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BLOCK HYBRID METHOD FOR ACCURATE SOLUTIONS OF HIGHER-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

This study presents a novel one-step block hybrid method for the direct numerical solution of second, third and fourth-order ordinary differential equations without reducing them to equivalent first-order systems. The method is formulated using power series interpolation and collocation techniques, leading to a continuous implicit scheme that is transformed into an explicit block form for efficient computation. Theoretical analysis shows that the method satisfies key properties of numerical algorithms, including consistency, zero-stability, and convergence, with a uniform order of six. The region of absolute stability is established using the Boundary Locus Method, confirming the method's strong stability characteristics. To validate its performance, the method is applied to several dynamic, linear and oscillatory initial value problems. The numerical results demonstrate excellent agreement between computed and exact solutions, with significantly smaller errors compared to existing methods in the literature. The findings indicate that the proposed method is highly accurate, computationally efficient, and robust, making it a reliable tool for solving complex higher-order differential equations encountered in science and engineering.

Keywords: One-step block method; higher-order ODEs; hybrid method; numerical stability; interpolation and collocation; initial value problem; absolute stability; convergence analysis.

INTRODUCTION

The field of science and engineering depends on mathematical models which include differential equation systems because these models help scientists and engineers forecast physical phenomena (Abdelrahim, 2021, Adewale & Sabo, 2024). The models produce equations which show how different quantities develop throughout time and space which engineers use to study moving systems and heat transfer and fluid dynamics and electrical circuits to create better designs and new technologies (Kuboye, 2015, Abdulrahim & Omar, 2017). Differential equations find applications in various domains because they serve as modeling tools for intricate systems that exist in scientific research and economic analysis and medical studies and psychological research and operations management and biological investigation and anthropological assessment (Kuboye, 2015; Abolarin et al., 2022). The natural sciences and engineering fields use ordinary differential equations because this mathematical method enables scientists to convert complex Newtonian systems into manageable equations which result in accurate predictions across diverse fields that include mechanics and astronomy and biology and aerospace engineering (Atabo & Adee, 2021, Duromola, 2022).

This study consider the direct solution of higher order initial value problems for ordinary differential equations of the form:

$$y^{(\lambda)}(\tau)=f\left(\tau, y, y^{(1)}, \cdots, y^{(\lambda-1)}\right), y(a_0)=\tau_0, y'(a_1)=\tau_1, \cdots, y^{(\lambda-1)}(a_\mu)=\tau_\mu \tag{1}$$

Specifically the second, third and fourth order initial value problems of the form

$$\left.\begin{array}{l} y''(\tau)=f(\tau, y, y'), y(a_0)=\tau_0, y'(a_1)=\tau_1 \\ y'''(\tau)=f(\tau, y, y', y''), y(a_0)=\tau_0, y'(a_1)=\tau_1, y''(a_2)=\tau_2 \\ y''''(\tau)=f(\tau, y, y', y'', y'''), y(a_0)=\tau_0, y'(a_1)=\tau_1, y''(a_2)=\tau_2, y'''(a_3)=\tau_3 \end{array}\right\} \tag{2}$$

Higher-order ordinary differential equations (ODEs) standard solution method requires researchers to convert ODEs into first-order ODE systems which they will solve through numerical methods. The process requires many computational resources while it increases system complexity and creates risks of major errors and operational difficulties. Researchers are developing a block hybrid method which will allow them to solve higher-order initial value problems by using direct methods that do not require conversion into first-order systems.

Ramos et al. (2020), Tumba et al. (2021), Raymond et al. (2021), and Skwame et al. (2024) each contribute to advancing numerical methods for solving ordinary differential equations (ODEs) through innovative block approaches tailored to higher-order initial value problems. Ramos et al. (2020) analyzed k-step linear block methods for second-order IVPs, identifying efficient formulations using collocation and interpolation that reduce computational time while maintaining accuracy. Tumba et al. (2021) introduced a half-step implicit hybrid block method of order four for third-order ODEs, demonstrating its consistency, zero-stability, and convergence, and overcoming limitations associated with first-order reduction and predictor-corrector schemes. Similar to this research, Raymond et al. (2021) created a four-step hybrid block method which directly solves fourth-order ODEs, providing improved accuracy together with decreased computational requirements through off-grid evaluations and power series-based interpolation. The work of Skwame et al. (2024) introduced a new one-step block method with eight partitions which solves second to fourth order oscillatory differential equations by improving both efficiency and convergence across different physical applications. The works demonstrate that block methods have become more effective in solving complex high-order ordinary differential equations which scientists and engineers encounter in their respective fields.

MATERIALS AND METHODS

Consider the approximate solution of power series in the

$$y(\tau)=\sum_{j=0}^{\eta+\nu}\sigma_j\tau^j \tag{3}$$

where $\sigma_j's$ are parameters to be determined, $\sigma\in[a,b]$, η and ν are the respective number of distinct collocation and interpolation points.

Let the solution of equation (3) be sought on the partition $\pi_N:\sigma=\tau_0<\tau_1<\tau_2<\cdots\tau_n<\tau_{n+1}<\tau_N=b$ on the interval $[a,b]$ with a constant step size h, given by $h=\tau_n-\tau_{n-1}$, where $n=0,1,2,\ldots,N$.

Using (3) with $\eta=4$ and $\nu=6$, the polynomial of degree $\eta+\nu-1$ as follows

$$y(\tau)=\sum_{j=0}^{9}\sigma_j\tau^j \tag{4}$$

Differentiate (4) four times, we have

$$y''''(\tau)=\sum_{j=0}^{9}j(j-1)(j-2)(j-3)\sigma_j\tau^{j-4} \tag{5}$$

Substitute (5) in to (1), we have

$$\sum_{j=0}^{9}j(j-1)(j-2)(j-3)\sigma_j\tau^{j-4}=f(\tau,y,y',y'',y''') \tag{6}$$

Now interpolating equation (4) at $\sigma_{n+\eta},\eta=0,\frac{1}{5},\frac{2}{5},\frac{3}{5}$ and collocating (6) at $\sigma_{n+\nu},\nu=0\left(\frac{1}{5}\right)1$ to give a system of nonlinear equation in a matrix form as

$$\Omega X=Z \tag{7}$$

where

$$\Omega=\begin{bmatrix}
1 & \tau_n & \tau_n^2 & \tau_n^3 & \tau_n^4 & \tau_n^5 & \tau_n^6 & \tau_n^7 & \tau_n^8 & \tau_n^9\\
1 & \tau_{n+\frac{1}{5}} & \tau^2_{n+\frac{1}{5}} & \tau^3_{n+\frac{1}{5}} & \tau^4_{n+\frac{1}{5}} & \tau^5_{n+\frac{1}{5}} & \tau^6_{n+\frac{1}{5}} & \tau^7_{n+\frac{1}{5}} & \tau^8_{n+\frac{1}{5}} & \tau^9_{n+\frac{1}{5}}\\
1 & \tau_{n+\frac{2}{5}} & \tau^2_{n+\frac{2}{5}} & \tau^3_{n+\frac{2}{5}} & \tau^4_{n+\frac{2}{5}} & \tau^5_{n+\frac{2}{5}} & \tau^6_{n+\frac{2}{5}} & \tau^7_{n+\frac{2}{5}} & \tau^8_{n+\frac{2}{5}} & \tau^9_{n+\frac{2}{5}}\\
1 & \tau_{n+\frac{3}{5}} & \tau^2_{n+\frac{3}{5}} & \tau^3_{n+\frac{3}{5}} & \tau^4_{n+\frac{3}{5}} & \tau^5_{n+\frac{3}{5}} & \tau^6_{n+\frac{3}{5}} & \tau^7_{n+\frac{3}{5}} & \tau^8_{n+\frac{3}{5}} & \tau^9_{n+\frac{3}{5}}\\
0 & 0 & 0 & 0 & 24 & 120\tau_n & 360\tau_n^2 & 840\tau_n^3 & 1680\tau_n^4 & 3024\tau_n^5\\
0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{1}{5}} & 360\tau^2_{n+\frac{1}{5}} & 840\tau^3_{n+\frac{1}{5}} & 1680\tau^4_{n+\frac{1}{5}} & 3024\tau^5_{n+\frac{1}{5}}\\
0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{2}{5}} & 360\tau^2_{n+\frac{2}{5}} & 840\tau^3_{n+\frac{2}{5}} & 1680\tau^4_{n+\frac{2}{5}} & 3024\tau^5_{n+\frac{2}{5}}\\
0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{3}{5}} & 360\tau^2_{n+\frac{3}{5}} & 840\tau^3_{n+\frac{3}{5}} & 1680\tau^4_{n+\frac{3}{5}} & 3024\tau^5_{n+\frac{3}{5}}\\
0 & 0 & 0 & 0 & 24 & 120\tau_{n+\frac{4}{5}} & 360\tau^2_{n+\frac{4}{5}} & 840\tau^3_{n+\frac{4}{5}} & 1680\tau^4_{n+\frac{4}{5}} & 3024\tau^5_{n+\frac{4}{5}}\\
0 & 0 & 0 & 0 & 24 & 120\tau_{n+1} & 360\tau_{n+1}^2 & 840\tau_{n+1}^3 & 1680\tau_{n+1}^4 & 3024\tau_{n+1}^5
\end{bmatrix}$$

$$X=\begin{bmatrix}\sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9\end{bmatrix}^T$$

$$Z=\begin{bmatrix}y_n & y_{n+\frac{1}{5}} & y_{n+\frac{2}{5}} & y_{n+\frac{3}{5}} & f_n & f_{n+\frac{1}{5}} & f_{n+\frac{2}{5}} & f_{n+\frac{3}{5}} & f_{n+\frac{4}{5}} & f_{n+1}\end{bmatrix}^T$$

The unknown values of $\sigma_j's,\ j=0(1)9$ in (7) can be obtained by using Gaussian elimination method and these values are substituted back into equation (4) to produce a continuous implicit scheme with derivatives of the form:

$$y(\xi)=\alpha_0(\xi)y_n+\alpha_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}}+\alpha_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}}+\alpha_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}}+h^4\left[\beta_0(\xi)f_n+\beta_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}}+\beta_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}}+\beta_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}}+\beta_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}}+\beta_1(\xi)f_{n+1}\right] \tag{8}$$

The coefficient of $\alpha_0,\alpha_{\frac{1}{5}},\alpha_{\frac{2}{5}},\alpha_{\frac{3}{5}},y_n,y_{n+\frac{1}{5}},y_{n+\frac{2}{5}},y_{n+\frac{3}{5}},f_n,f_{n+\frac{1}{5}},f_{n+\frac{2}{5}},f_{n+\frac{3}{5}},f_{n+\frac{4}{5}},f_{n+1}$ are obtained in terms of ξ as

$$\left.\begin{aligned}
\alpha_0 &= 1-\frac{55}{6}\tau+25\tau^2-\frac{125}{6}\tau^3\\
\alpha_{\frac{1}{5}} &= 15\tau-\frac{125}{2}\tau^2+\frac{125}{2}\tau^3\\
\alpha_{\frac{2}{5}} &= -\frac{15}{2}\tau+50\tau^2-\frac{125}{2}\tau^3\\
\alpha_{\frac{3}{5}} &= \frac{3}{5}\tau-\frac{25}{2}\tau^2+\frac{125}{6}\tau^3\\
\beta_0 &= -\frac{937}{12600000}\tau+\frac{1411}{1008000}\tau^2-\frac{19151}{1814400}\tau^3+\frac{1}{24}\tau^4-\frac{137}{1440}\tau^5+\frac{25}{192}\tau^6-\frac{425}{4032}\tau^7+\frac{125}{2688}\tau^8-\frac{625}{1008}\tau^9\\
\beta_{\frac{1}{5}} &= -\frac{19}{13125}\tau+\frac{3091}{216000}\tau^2-\frac{73967}{1814400}\tau^3+\frac{5}{24}\tau^5-\frac{385}{864}\tau^6+\frac{1775}{4032}\tau^7-\frac{125}{576}\tau^8+\frac{3125}{1008}\tau^9\\
\beta_{\frac{2}{5}} &= -\frac{599}{1260000}\tau+\frac{2831}{1512000}\tau^2+\frac{1261}{181440}\tau^3-\frac{5}{25}\tau^5+\frac{535}{864}\tau^6-\frac{1475}{2016}\tau^7+\frac{1625}{4032}\tau^8-\frac{3125}{504}\tau^9\\
\beta_{\frac{3}{5}} &= -\frac{1}{90000}\tau+\frac{143}{126000}\tau^2-\frac{7439}{90720}\tau^3+\frac{5}{96}\tau^5-\frac{65}{144}\tau^6+\frac{175}{288}\tau^7-\frac{125}{336}\tau^8+\frac{3125}{504}\tau^9\\
\beta_{\frac{4}{5}} &= \frac{3}{280000}\tau-\frac{1391}{3024000}\tau^2+\frac{5549}{1814400}\tau^3-\frac{5}{96}\tau^5+\frac{305}{1728}\tau^6-\frac{1025}{4032}\tau^7+\frac{1375}{8064}\tau^8-\frac{3125}{1008}\tau^9\\
\beta_1 &= -\frac{1}{450000}\tau+\frac{23}{302400}\tau^2-\frac{883}{1814400}\tau^3+\frac{1}{120}\tau^5-\frac{25}{864}\tau^6+\frac{25}{576}\tau^7-\frac{125}{4032}\tau^8+\frac{625}{1008}\tau^9
\end{aligned}\right\}\qquad (9)$$

Evaluating (8) at non interpolating point to obtain the continuous form as

$$\left.\begin{aligned}
y_{n+\frac{4}{5}} &= -y_n+4y_{n+\frac{1}{5}}-6y_{n+\frac{2}{5}}+4y_{n+\frac{3}{5}}+h^4\left(-\frac{1}{450000}f_n+\frac{31}{112500}f_{n+\frac{1}{5}}+\frac{79}{75000}f_{n+\frac{2}{5}}+\frac{31}{112500}f_{n+\frac{3}{5}}-\frac{1}{450000}f_{n+\frac{4}{5}}\right)\\
y_{n+1} &= -4y_n+15y_{n+\frac{1}{5}}-20y_{n+\frac{2}{5}}+10y_{n+\frac{3}{5}}+h^4\left(-\frac{1}{112500}f_n+\frac{11}{1000}f_{n+\frac{1}{5}}+\frac{101}{22500}f_{n+\frac{2}{5}}+\frac{97}{45000}f_{n+\frac{3}{5}}+\frac{1}{3750}f_{n+\frac{4}{5}}-\frac{1}{450000}f_{n+1}\right)
\end{aligned}\right\}\qquad (10)$$

The first, second and third derivative of (8) is given by

$$y'(\xi)=\alpha'_0(\xi)y_n+\alpha'_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}}+\alpha'_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}}+\alpha'_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}}+h^4\left[\beta'_0(\xi)f_n+\beta'_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}}+\beta'_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}}+\beta'_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}}+\beta'_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}}+\beta'_1(\xi)f_{n+1}\right]\qquad (11)$$

$$y''(\xi)=\alpha''_0(\xi)y_n+\alpha''_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}}+\alpha''_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}}+\alpha''_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}}+h^4\left[\beta''_0(\xi)f_n+\beta''_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}}+\beta''_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}}+\beta''_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}}+\beta''_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}}+\beta''_1(\xi)f_{n+1}\right]\qquad (12)$$

$$y'''(\xi)=\alpha'''_0(\xi)y_n+\alpha'''_{\frac{1}{5}}(\xi)y_{n+\frac{1}{5}}+\alpha'''_{\frac{2}{5}}(\xi)y_{n+\frac{2}{5}}+\alpha'''_{\frac{3}{5}}(\xi)y_{n+\frac{3}{5}}+h^4\left[\beta'''_0(\xi)f_n+\beta'''_{\frac{1}{5}}(\xi)f_{n+\frac{1}{5}}+\beta'''_{\frac{2}{5}}(\xi)f_{n+\frac{2}{5}}+\beta'''_{\frac{3}{5}}(\xi)f_{n+\frac{3}{5}}+\beta'''_{\frac{4}{5}}(\xi)f_{n+\frac{4}{5}}+\beta'''_1(\xi)f_{n+1}\right]\qquad (13)$$

Evaluating equations (11) to (12) of continuous scheme at all the points $0\left(\frac{1}{5}\right)1$ to gives discrete scheme and its derivatives can be written explicit in the block form as

$$\left.\begin{aligned}
y'_n &= -\frac{55}{6}y_n+15y_{n+\frac{1}{5}}-15y_{n+\frac{2}{5}}+\frac{3}{5}y_{n+\frac{3}{5}}+h^3\left(-\frac{937}{12600000}f_n-\frac{19}{13125}f_{n+\frac{1}{5}}-\frac{599}{1260000}f_{n+\frac{2}{5}}-\frac{1}{90000}f_{n+\frac{3}{5}}+\frac{3}{280000}f_{n+\frac{4}{5}}-\frac{1}{450000}f_{n+1}\right)\\
y'_{n+\frac{1}{5}} &= -\frac{5}{3}y_n-\frac{5}{2}y_{n+\frac{1}{5}}+5y_{n+\frac{2}{5}}+\frac{5}{6}y_{n+\frac{3}{5}}+h^3\left(-\frac{1}{675000}f_n+\frac{2809}{7560000}f_{n+\frac{1}{5}}+\frac{43}{126000}f_{n+\frac{2}{5}}-\frac{229}{3780000}f_{n+\frac{3}{5}}+\frac{1}{54000}f_{n+\frac{4}{5}}-\frac{11}{4200000}f_{n+1}\right)\\
y'_{n+\frac{2}{5}} &= -\frac{5}{6}y_n-5y_{n+\frac{1}{5}}+\frac{5}{2}y_{n+\frac{2}{5}}+\frac{5}{3}y_{n+\frac{3}{5}}+h^3\left(\frac{169}{37800000}f_n-\frac{311}{1260000}f_{n+\frac{1}{5}}-\frac{353}{756000}f_{n+\frac{2}{5}}+\frac{1}{16875}f_{n+\frac{3}{5}}-\frac{7}{360000}f_{n+\frac{4}{5}}+\frac{53}{18900000}f_{n+1}\right)\\
y'_{n+\frac{3}{5}} &= -\frac{5}{3}y_n+\frac{15}{2}y_{n+\frac{1}{5}}-15y_{n+\frac{2}{5}}+\frac{55}{6}y_{n+\frac{3}{5}}+h^3\left(-\frac{41}{6300000}f_n+\frac{173}{360000}f_{n+\frac{1}{5}}+\frac{11}{7500}f_{n+\frac{2}{5}}+\frac{61}{1260000}f_{n+\frac{3}{5}}+\frac{17}{1260000}f_{n+\frac{4}{5}}-\frac{11}{4200000}f_{n+1}\right)\\
y'_{n+\frac{4}{5}} &= -\frac{55}{6}y_n+35y_{n+\frac{1}{5}}-\frac{95}{2}y_{n+\frac{2}{5}}+\frac{65}{3}y_{n+\frac{3}{5}}+h^3\left(-\frac{671}{37800000}f_n+\frac{169}{67500}f_{n+\frac{1}{5}}+\frac{12821}{1260000}f_{n+\frac{2}{5}}+\frac{7447}{1890000}f_{n+\frac{3}{5}}+\frac{509}{7560000}f_{n+\frac{4}{5}}-\frac{1}{450000}f_{n+1}\right)\\
y'_{n+1} &= -\frac{65}{3}y_n+\frac{155}{2}y_{n+\frac{1}{5}}+95y_{n+\frac{2}{5}}+\frac{235}{6}y_{n+\frac{3}{5}}+h^3\left(\frac{31}{675000}f_n+\frac{1663}{280000}f_{n+\frac{1}{5}}+\frac{6847}{270000}f_{n+\frac{2}{5}}+\frac{60863}{3780000}f_{n+\frac{3}{5}}+\frac{2473}{630000}f_{n+\frac{4}{5}}+\frac{2041}{37800000}f_{n+1}\right)
\end{aligned}\right\}\qquad (14)$$

$$\left.\begin{aligned}
y''_{n} &= 50y_n + 125y_{n+\frac{1}{5}} + 100y_{n+\frac{2}{5}} - 25y_{n+\frac{3}{5}} + h^2\left(\frac{1411}{50400}f_n + \frac{30921}{108000}f_{n+\frac{1}{5}} + \frac{2831}{756000}f_{n+\frac{2}{5}} + \frac{143}{63000}f_{n+\frac{3}{5}} - \frac{1391}{1512000}f_{n+\frac{4}{5}} + \frac{23}{151200}f_{n+1}\right)\\
y''_{n+\frac{1}{5}} &= 25y_n - 50y_{n+\frac{1}{5}} + 25y_{n+\frac{2}{5}} + h^2\left(-\frac{73}{756000}f_n - \frac{1601}{504000}f_{n+\frac{1}{5}} + \frac{1}{189000}f_{n+\frac{2}{5}} - \frac{11}{108000}f_{n+\frac{3}{5}} + \frac{11}{252000}f_{n+\frac{4}{5}} - \frac{11}{1512000}f_{n+1}\right)\\
y''_{n+\frac{2}{5}} &= 25y_{n+\frac{1}{5}} - 50y_{n+\frac{2}{5}} + 25y_{n+\frac{3}{5}} + h^2\left(\frac{11}{151200}f_n - \frac{53}{378000}f_{n+\frac{1}{5}} - \frac{773}{252000}f_{n+\frac{2}{5}} - \frac{53}{37800}f_{n+\frac{3}{5}} + \frac{11}{1512000}f_{n+\frac{4}{5}}\right)\\
y''_{n+\frac{3}{5}} &= 25y_n + 100y_{n+\frac{1}{5}} - 125y_{n+\frac{2}{5}} + 50y_{n+\frac{3}{5}} + h^2\left(-\frac{1}{18000}f_n + \frac{10427}{1512000}f_{n+\frac{1}{5}} + \frac{9901}{378000}f_{n+\frac{2}{5}} + \frac{107}{28000}f_{n+\frac{3}{5}} - \frac{37}{189000}f_{n+\frac{4}{5}} + \frac{11}{1512000}f_{n+1}\right)\\
y''_{n+\frac{4}{5}} &= -50y_n + 175y_{n+\frac{1}{5}} - 200y_{n+\frac{2}{5}} + 75y_{n+\frac{3}{5}} + h^2\left(-\frac{179}{1512000}f_n + \frac{3469}{252000}f_{n+\frac{1}{5}} + \frac{6421}{108000}f_{n+\frac{2}{5}} + \frac{3791}{94500}f_{n+\frac{3}{5}} + \frac{121}{33600}f_{n+\frac{4}{5}} + \frac{23}{151200}f_{n+1}\right)\\
y''_{n+1} &= -75y_n + 250y_{n+\frac{1}{5}} - 275y_{n+\frac{2}{5}} + 100y_{n+\frac{3}{5}} + h^2\left(-\frac{11}{756000}f_n + \frac{29689}{1512000}f_{n+\frac{1}{5}} + \frac{499}{5250}f_{n+\frac{2}{5}} + \frac{58271}{756000}f_{n+\frac{3}{5}} + \frac{4561}{108000}f_{n+\frac{4}{5}} + \frac{271}{100800}f_{n+1}\right)
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
y'''_{n} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{19151}{302400}f_n - \frac{73967}{302400}f_{n+\frac{1}{5}} + \frac{1261}{30240}f_{n+\frac{2}{5}} - \frac{7439}{151200}f_{n+\frac{3}{5}} + \frac{5549}{302400}f_{n+\frac{4}{5}} - \frac{883}{302400}f_{n+1}\right)\\
y'''_{n+\frac{1}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(\frac{799}{302400}f_n - \frac{14033}{302400}f_{n+\frac{1}{5}} - \frac{10453}{151200}f_{n+\frac{2}{5}} + \frac{2683}{151200}f_{n+\frac{3}{5}} + \frac{1717}{302400}f_{n+\frac{4}{5}} + \frac{251}{302400}f_{n+1}\right)\\
y'''_{n+\frac{2}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{67}{60480}f_n + \frac{12721}{302400}f_{n+\frac{1}{5}} + \frac{11009}{151200}f_{n+\frac{2}{5}} - \frac{547}{30240}f_{n+\frac{3}{5}} + \frac{1517}{302400}f_{n+\frac{4}{5}} - \frac{211}{302400}f_{n+1}\right)\\
y'''_{n+\frac{3}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{127}{302400}f_n + \frac{1763}{60480}f_{n+\frac{1}{5}} + \frac{27851}{151200}f_{n+\frac{2}{5}} + \frac{14107}{151200}f_{n+\frac{3}{5}} - \frac{2389}{302400}f_{n+\frac{4}{5}} + \frac{251}{302400}f_{n+1}\right)\\
y'''_{n+\frac{4}{5}} &= -125y_n + 375y_{n+\frac{1}{5}} - 375y_{n+\frac{2}{5}} + 125y_{n+\frac{3}{5}} + h\left(-\frac{67}{60480}f_n + \frac{12049}{302400}f_{n+\frac{1}{5}} + \frac{22433}{151200}f_{n+\frac{2}{5}} + \frac{35569}{151200}f_{n+\frac{3}{5}} + \frac{4873}{60480}f_{n+\frac{4}{5}} - \frac{883}{302400}f_{n+1}\right)
\end{aligned}\right\} \quad (16)$$

Equations (10), (14) to (16) are combined together in matrix form and by using matrix inversion to gives the following new schemes as explicit block hybrid method as

$$\begin{aligned}
y_{n+\frac{1}{5}} = {} & y_n + \frac{1}{5}h^1 y'_n + \frac{1}{50}h^2 y''_n + \frac{1}{750}h^3 y'''_n + \frac{3509}{81000000}h^4 f_n + \frac{9809}{226800000}h^4 f_{n+\frac{1}{5}}\\
& - \frac{251}{7087500}h^4 f_{n+\frac{2}{5}} + \frac{2543}{113400000}h^4 f_{n+\frac{3}{5}} - \frac{133}{16200000}h^4 f_{n+\frac{4}{5}} + \frac{1469}{1134000000}h^4 f_{n+1}
\end{aligned}$$

$$\begin{aligned}
y_{n+\frac{2}{5}} = {} & y_n + \frac{2}{5}h^1 y'_n + \frac{2}{25}h^2 y''_n + \frac{4}{375}h^3 y'''_n + \frac{4264}{8859375}h^4 f_n + \frac{1592}{1771875}h^4 f_{n+\frac{1}{5}}\\
& - \frac{982}{1771875}h^4 f_{n+\frac{2}{5}} + \frac{88}{253125}h^4 f_{n+\frac{3}{5}} - \frac{32}{253125}h^4 f_{n+\frac{4}{5}} + \frac{176}{8859375}h^4 f_{n+1}
\end{aligned}$$

$$\begin{aligned}
y_{n+\frac{3}{5}} = {} & y_n + \frac{3}{5}h^1 y'_n + \frac{9}{50}h^2 y''_n + \frac{9}{250}h^3 y'''_n + \frac{1593}{875000}h^4 f_n + \frac{1809}{400000}h^4 f_{n+\frac{1}{5}}\\
& - \frac{189}{100000}h^4 f_{n+\frac{2}{5}} + \frac{1917}{1400000}h^4 f_{n+\frac{3}{5}} - \frac{351}{700000}h^4 f_{n+\frac{4}{5}} + \frac{1107}{14000000}h^4 f_{n+1}
\end{aligned}$$

$$\begin{aligned}
y_{n+\frac{4}{5}} = {} & y_n + \frac{4}{5}h^1 y'_n + \frac{8}{25}h^2 y''_n + \frac{32}{375}h^3 y'''_n + \frac{40448}{8859375}h^4 f_n + \frac{3328}{253125}h^4 f_{n+\frac{1}{5}}\\
& - \frac{5888}{1771875}h^4 f_{n+\frac{2}{5}} + \frac{6656}{1771875}h^4 f_{n+\frac{3}{5}} - \frac{2272}{1771875}h^4 f_{n+\frac{4}{5}} + \frac{256}{1265625}h^4 f_{n+1}
\end{aligned}$$

$$\begin{aligned}
y_{n+1} = {} & y_n + h^1 y'_n + \frac{1}{2}h^2 y''_n + \frac{1}{6}h^3 y'''_n + \frac{239}{25920}h^4 f_n + \frac{2105}{72576}h^4 f_{n+\frac{1}{5}} - \frac{5}{1296}h^4 f_{n+\frac{2}{5}}\\
& + \frac{335}{36288}h^4 f_{n+\frac{3}{5}} - \frac{85}{36288}h^4 f_{n+\frac{4}{5}} + \frac{149}{362880}h^4 f_{n+1}
\end{aligned}$$

$$y'_{n+\frac{1}{5}} = y'_n + \frac{1}{5}h^1 y''_n + \frac{1}{50}h^2 y'''_n + \frac{3929}{5040000}h^3 f_n + \frac{199}{201600}h^3 f_{n+\frac{1}{5}} - \frac{1931}{2520000}h^3 f_{n+\frac{2}{5}}$$

$$+\frac{173}{360000}\,h^3 f_{n+\frac{3}{5}}-\frac{883}{5040000}\,h^3 f_{n+\frac{4}{5}}+\frac{139}{5040000}\,h^3 f_{n+1}$$

$$y'_{n+\frac{2}{5}}=y'_n+\frac{2}{5}\,h^1 y''_n+\frac{2}{25}\,h^2 y'''_n+\frac{317}{78750}\,h^3 f_n+\frac{367}{39375}\,h^3 f_{n+\frac{1}{5}}-\frac{38}{7875}\,h^3 f_{n+\frac{2}{5}}+$$

$$\frac{122}{39375}\,h^3 f_{n+\frac{3}{5}}-\frac{89}{78750}\,h^3 f_{n+\frac{4}{5}}+\frac{1}{5625}\,h^3 f_{n+1}$$

$$y'_{n+\frac{3}{5}}=y'_n+\frac{3}{5}\,h^1 y''_n+\frac{9}{50}\,h^2 y'''_n+\frac{783}{80000}\,h^3 f_n+\frac{16119}{560000}\,h^3 f_{n+\frac{1}{5}}-\frac{2187}{280000}\,h^3 f_{n+\frac{2}{5}}$$

$$+\frac{423}{56000}\,h^3 f_{n+\frac{3}{5}}-\frac{1539}{560000}\,h^3 f_{n+\frac{4}{5}}+\frac{243}{560000}\,h^3 f_{n+1}$$

$$y'_{n+\frac{4}{5}}=y'_n+\frac{4}{5}\,h^1 y''_n+\frac{8}{25}\,h^2 y'''_n+\frac{712}{39375}\,h^3 f_n+\frac{2336}{39375}\,h^3 f_{n+\frac{1}{5}}-\frac{32}{5625}\,h^3 f_{n+\frac{2}{5}}$$

$$+\frac{704}{39375}\,h^3 f_{n+\frac{3}{5}}-\frac{8}{1575}\,h^3 f_{n+\frac{4}{5}}+\frac{32}{39375}\,h^3 f_{n+1}$$

$$y'_{n+1}=y'_n+1\,h^1 y''_n+\frac{1}{2}\,h^2 y'''_n+\frac{233}{8064}\,h^3 f_n$$

$$+\frac{815}{8064}\,h^3 f_{n+\frac{1}{5}}+\frac{5}{4032}\,h^3 f_{n+\frac{2}{5}}+\frac{155}{4032}\,h^3 f_{n+\frac{3}{5}}$$

$$-\frac{5}{1152}\,h^3 f_{n+\frac{4}{5}}+\frac{11}{8064}\,h^3 f_{n+1}$$

$$y''_{\mathrm{n}+\frac{1}{5}}=y''_n+\frac{1}{5}\,h^1 y'''_n+\frac{1231}{126000}\,h^2 f_n+\frac{863}{50400}\,h^2 f_{n+\frac{1}{5}}-\frac{761}{63000}\,h^2 f_{n+\frac{2}{5}}$$

$$+\frac{941}{126000}\,h^2 f_{n+\frac{3}{5}}$$

$$-\frac{341}{126000}\,h^2 f_{n+\frac{4}{5}}+\frac{107}{252000}\,h^2 f_{n+1}$$

$$y''_{\mathrm{n}+\frac{2}{5}}=y''_n+\frac{2}{5}\,h^1 y'''_n+\frac{71}{3150}\,h^2 f_n+\frac{544}{7875}\,h^2 f_{n+\frac{1}{5}}-\frac{37}{1575}\,h^2 f_{n+\frac{2}{5}}+\frac{136}{7875}\,h^2 f_{n+\frac{3}{5}}$$

$$-\,h^2 f_{n+\frac{4}{5}}+\frac{101}{15750}\,h^2 f_{n+1}$$

$$y''_{\mathrm{n}+\frac{3}{5}}=y''_n+\frac{3}{5}\,h^1 y'''_n+\frac{123}{3500}\,h^2 f_n+\frac{3501}{28000}\,h^2 f_{n+\frac{1}{5}}-\frac{9}{3500}\,h^2 f_{n+\frac{2}{5}}+\frac{87}{2800}\,h^2 f_{n+\frac{3}{5}}$$

$$-\frac{9}{875}\,h^2 f_{n+\frac{4}{5}}+\frac{9}{5600}\,h^2 f_{n+1}$$

$$y''_{\mathrm{n}+\frac{4}{5}}=y''_n+\frac{4}{5}\,h^1 y'''_n+\frac{376}{7875}\,h^2 f_n+\frac{1424}{7875}\,h^2 f_{n+\frac{1}{5}}+\frac{176}{7875}\,h^2 f_{n+\frac{2}{5}}+\frac{608}{7875}\,h^2 f_{n+\frac{3}{5}}$$

$$-\frac{16}{1575}\,h^2 f_{n+\frac{4}{5}}+\frac{16}{7875}\,h^2 f_{n+1}$$

$$y''_{\mathrm{n}+1}=y''_n+h^1 y'''_n+\frac{61}{1008}\,h^2 f_n+\frac{475}{2016}\,h^2 f_{n+\frac{1}{5}}+\frac{25}{504}\,h^2 f_{n+\frac{2}{5}}+\frac{125}{1008}\,h^2 f_{n+\frac{3}{5}}$$

$$+\frac{25}{1008}\,h^2 f_{n+\frac{4}{5}}+\frac{11}{2016}\,h^2 f_{n+1}$$

$$
\begin{aligned}
y'''_{n+\frac{1}{5}} = {} & y'''_{n} + \frac{19}{288} h^{1} f_{n} + \frac{1427}{7200} h^{1} f_{n+\frac{1}{5}} \\
& - \frac{133}{1200} h^{1} f_{n+\frac{2}{5}} + \frac{241}{3600} h^{1} f_{n+\frac{3}{5}} - \frac{173}{7200} h^{1} f_{n+\frac{4}{5}} \\
& + \frac{3}{800} h^{1} f_{n+1} \\
y'''_{n+\frac{2}{5}} = {} & y'''_{n} + \frac{14}{225} h^{1} f_{n} + \frac{43}{150} h^{1} f_{n+\frac{1}{5}} + \frac{7}{225} h^{1} f_{n+\frac{2}{5}} + \frac{7}{225} h^{1} f_{n+\frac{3}{5}} - \frac{1}{75} h^{1} f_{n+\frac{4}{5}} \\
& + \frac{1}{450} h^{1} f_{n+1} \\
y'''_{n+\frac{3}{5}} = {} & y'''_{n} + \frac{51}{800} h^{1} f_{n} + \frac{219}{800} h^{1} f_{n+\frac{1}{5}} + \frac{57}{400} h^{1} f_{n+\frac{2}{5}} + \frac{57}{400} h^{1} f_{n+\frac{3}{5}} - \frac{21}{800} h^{1} f_{n+\frac{4}{5}} \\
& + \frac{3}{800} h^{1} f_{n+1} \\
y'''_{n+\frac{4}{5}} = {} & y'''_{n} + \frac{14}{225} h^{1} f_{n} + \frac{64}{225} h^{1} f_{n+\frac{1}{5}} + \frac{8}{75} h^{1} f_{n+\frac{2}{5}} + \frac{64}{225} h^{1} f_{n+\frac{3}{5}} + \frac{14}{225} h^{1} f_{n+\frac{4}{5}} \\
y'''_{n+1} = {} & y'''_{n} + \frac{19}{288} h^{1} f_{n} + \frac{25}{96} h^{1} f_{n+\frac{1}{5}} + \frac{25}{144} h^{1} f_{n+\frac{2}{5}} + \frac{25}{144} h^{1} f_{n+\frac{3}{5}} + \frac{25}{96} h^{1} f_{n+\frac{4}{5}} \\
& + \frac{19}{288} h^{1} f_{n+1}
\end{aligned} \tag{17}
$$

Basic Properties of the New Method

The basic properties of one-step block hybrid methods (were numerically analyzed. These properties include; order, error constant, consistency and zero-stability, which reveal the nature of convergence of the methods. The region of absolute stability of the methods will also been obtained.

Order and Error Constant of the New Method

Consider the linear operator associated in L associated with the new method in be defined as

$$L[y(x);h] = \sum_{j=0}^{k} \left\{ \alpha_j y(x_n + jh) - \alpha_{vi} y(x_n + vih) - h^d \beta_j y^d(x_n + jh) - h^d \beta_{vi} y^d(x_n + vih) \right\} \tag{18}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. We expand $y(x_n + jh)$ and $y^d(x_n + jh)$ using a Taylor series about x_n and collecting like terms in h and y to obtain the expression;

$$\ell\{y(x);h\} = C_0 y(x) + C_1 y'(x) + \ldots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + \ldots \tag{19}$$

We consider the linear operator $L[y(t_n);h]$ of the new method with the corollary 1 and 2 below to determining the order and error constant of the new method (Adewale & Sabo, 2024).

Corollary 1

The linear operator $L[y(t_n);h]$ associate with the local truncation error of the method in equation is $C_{06} h^{06} y^{06}(t_n) + 0(h^{10})$

Proof

The linear difference operators associated with the new method are given

$$
\left.
\begin{aligned}
L[y(t_n);h] &= y\left(t_n + \frac{1}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n + \frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n + \frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n + \frac{3}{5}h\right) + h^4 \sum_{i=0}^{\eta} \left(\beta_i(t) f_{n+i} + \beta_\eta(t) f_{n+\eta}\right)\right) \\
L[y(t_n);h] &= y\left(t_n + \frac{2}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n + \frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n + \frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n + \frac{3}{5}h\right) + h^4 \sum_{i=0}^{\eta} \left(\beta_i(t) f_{n+i} + \beta_\eta(t) f_{n+\eta}\right)\right) \\
L[y(t_n);h] &= y\left(t_n + \frac{3}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n + \frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n + \frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n + \frac{3}{5}h\right) + h^4 \sum_{i=0}^{\eta} \left(\beta_i(t) f_{n+i} + \beta_\eta(t) f_{n+\eta}\right)\right) \\
L[y(t_n);h] &= y\left(t_n + \frac{4}{5}h\right) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n + \frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n + \frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n + \frac{3}{5}h\right) + h^4 \sum_{i=0}^{\eta} \left(\beta_i(t) f_{n+i} + \beta_\eta(t) f_{n+\eta}\right)\right) \\
L[y(t_n);h] &= y(t_n + h) - \left(\alpha_0(x_n) + \alpha_{\frac{1}{5}}\left(x_n + \frac{1}{5}h\right) + \alpha_{\frac{2}{5}}\left(t_n + \frac{2}{5}h\right) + \alpha_{\frac{3}{5}}\left(t_n + \frac{3}{5}h\right) + h^4 \sum_{i=0}^{\eta} \left(\beta_i(t) f_{n+i} + \beta_\eta(t) f_{n+\eta}\right)\right)
\end{aligned}
\right\} \tag{20}
$$

Corollary 2

The local truncation error of the new method is assume $y(t)$ to be sufficiently differentiable and expanding $y(t_n + qh)$ and $y(t_n + jh)$ about t_n using Taylor series, have

$$L_{\frac{1}{5}}[y(t_n);h] = (-6.5552\times 10^{-11}),\quad L_{\frac{2}{5}}[y(t_n);h] = (-9.8969\times 10^{-10}),\quad L_{\frac{3}{5}}[y(t_n);h] = (-3.9703\times 10^{-09}),$$

$$L_{\frac{4}{5}}[y(t_n);h] = (-1.0171\times 10^{-08}),\quad L_{1}[y(t_n);h] = (-2.0723\times 10^{-08})$$

Proof

Expand equation (20) using corollary 2 and then collect the like terms to the power of h gives

$$L_{\frac{1}{5}}[y(t_n);h] = (-6.5552\times 10^{-11})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{\frac{2}{5}}[y(t_n);h] = (-9.8969\times 10^{-10})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{\frac{3}{5}}[y(t_n);h] = (-3.9703\times 10^{-09})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{\frac{4}{5}}[y(t_n);h] = (-1.0171\times 10^{-08})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

$$L_{1}[y(t_n);h] = (-2.0723\times 10^{-08})C_{06}h^{06}y^{06}(t_n) + 0(h^{10})$$

Consistency of the New Method

The numerical method is said to be consistent, if its order is greater than or equal to zero. The new method is consistent since it of uniform order six (Sunday, 2018).

Zero Stability of the New Method

Definition 1**:** A numerical method is said to be zero-stable if the roots $z_s,\ s = 1,2,\ldots,n$ of the first characteristic polynomial $\overline{\rho}(z)$, defined by

$$\overline{\rho}(z) = \det\left[zA^{(0)} - E\right] \tag{21}$$

satisfies $|z_s| \le 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding the order of the differential equation as $h \to 0$. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}\ and\ E$. The main consequence of zero-stability is to control the propagation of the error as the integration proceeds (Adewale & Sabo, 2024).

To determine the zero-stability of the new method, we applying definition 1 on the new method, with the first characteristic polynomial given by

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} z & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & -1 \\ 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & z-1 \end{bmatrix} \right| = z^4(z-1)$$

Solving for z in

$$z^4(z-1) \tag{22}$$

Gives $z = 0, 0, 0, 1$. Hence, the new method is zero-stable.

Convergence of the New Method

The new method is convergent, since it satisfied consistent and zero stable according to Dhalquist theorem (Sunday, 2018).

Region of Absolute Stability (RAS) of the New Method

Definition 2**:** The region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y^d = -\lambda^d y$ satisfy $y_j \to 0\ as\ j \to \infty$ for any initial condition (Adewale & Sabo, 2024). To determine the regions of absolute stability of K-step method, a method that requires neither the computation of roots of a polynomial nor solving of simultaneous inequalities was adopted. This method is called the Boundary Locus Method (BLM).

Applying the Boundary Locus Method on the new method, we obtained the stability polynomial a

$$\bar{h}(w)=\left(-\frac{1}{56250}w^4-\frac{4}{84375}w^5\right)h^5+\left(-\frac{79}{112500}w^4+\frac{1121}{1012500}w^5\right)h^4+\left(-\frac{749}{45000}w^4-\frac{17857}{1215000}w^5\right)h^3+\left(-\frac{81}{500}w^4+\frac{781}{6750}w^5\right)h^2+\left(-\frac{1}{2}w^4-\frac{1}{2}w^5\right)h-w^4+w^5$$

(23)

Using the stability polynomial (23), the region of absolute stability of the new method is obtained as

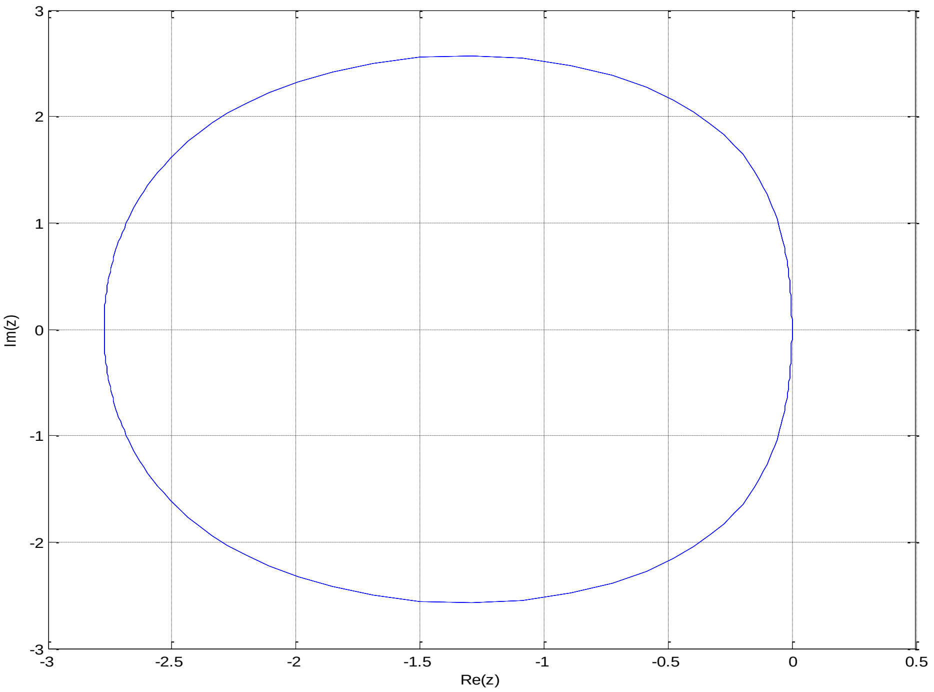

Figure 1: Showing an A_α-stable Region of Absolute Stability of the New Method

Numerical Examples

The new block hybrid method was used to implement on second, third and fourth order initial value problems of ordinary differential equation of the form (1) without the need to reduce it to an equivalent system of first order ODEs. The absolute error of each approximate solution was calculated and then compared to the results from existing methods that were specifically developed by Fasasi (2018) Adoghe and Omole (2019) Skwame et al (2019) Skwame et al (2020) Sabo et al (2021) Atabo & Adee (2021) and Raymond et al (2023).

The following acronym were used in the tables and figures below:

ES: Means Exact Solution
CS: Means Computed Solution in the new method
ENM: Means Error in new method
EF18: Means Error in Fasasi, (2018),
ESe19: Means Error in Skwame et al. (2019),
ESe20: Means Error in Skwame et al. (2020),
ESe21: Means Error in Sabo et al. (2021),
EAA21: Means Error in Atabo & Adee, (2021),
ERe23: Means Error in Raymond et al. (2023).

Example 1: Consider the Second Order Dynamic Problem

A 10 kilogram mass is attached to a spring having a spring constant of $140\ N/M$. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(\tau)=5\sin t$. Find the subsequent motion of the mass $(\tau:0.10\le\tau\le1.00)$ if the force due to air resistance is $90(y'(\tau))N$.

Applying the procedure, where $m=10, k=140, a=90\ and\ F(\tau)=5\sin\tau$, example 1 reduces to

$$dsolver\left(\left\{y''(\tau)+9y'(\tau)+14y(\tau)=\frac{1}{2}\sin(\tau),\ y(0)=0,\ y'(0)=-1\right\}\right) \tag{24}$$

with the exact solution of (24) is given by,

$$y(\tau)=\frac{1}{500}\left(-90e^{-2\tau}+99e^{-7\tau}+13\sin\tau-9\cos\tau\right) \tag{25}$$

Source [Skwame et al., (2020), Sabo, et al., (2021)].

Table 1: Numerical Results for Example 1

τ	**ES**	**CS**	**ENM**	**ESe21**	**ESe20**
0.1	-0.06436205154552458248	-0.06436205154552458248	4.0884(-11)	2.0453(-10)	4.4268(-09)
0.2	-0.08430720522644774945	-0.08430720522644774945	3.2076(-11)	4.8485(-10)	2.2383(-08)
0.3	-0.08405225313390041905	-0.08405225313390041905	1.4792(-11)	6.6174(-10)	3.5865(-08)
0.4	-0.07529304213333374810	-0.07529304213333374810	1.0639(-12)	7.2649(-10)	4.2157(-08)
0.5	-0.06357063960355798563	-0.06357063960355798563	7.2717(-12)	7.1295(-10)	4.2895(-08)
0.6	-0.05142117069384508163	-0.05142117069384508163	1.1317(-11)	6.5550(-10)	4.0288(-08)
0.7	-0.03993052956438697070	-0.03993052956438697070	1.2584(-11)	5.7884(-10)	3.6051(-08)
0.8	-0.02949865862803573900	-0.02949865862803573900	1.2276(-11)	4.9808(-10)	3.1287(-08)
0.9	-0.02021269131259124546	-0.02021269131259124546	1.1192(-11)	4.2140(-10)	2.6618(-08)
1.0	-0.01202699425403169607	-0.01202699425403169607	9.8104(-12)	3.5257(-10)	2.2352(-08)

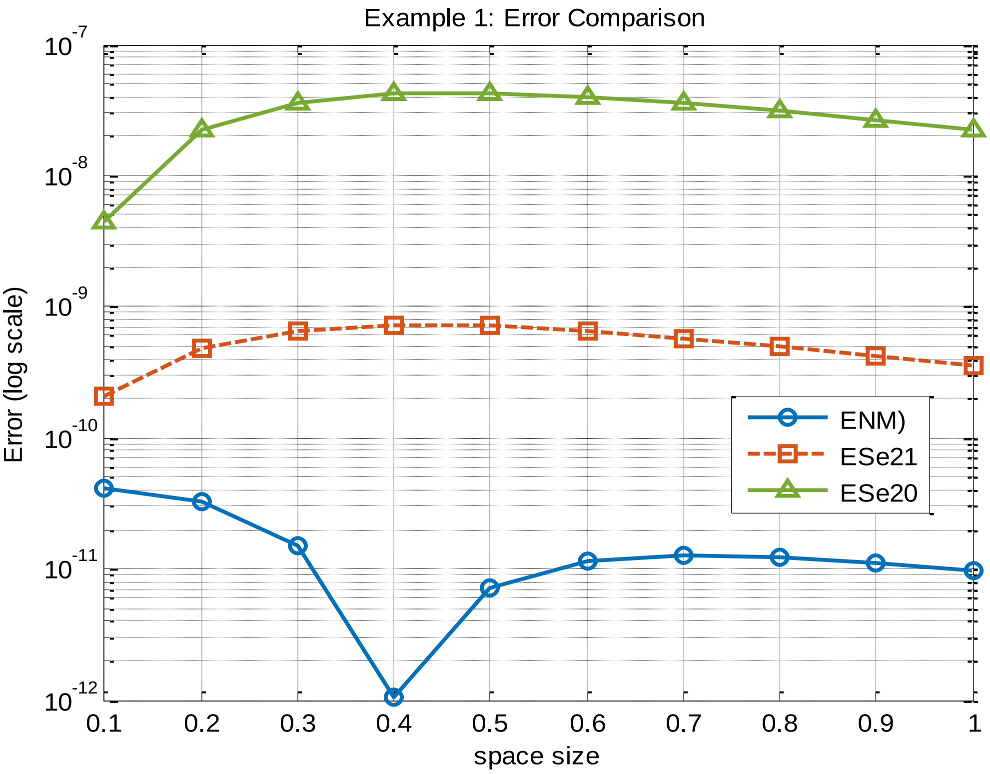

Figure 2: Graphical Curve of Table 1

Example 2: Consider the Highly non- Stiff Third Order Linear Problem

$$y'''(\tau)=3\cos(\tau)=0, \quad y(0)=1,\ y'(0)=0,\ y''(0)=2 \tag{26}$$

with the exact solution given by

$$y(\tau)=(\tau)^2-3\sin(\tau)+3\tau+1 \tag{27}$$

Source: [Fasasi (2018), Skwame *et al.* (2019)].

Table 2: Numerical Results for Example 2

τ	**ES**	**CS**	**ENM**	**ESe19**	**EF18**
0.1	1.01049975005951554310	1.01049975005951558087	3.7800(-17)	1.9700(-16)	0.0000(-00)
0.2	1.04399200761481635360	1.04399200761481636570	1.2100(-17)	1.2639(-15)	2.2205(-16)
0.3	1.10343938001598127470	1.10343938001598120254	7.2200(-17)	4.0627(-15)	8.8818(-16)
0.4	1.19174497307404852500	1.19174497307404850000	2.5000(-17)	9.4370(-15)	1.5543(-15)
0.5	1.31172338418739099920	1.31172338418739075670	2.4250(-16)	1.8205(-14)	2.8866(-15)
0.6	1.46607257981489392840	1.46607257981489399776	6.9400(-17)	3.1152(-14)	5.3291(-15)
0.7	1.65734693828692683900	1.65734693828692442654	2.4125(-15)	4.9021(-14)	7.5495(-15)
0.8	1.88793172730143171510	1.88793172730143937151	7.6564(-15)	7.2504(-14)	1.0436(-14)
0.9	2.16001927111754983460	2.16001927111754588000	3.9546(-15)	1.0224(-13)	1.4211(-14)
1.0	2.47558704557631048000	2.47558704557631388690	3.4069(-15)	1.3880(13)	1.8208(-14)

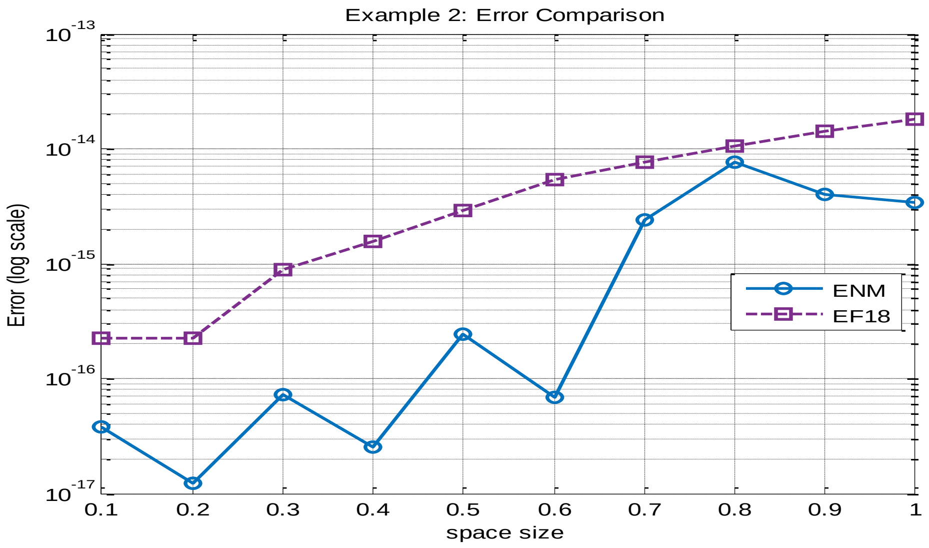

Figure 2: Graphical Curve of Table 2

Example 3: Consider the System of Oscillatory Differential Equation in a Ship Dynamics

$$y''''(\tau) = -\left[3y''(\tau) + \left(2 + \psi \cos(\Lambda \tau)\right)\right],\ y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0 \tag{28}$$

Where $\psi = 0$ for the existence of the theoretical solution,

$$y(\tau) = 2\cos\tau - \cos\left(\tau\sqrt{2}\right) \tag{29}$$

The oscillatory differential equation is applied to solve a ship dynamics. A sinusoidal wave of frequency ψ passes along a ship or offshore structure. The case study by [Raymond et al. 2023, Atabo & Adee, 2021].

Table 3: Numerical Results for Example 3

τ	**ES**	**CS**	**ENM**	**RKe23**	**EAA21**
0.003125	0.99999999999205272181	0.99999999999205272181	0.0000(00)	9.9999(-19)	0.0000(00)
0.003125	0.99999999987284392123	0.99999999987284392123	0.0000(00)	9.3300(-18)	1.1102(-16)
0.009375	0.99999999935627549414	0.99999999935627549414	0.0000(00)	6.3100(-17)	0.0000(00)
0.001250	0.99999999796552658062	0.99999999796552658062	0.0000(00)	2.2893(-16)	2.2205(-16)
0.015625	0.99999999503306753347	0.99999999503306753347	0.0000(00)	6.0454(-16)	7.7716(-16)
0.018750	0.99999998970067947569	0.99999998970067947569	0.0000(00)	1.3188(-15)	2.8867(-15)
0.021875	0.99999998091947944412	0.99999998091947944412	0.0000(00)	2.5316(-15)	8.5487(-15)
0.025000	0.99999996744995111889	0.99999996744995111889	0.0000(00)	4.4336(-15)	2.1871(-14)
0.028125	0.99999994786198113959	0.99999994786198113959	0.0000(00)	7.2464(-15)	4.9516(-14)
0.031250	0.99999992053490100516	0.99999992053490100516	0.0000(00)	1.1222(-14)	1.0358(-14)

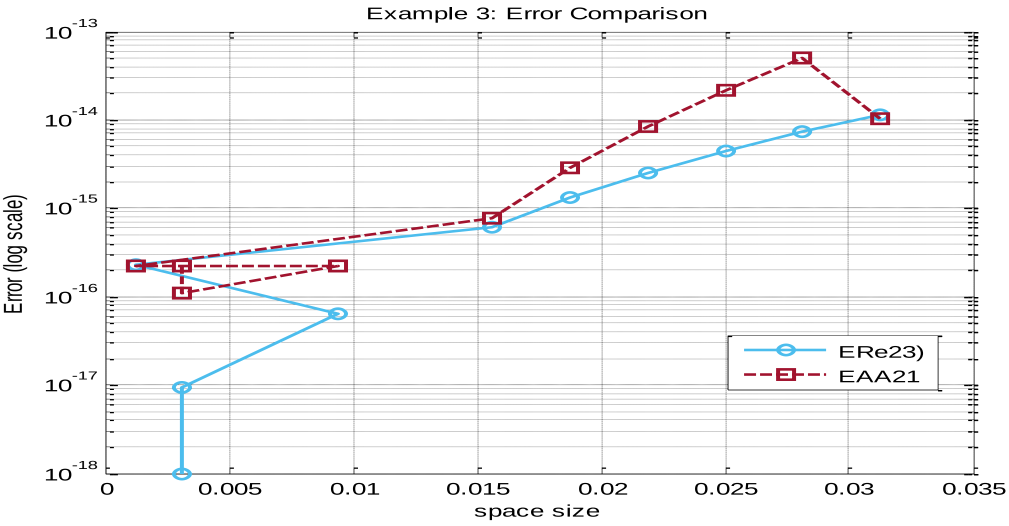

Figure 3: Graphical Curve of Table 3

RESULTS AND DISCUSSION

The newly developed block hybrid method was assessed using three different categories of initial value problems: second, third and fourth-order ordinary differential equations (ODEs). In the first test case, a second-order dynamic system involving a mass-spring setup with air resistance and an external force was considered. The results, presented in Table 1, show that the computed solutions (CS) aligned perfectly with the exact solutions (ES), indicating a high level of numerical accuracy. This consistency was also clearly illustrated in Figure 1, where the curve of the computed solution perfectly traced the exact solution, confirming the method's ability to accurately resolve second-order dynamics without transformation into a first-order system.

In the second example, a third-order linear non-stiff problem was solved to evaluate the method's performance on more complex equations. Table 2 demonstrates that the computed solutions continued to match the exact solutions very closely, even as the equation's order increased. The results validate the method's robustness and stability when applied to higher-order problems. Figure 2 visually supports this finding, with no observable deviation between the exact and computed solution curves, highlighting the method's precision and computational reliability.

The third case addressed a fourth-order oscillatory differential equation, a type often used in modeling wave behavior around ships and offshore structures. The results in Table 3 show that the computed solutions matched the exact values exactly throughout the tested interval. This accuracy was reaffirmed in Figure 3, where the solution curve followed the theoretical trajectory flawlessly. These results showcase the method's effectiveness in capturing periodic behavior accurately, which is critical in maritime and structural applications involving dynamic oscillatory forces.

Across all three examples, the tabulated data and graphical results consistently confirmed the precision of the new hybrid block method. It performed exceptionally well on problems of increasing order and complexity, including oscillatory systems. The method's ability to deliver exact numerical solutions without converting higher-order ODEs to first-order systems demonstrates its efficiency and practicality. This confirms its value as a versatile and reliable computational tool for solving complex differential equations in scientific and engineering contexts.

CONCLUSION

This study developed a one-step block hybrid method for the direct solution of second, third and fourth-order ordinary differential equations without reducing them to equivalent first-order systems. The method was formulated using power series interpolation and collocation techniques, leading to a continuous implicit scheme that was transformed into an explicit block representation. Theoretical analysis established that the method possesses uniform order six, is consistent, zero-stable, and convergent in line with the Dahlquist theorem. Furthermore, the Boundary Locus Method was employed to determine its region of absolute stability, confirming its A_α-stable property. These theoretical results demonstrate that the proposed method satisfies the fundamental requirements of an efficient and reliable numerical scheme.

The performance of the new method was validated through numerical experiments involving second-order dynamic systems, third-order non-stiff problems and fourth-order oscillatory equations arising in practical applications such as mass–spring systems and ship dynamics. In all cases, the computed solutions closely matched the exact solutions, with errors consistently smaller than those produced by several existing methods in the literature. The graphical and tabulated results confirmed its high accuracy, computational efficiency, and robustness across increasing orders of complexity. Therefore, the proposed block hybrid method provides a powerful and practical computational tool for solving higher-order initial value problems encountered in science and engineering.

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