entropy and order statistics. The model was subjected to a real data application and found to fit better than a two-parameters odd generalized exponential-exponential (OGE-E) distribution developed by (Maiti & Pramanik, 2016). Recently, Mohammed & Yahaya (2019) introduce exponentiated transmuted inverse-exponential distribution which extends the classical inverted exponential distribution. The model is appeared to be flexible when it was applied to real data set.

In this paper, we extend the ED to three-parameters flexible distribution called generalized odd generalized exponential-exponential (GOGEE) distribution. Unlike most of the recent extension of the ED, GOGEE model has decreasing-increasing-decreasing failure rate in addition to the four standard failure rates. Thus, the new model is a good alternative to many of the generalized exponential distribution. The model is constructed using a two-parameters generalized odd generalized exponential (GOGE-G) family of distribution introduced by (Alizadeh et al., 2017). This family of distributions (generator), has a wide advantage, as flexible distributions can be defined with both monotone and non-monotone FR, even though the baseline FR may be monotonic or constant. A comprehensive description of its general structural properties is derived. For a random variable X, the probability density function (PDF) and cumulative distribution function (CDF) of the GOGE-G class are given, respectively by;

$$f(x) = \frac{\alpha\vartheta g(x;\psi)G(x;\psi)^{\alpha-1}}{(1-G(x;\psi)^{\alpha})^2}\exp\left[-\left\{\frac{G(x)^{\alpha}}{1-G(x)^{\alpha}}\right\}\right]\left(1-\exp\left[-\left\{\frac{G(x)^{\alpha}}{1-G(x)^{\alpha}}\right\}\right]\right)^{\vartheta-1} \tag{1}$$

and

$$F(x) = \left(1-\exp\left[-\left\{\frac{G(x)^{\alpha}}{1-G(x)^{\alpha}}\right\}\right]\right)^{\vartheta} \tag{2}$$

where $\alpha > 0$ and $\vartheta > 0$ are two additional shape parameters to any baseline distribution, and $x \in \mathbb{R}$, is the support of $g(x,\psi)$, while $G(x;\psi)$ and $g(x;\psi)$ are the CDF and PDF of the baseline distribution, which are taken for this study as the CDF and PDF of the ED as given in Equations (3) and (4) with scale parameter $\lambda > 0$.

$$G(x) = 1 - e^{-\lambda x} \tag{3}$$

and

$$g(x) = \lambda e^{-\lambda x} \tag{4}$$

Model Description

Definition 1. A random variable X is said to follow a generalized odd generalized exponential-exponential (GOGE-E) distribution if its PDF has the form:

$$f(x) = \frac{\alpha\vartheta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left[1-\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{-2}}{\left(1-\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]\right)^{1-\vartheta}}\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right] \tag{5}$$

The CDF of the GOGE-E distribution is

$$F(x) = \left(1-\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]\right)^{\vartheta} \tag{6}$$

where $\boldsymbol{\rho} = (\alpha,\vartheta,\lambda)'$ is a vector of the distribution parameters. The plot of the GOGE-E distribution PDF is displayed in Figures 1 for different values of the parameters α, ϑ and λ. It is noted that the proposed model density has a decreasing and unimodal shapes. The survival, failure rate, reverse failure (FR) rate functions of the GOGE-E distribution are defined respectively, as:

$$S(x) = 1-\left(1-\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]\right)^{\vartheta} \tag{7}$$

$$h(x) = \frac{\alpha\vartheta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}\left[1-\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{-2}\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]}{\left(1-\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]\right)^{1-\vartheta}-\left(1-\exp\left[-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]\right)} \tag{8}$$

and

$$r(x) = \frac{\alpha\vartheta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}}{\left[1-\left(1-e^{-\lambda x}\right)^{\alpha}\right]^{2}}\left[\exp\left[\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}\right]-1\right]^{-1} \tag{9}$$

Figure 2 present the FR function plots of the GOGE-E distribution for different values of the parameters α, ϑ and λ. The plots reveal the potential of the new model, that, it can be used to models lifetime data with distinct FR shapes, including the decreasing, increasing, bathtub, up-side-down bathtub (unimodal), and decreasing-increasing-decreasing shapes. This is important as most of generalizations (modifications or extensions) of the classical lifetime distributions have only the bathtub or unimodal FR shapes, which limits their ability to provide a better fit in modelling some problems with different FR shapes.

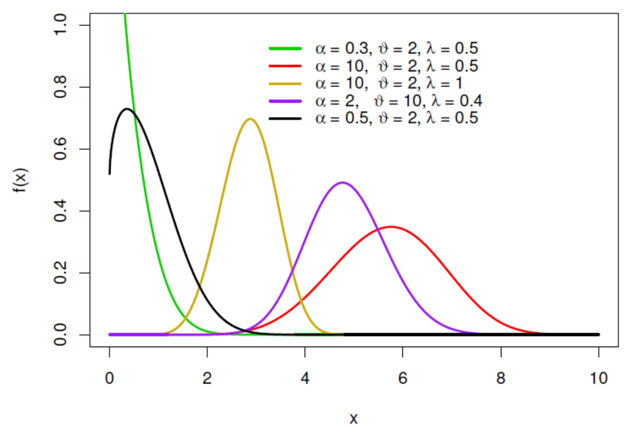

Figure 1. Density function plot for some parameters' values

Statistical Properties of the GOGE-E Distribution

In this section, we derived the statistical properties of the distribution, including the quantile function, moments, mean deviations and Renyi entropy. The distribution of the order statistics of the GOGEE random variable is as well defined. We determine an approximation for the values of the mean, variance, skewness, kurtosis and mean deviations of X using Monte Carlo simulation technique.

Quantile Function

For the GOGE-E distribution, the quantile function, say $Q(u)$ is provided in the following proposition.

Proposition 3.1: *For a non-negative continuous random variable X that follows the GOGE-E distribution, the quantile function $Q(u)$ is given by*

$$Q(u) = x_u = \lambda^{-1} ln\left[1 - \left(1 + \left\{ln\left(\frac{1}{1-u^{1/\vartheta}}\right)\right\}^{-1}\right)^{-\frac{1}{\alpha}}\right]^{-1} \quad (10)$$

The quantile function reduced to median (M) when $u = \frac{1}{2}$.

Quantile Based Skewness and Kurtosis

Skewness measures the degree of asymmetry while kurtosis measures the degree of tail heaviness. Based on the quantile function $Q(u)$, the Bowley measure of skewness and Moor's measure of kurtosis defined by (Bowley, 1920) and (Moors, 1988), are respectively, given by

$$Sk = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \text{ and } K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

3.2 Useful expansion

Here, we provide the CDF and PDF representations using theory binomial and series expansions as follows:

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i z^i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b) z^i}{i!\,\Gamma(b-i)} \qquad \text{for } |z| < 1 \text{ and } b > 0$$

and

$$e^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j}$$

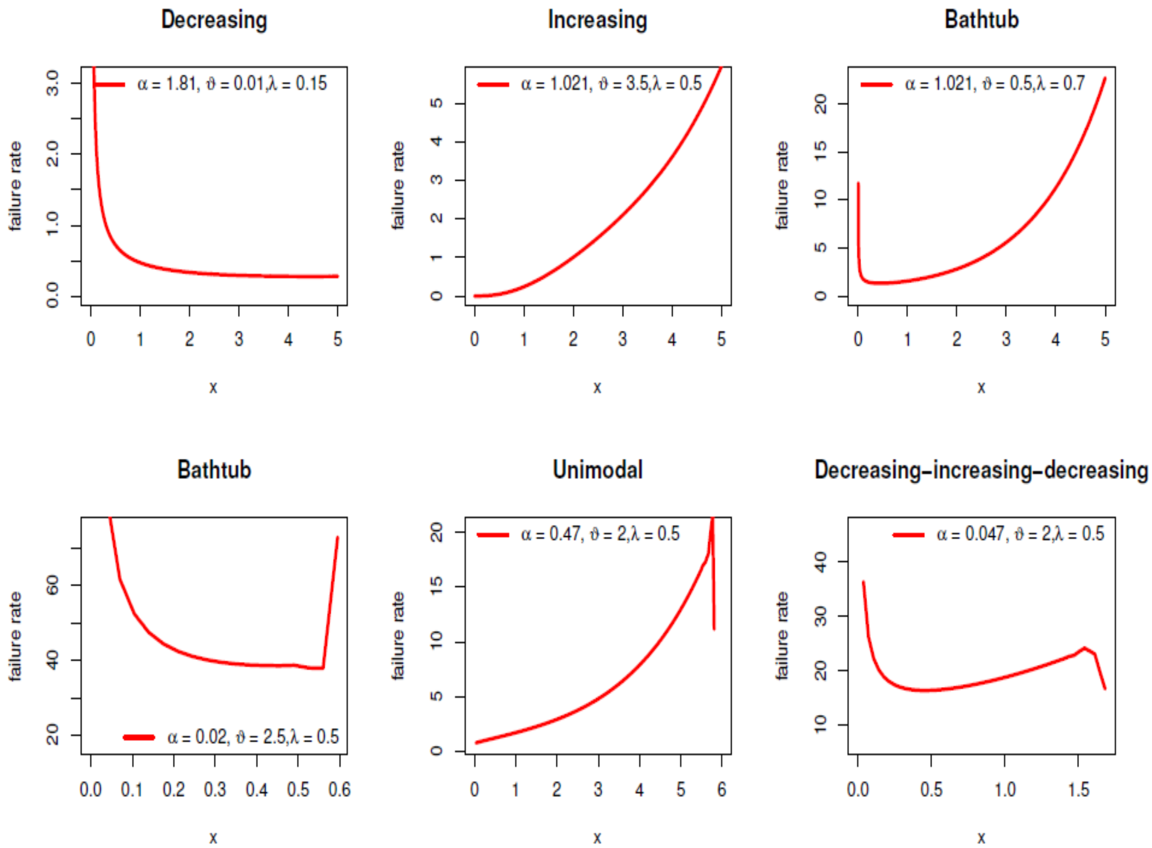

Figure 2. failure rate function plots at various of the parameters

Then, the CDF (6) and PDF (5) representations are given as

$$F(x;\boldsymbol{\rho}) = \left[1 - e^{-\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}}\right]^{\vartheta} = \sum_{i=0}^{\infty} \frac{\Gamma(\vartheta-1)(-1)^i}{i!\,\Gamma(\vartheta-i-1)} e^{-i\left\{\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{1-\left(1-e^{-\lambda x}\right)^{\alpha}}\right\}} \quad (11)$$

and

$$f(x;\boldsymbol{\rho}) = K_{i,j,k,l}\,\alpha\lambda e^{-(l+1)\lambda x} \quad (12)$$

where

$$K_{i,j,k,l} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(\vartheta+1)\Gamma(j+3)\Gamma\left(\alpha(k+j+1)\right)(-1)^{i+j+l}(i+1)^j}{i!\,j!\,\Gamma(\vartheta-i-1)\,k!\,\Gamma(j+3-k)\,l!\,\Gamma(\alpha(k+j+1)-l)}$$

Moments

Preposition 3.2: *Let X be a random variable with the GOGE-E density function (5). Then the r^{th} moment about origin of X is given by*

$$\mu_r' = \frac{\alpha\lambda\Gamma(r+1)}{\left((l+1)\lambda\right)^{r+1}} K_{i,j,k,l} \quad (13)$$

where,

$K_{i,j,k,l} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(\vartheta+1)\Gamma(j+3)\Gamma(\alpha(k+j+1))(-1)^{i+j+l}(i+1)^{j}}{i!j!\Gamma(\vartheta-i-1)k!\Gamma(j+3-k)l!\Gamma(\alpha(k+j+1)-l)}, r = 1,2,\ldots$

Proof: Using the definition of moment about origin and the density (12). The moment is

$$\mu_r' = E(X^r) = K_{i,j,k,l}\alpha\lambda\int_0^{\infty} x^r e^{-(l+1)\lambda x}dx \quad (14)$$

Letting $y = \lambda(l+1)x$, implies

$$\mu_r' = E(X^r) = \frac{\alpha\lambda}{((l+1)\lambda)^{r+1}}K_{i,j,k,l}\int_0^{\infty} y^r e^{-y}dy = \frac{\alpha\lambda\Gamma(r+1)}{((l+1)\lambda)^{r+1}}K_{i,j,k,l} \quad (15)$$

hence, we completed the proof.

In particular, we can determine the first four moments (for $r = 1,\ldots,4$) to calculate the variance (σ^2), skewness ($\sqrt{\beta_1}$) and kurtosis (β_2) using some known results.

3.4 Mean Deviations

In what follows, we defined the mean deviation about the mean (γ_1) and mean deviation about the median (γ_2) of GOGEE distribution. Using the median (M) and the mean (μ_1) from Equation (8) and (13), at $u = 0.5$, respectively. The mean deviations about the mean and median are defined as

$$\gamma_1 = \int_0^{\infty}|x-\mu_1|f(x)dx = 2\mu_1 F(\mu_1) - 2\Omega(\mu_1) \quad (16)$$

and

$$\gamma_2 = \int_0^{\infty}|x-M|f(x)dx = \mu_1 - 2\Omega(M) \quad (17)$$

respectively, where $\Omega = \int_0^{t} xf(x)dx$ – the first lower incomplete moment of X.

To investigate the properties of the model, we conduct a Monte Carlo simulation for $N = 1000$ samples each of size $n = 200$ from the GOGEE(α,ϑ,λ) distribution, with $\rho = (\alpha_0,\vartheta_0,1.5)^T$ - the vector of parameters, where $\alpha = 0.5, 1, 1.5, 2.5$ and 4, and $\vartheta = 0.5, 1, 1{,}5, 2.5$ and 5. Table 1 listed the numerical results for the mean, variance, skewness, kurtosis, mean deviation about the mean and mean deviation about the median with their standard deviations (SDs) in parenthesis. It is note that, the mean increases with either increasing value of α or ϑ with variant. In contrast to the skewness and kurtosis, the variance, mean deviation about the mean and mean deviation about the median all increases when either α or ϑ increase with consistent decrease in standard deviations. Additionally, the mean deviation about mean and the mean deviation about the median tends to be near when either α or ϑ increases.

Entropy and Order Statistics pf the Distribution

Rényi **entropy**

Renyi entropy is a generalization of a Shannon entropy. It is a measure of randomness or molecular disorder of a system and is widely applied in areas such as physics, molecular imaging of tumors.

Proposition 3.3: *Let the random variable X have the density function given by (5). Then the Renyi entropy X is given by*

$$I_\gamma(X) = \frac{\gamma}{1-\gamma}\log(\alpha\vartheta\lambda) + \frac{1}{1-\gamma}\log\left\{\sum_{k,\ell}^{\infty}\frac{\varphi_{k,\ell}}{j!(\gamma+\ell)\lambda}\right\} \quad (18)$$

where $\varphi_{k,\ell} = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+\ell}\Gamma(\gamma(\vartheta-1)+1)\Gamma(j+k+2)\Gamma(\alpha(\gamma+j+k)-(\gamma-1))}{i!k!\ell!\Gamma(\gamma(\vartheta-1)-(i-1))\Gamma(j+2)\Gamma(\alpha(\gamma+j+k)-(\gamma+\ell-1))}$.

Table 1. Mean, variance, skewness, kurtosis and mean deviations; $\lambda = 1.5$ and different values of α and θ. Monte Carlo error is between parentheses.

$\vartheta \rightarrow$	0.5	1.0	1.5	2.5	5.0
$\alpha = 0.5$					
Mean (μ)	0.1112 (0.0115)	0.1860 (0.0125)	0.2320 (0.0130)	0.3297 (0.0138)	0.4213 (0.0120)
Variance (σ^2)	0.0274 (0.0053)	0.0333 (0.0044)	0.0362 (0.0042)	0.0359 (0.0037)	0.0307 (0.0029)
Skewness ($\sqrt{\beta_1}$)	2.1464 (0.2887)	1.2294 (0.1996)	0.9804 (0.1636)	0.6450 (0.1489)	0.4530 (0.1337)
Kurtosis (β_2)	7.9869 (1.8844)	4.2511 (0.8661)	3.5572 (0.5818)	3.0457 (0.4167)	2.7974 (0.3072)
γ_1	0.4769 (0.0477)	0.5817 (0.0380)	0.6098 (0.0373)	0.6098 (0.0345)	0.5668 (0.0295)
γ_2	0.4147 (0.0418)	0.5573 (0.0360)	0.5948 (0.0353)	0.6033 (0.0340)	0.5633 (0.0291)
$\alpha = 1.0$					
Mean (μ)	0.2635 (0.0178)	0.4006 (0.0205)	0.4906 (0.0195)	0.6415 (0.0179)	0.7360 (0.0161)
Variance (σ^2)	0.0704 (0.0082)	0.0814 (0.0079)	0.0756 (0.0066)	0.0648 (0.0059)	0.0516 (0.0048)
Skewness ($\sqrt{\beta_1}$)	1.1745 (0.1612)	0.7150 (0.1282)	0.4720 (0.1261)	0.2220 (0.1509)	0.1648 (0.1274)
Kurtosis (β_2)	3.8114 (0.5944)	2.9233 (0.3539)	2.6089 (0.3059)	2.7879 (0.3403)	2.6346 (0.1829)
γ_1	0.8540 (0.0542)	0.9307 (0.0501)	0.9028 (0.0448)	0.8150 (0.0416)	0.7360 (0.0382)
γ_2	0.8102 (0.0513)	0.9200 (0.0486)	0.8971 (0.0441)	0.8135 (0.0415)	0.7342 (0.0382)
$\alpha = 1.5$					
Mean (μ)	0.3804 (0.0238)	0.5752 (0.0232)	0.6842 (0.0211)	0.8555 (0.0195)	0.9797 (0.0176)
Variance (σ^2)	0.1063 (0.0112)	0.1109 (0.0102)	0.0927 (0.0081)	0.0763 (0.0074)	0.0643 (0.0064)
Skewness ($\sqrt{\beta_1}$)	0.9426 (0.1414)	0.4493 (0.1383)	0.2682 (0.1321)	0.2693 (0.1754)	0.1999 (0.1492)
Kurtosis (β_2)	3.2072 (0.4457)	2.6408 (0.3163)	2.6356 (0.2262)	2.9698 (0.4477)	2.8568 (0.2914)
γ_1	1.0585 (0.0638)	1.0960 (0.0545)	0.9906 (0.0482)	0.8870 (0.0455)	0.8123 (0.0434)
γ_2	1.0294 (0.0612)	1.0919 (0.0540)	0.9893 (0.0480)	0.8852 (0.0452)	0.8111 (0.0434)
$\alpha = 2.5$					
Mean (μ)	0.6198 (0.0304)	0.8047 (0.0264)	0.9581 (0.0242)	1.1500 (0.0213)	1.2875 (0.0187)
Variance (σ^2)	0.1738 (0.0145)	0.1344 (0.0121)	0.1192 (0.0110)	0.0885 (0.0087)	0.0721 (0.0075)
Skewness ($\sqrt{\beta_1}$)	0.5611 (0.1205)	0.3160 (0.1238)	0.1554 (0.1422)	0.0493 (0.1513)	0.0323 (0.1713)

Kurtosis (β_2)	2.5670 (0.2499)	2.5910 (0.2244)	2.6983 (0.2275)	2.8211 (0.2967)	2.8985 (0.3276)
γ_1	1.3783 (0.0644)	1.1910 (0.061)	1.1210 (0.0567)	0.9518 (0.0512)	0.8577 (0.0475)
γ_2	1.3662 (0.0639)	1.1871 (0.0604)	1.1190 (0.0567)	0.9499 (0.0513)	0.8563 (0.0475)
$\alpha = 4.0$					
Mean (μ)	0.7999 (0.0328)	1.0807 (0.0296)	1.2281 (0.0267)	1.4204 (0.0216)	1.5827 (0.0176)
Variance (σ^2)	0.2065 (0.0185)	0.1721 (0.0144)	0.1387 (0.0126)	0.0956 (0.0095)	0.0668 (0.0063)
Skewness ($\sqrt{\beta_1}$)	0.5281 (0.1190)	0.0888 (0.1190)	-0.0082 (0.1259)	-0.0928 (0.1731)	0.0433 (0.1285)
Kurtosis (β_2)	2.6008 (0.2450)	2.4546 (0.1876)	2.6086 (0.1888)	2.987 (0.3021)	2.6579 (0.2053)
γ_1	1.4921 (0.0768)	1.3607 (0.0662)	1.2050 (0.0618)	0.9842 (0.0518)	0.8317 (0.0442)
γ_2	1.4837 (0.0753)	1.3579 (0.066)	1.2029 (0.0618)	0.9808 (0.0517)	0.8306 (0.0442)

Proof. By the definition of the Rényi entropy, we have

$$I_\gamma(X) = \frac{1}{1-\gamma} log\left(\int_{-\infty}^{\infty} f^\gamma(x)dx\right),\ \gamma > 0 \text{ and } \gamma \neq 1$$

After some algebra, we write the PDF (5) as

$$f^\gamma(x) = (\alpha\vartheta\lambda)^\gamma \sum_{k,\ell}^{\infty} \frac{\varphi_{k,\ell}}{j!} e^{-(\gamma+\ell)\lambda x} \quad (19)$$

where

$$\varphi_{k,\ell} = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+\ell}\Gamma(\gamma(\vartheta-1)+1)\Gamma(j+k+2)\Gamma\left(\alpha(\gamma+j+k)-(\gamma-1)\right)}{i!k!\ell!\Gamma\left(\gamma(\vartheta-1)-(i-1)\right)\Gamma(j+2)\Gamma\left(\alpha(\gamma+j+k)-(\gamma+\ell-1)\right)}.$$

we get the required proof by integrating (19).

Parameter Estimation

The estimation of the parameters of the GOGE-E distribution is done using the method of maximum likelihood estimation. Let $x_1, x_2 \cdots, x_n$ be a random sample from the GOGE-E distribution with unknown parameter vector $\boldsymbol{\rho} = (\alpha, \vartheta, \lambda)'$. The total log-likelihood function for $\boldsymbol{\rho}$ is obtained from $f(x)$ as follows.

$$\ell(\boldsymbol{\rho}) = n\ln\alpha + n\ln\vartheta + n\ln\lambda - \lambda\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\lambda x_i}\right) - \sum_{i=1}^{n}\omega_1$$
$$+(\vartheta-1)\sum_{i=1}^{n}\ln(1-\omega_2) - 2\sum_{i=1}^{n}\ln\left(1-\left(1-e^{-\lambda x_i}\right)^{\alpha}\right) \quad (20)$$

where $\omega_1 = \left((1-e^{-\lambda x_i})^{-\alpha}-1\right)^{-1}$ and $\omega_2 = \exp[-\omega_1]$. The estimate $\hat{\boldsymbol{\rho}} = \left(\hat{\alpha}, \hat{\vartheta}, \hat{\lambda}\right)'$ of $\boldsymbol{\rho} = (\alpha, \vartheta, \lambda)'$ is determined by maximizing the log-likelihood function (25) with respect to α, ϑ and λ respectively. Thus, we have following score functions.

$$\frac{\partial\ell(\rho)}{\partial\vartheta} = \frac{n}{\vartheta} + \sum_{i=1}^{n}\ln(1-\omega_2) \quad (21)$$

$$\frac{\partial\ell(\rho)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1-e^{-\lambda x_i}\right) + \sum_{i=1}^{n}\frac{\partial\omega_1}{\partial\alpha}\left[1-(\vartheta-1)(\omega_2-1)^{-1}\right]$$
$$+2\sum_{i=1}^{n}\omega_1\ln\left(1-e^{-\lambda x_i}\right) \quad (22)$$

and

$$\frac{\partial \ell(\rho)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha - 1)\sum_{i=1}^{n} x\left(e^{\lambda x} - 1\right)^{-1} - \alpha \sum_{i=1}^{n} \frac{\partial \omega_1}{\partial \lambda}\left[1 - (\vartheta - 1)(\omega_2 - 1)^{-1}\right] \quad (23)$$

where $\frac{\partial \omega_1}{\partial \lambda} = \frac{\alpha x e^{-\lambda x}\left(1-e^{-\lambda x_i}\right)^{-(\alpha+1)}}{\left[\left(1-e^{-\lambda x_i}\right)^{-\alpha}-1\right]^2}$ and $\frac{\partial \omega_1}{\partial \alpha} = -\frac{\left(1-e^{-\lambda x_i}\right)^{-\alpha}\ln\left(1-e^{-\lambda x_i}\right)}{\left[\left(1-e^{-\lambda x_i}\right)^{-\alpha}-1\right]^2}$,

Solving equations (21)-(23) analytically may be intractable. Thus, we adopt a numerical approach to obtain the maximum likelihood estimates (MLEs) of the parameters $\boldsymbol{\rho} = (\alpha, \vartheta, \lambda)'$ with a good set of initial values using R statistical package.

5. Simulation Results

This section presents a Monte Carlo simulation experiments to investigate the performance of the proposed maximum likelihood method described in the prior section for estimating the GOGEE

Table 2: Estimates, biases, MSEs, 95% confidence interval and interval width of the estimates

Sample size (n)	Parameters	Estimate	Bias	MSE	Confidence Interval		Width
					Lower	Upper	
50	$\alpha = 1.5$, $\vartheta = 1.0$, $\lambda = 0.5$	1.5309	0.0309	0.0621	1.4112	1.6507	0.2396
		1.0343	0.0343	0.0290	0.9797	1.0888	0.1091
		0.5126	0.0126	0.0041	0.5050	0.5203	0.0154
75		1.5163	0.0163	0.0589	1.4015	1.6312	0.2297
		1.0306	0.0306	0.0266	0.9803	1.0809	0.1006
		0.5064	0.0064	0.0028	0.5009	0.5119	0.0110
100		1.5156	0.0156	0.0575	1.4034	1.6277	0.2243
		1.0271	0.0271	0.0257	0.9781	1.0760	0.0978
		0.5042	0.0042	0.0024	0.4995	0.5089	0.0094
150		1.5030	0.0030	0.0560	1.3933	1.6127	0.2194
		1.0268	0.0268	0.0232	0.9827	1.0708	0.0881
		0.5010	0.0010	0.0019	0.4972	0.5047	0.0075
50	$\alpha = 2.0$, $\vartheta = 1.5$, $\lambda = 0.7$	2.0268	0.0268	0.0829	1.8658	2.1879	0.3221
		1.5351	0.0351	0.0432	1.4528	1.6173	0.1645
		0.7095	0.0095	0.0047	0.7005	0.7185	0.0180
75		2.0102	0.0102	0.0780	1.8576	2.1628	0.3052
		1.5350	0.0350	0.0381	1.4628	1.6072	0.1444
		0.7045	0.0045	0.0035	0.6976	0.7114	0.0138
100		2.0167	0.0167	0.0712	1.8776	2.1558	0.2782
		1.5259	0.0259	0.0365	1.4556	1.5963	0.1406
		0.7035	0.0035	0.0029	0.6980	0.7091	0.0111
150		2.0019	0.0019	0.0655	1.8735	2.1303	0.2569
		1.5270	0.0270	0.0317	1.4664	1.5876	0.1212
		0.7008	0.0008	0.0022	0.6965	0.7051	0.0086

model parameters. We used $N = 1000$ Monte Carlo samples of sizes $n = 50, 75, 100$ and 150 with some combination of the parameter values. The numerical results for the biases, mean square errors (MSEs), 95% confidence interval and interval widths of the average estimates (AEs) are presented in Tables 2 - 3. Following the conclusions derived from these results. (I) the performance of the estimates is reasonably good, that is, dispersion between the true and the estimated values are small. (II) the AEs approach to true values in all cases as the sample size increase. (III) the biases, MSEs and confidence interval widths decrease as the sample size increases. Therefore, for these experiments, we conclude that estimating the distribution parameters using the method of maximum likelihood is satisfactory.

Table 3: Estimates, biases, MSEs, 95% confidence interval and interval width of the estimates

Sample size (n)	Parameters	Estimate	Bias	MSE	Confidence Interval		Width
					Lower	Upper	
	$\alpha = 0.6$ $\vartheta = 0.3$ $\lambda = 0.7$	0.7041	0.1041	0.0570	0.6135	0.7946	0.1811
50		0.3003	0.0003	0.0167	0.2676	0.3330	0.0655
		0.8135	0.1135	0.0505	0.7397	0.8872	0.1474
		0.7011	0.1011	0.0513	0.6205	0.7816	0.1611
75		0.2945	-0.0055	0.0144	0.2664	0.3226	0.0563
		0.7884	0.0884	0.0395	0.7263	0.8505	0.1242
		0.6867	0.0867	0.0425	0.6182	0.7552	0.1370
100		0.2938	-0.0062	0.0118	0.2707	0.3168	0.0461
		0.7712	0.0712	0.0301	0.7221	0.8203	0.0981
		0.6702	0.0702	0.0330	0.6152	0.7251	0.1099
150		0.2941	-0.0059	0.0093	0.2760	0.3122	0.0362
		0.7531	0.0531	0.0214	0.7167	0.7896	0.0729
	$\alpha = 3.0$ $\vartheta = 3.0$ $\lambda = 2.0$	2.9949	-0.0051	0.1391	2.7224	3.2674	0.5450
50		3.0020	0.0020	0.1092	2.7880	3.2160	0.4280
		1.9966	-0.0034	0.0185	1.9604	2.0327	0.0723
		2.9861	-0.0139	0.1300	2.7317	3.2405	0.5088
75		3.0138	0.0138	0.0865	2.8445	3.1830	0.3385
		1.9941	-0.0059	0.0154	1.9640	2.0242	0.0602
		2.9892	-0.0108	0.1181	2.7580	3.2204	0.4623
100		3.0132	0.0132	0.0810	2.8547	3.1717	0.3170
		1.9935	-0.0065	0.0128	1.9685	2.0185	0.0500
		3.0073	0.0073	0.0926	2.8258	3.1887	0.3628
150		3.0036	0.0036	0.0695	2.8673	3.1398	0.2725
		1.9988	-0.0012	0.0091	1.9809	2.0168	0.0358

Applications

In this section, we evaluated the fitness of the GOGE-E distribution using two real datasets with other known distributions including exponential (E), odd generalized exponential-exponential (OGE-E) and new odd generalized exponential-exponential (NOGE-E) distributions. In this case, we used the two data sets, namely waiting time and randomly generated datasets. has been previously used by Ghitany et al., 2008 and Alqallaf et al., 2015. It represents the waiting time (measured in min) of 100 bank customers before service is being rendered, while for the second, an already known skewed distribution (Exponential distribution) with parameter alpha=1 was used to generate a data of size N= 150. A sample size of 100 was drawn using a method of simple random sampling (SRS) via R package. Additionally, some goodness of fit measures is used to compare the performance of the GOGE-E distribution with E, OGE-E and NOGE-E distributions. The MLEs, the values of the log-likelihood functions ($-\ell\ell$), Akaike Information Criterion (AIC), Consistent Akaike's Information Criterion (CAIC) and Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQ) of the distributions are presented below. The best distribution is the one with the smallest values of AIC, CAIC, BIC and HQIC.

Table 4: Estimated parameters for the waiting time data

Distribution	$\widehat{\alpha}$	$\widehat{\vartheta}$	$\widehat{\lambda}$
E	-	-	0.1012
OGE-E	1.7358	-	0.0409
NOGE-E	11.6550	1.9606	0.0118
GOGE-E	12.9570	0.1166	0.1086

Table 5: Goodness-of-fit statistics for waiting time dataset

Distribution	-ll	AIC	BIC	CAIC	HQIC
E	329.0209	660.0418	662.6469	660.0826	661.0961
OGE-E	323.9757	651.9513	657.1617	652.075	654.06
NOGE-E	317.5047	641.0094	648.8249	641.2594	644.1725
GOGE-E	255.4922	516.9844	524.7999	517.2344	520.1475

Table 6: Estimated parameters for the simulated data

Distribution	$\widehat{\alpha}$	$\widehat{\vartheta}$	$\widehat{\lambda}$
E	-	-	0.9660
OGE-E	4.6142	-	0.1763
NOGE-E	4.6656	0.9989	0.1746
GOGE-E	7.2791	0.1304	0.7940

Table 7: Goodness-of-fit statistics for simulated dataset

Distribution	-ll	AIC	BIC	CAIC	HQIC
E	103.4637	208.9273	211.5325	208.9681	209.9817
OGE-E	102.3628	208.7257	213.936	208.8494	210.8344
NOGE-E	102.3629	210.7258	218.5413	210.9758	213.8889
GOGE-E	37.69641	81.39281	89.20832	81.64281	84.55589

CONCLUSION

This article proposed a three-parameter model called generalized odd generalized exponential-exponential distribution. An explicit expression for some of the statistical properties, including the quantile function, moments, mean deviations and Rényi entropy are derived. We obtained some numerical values for the Mean, variance, skewness, kurtosis, and mean deviations via Monte Carlo simulation experiments. Method of maximum likelihood was employed to estimate the parameters of the model. The numerical results for the biases, mean square errors (MSEs), 95% confidence interval and interval widths of the average estimates (AEs) are presented. An application to real and simulated datasets indicated that the new model is superior to the fits than the other compared distributions.

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